# Invariant measures of infinite dimensional Hamiltonian systems and properties of Koopman groups

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# Plan

Our aim is to present an infinite dimentional Hamiltonian flow by a unitary group.
To this aim we

1. construct an analog of Lebesgue measure on a separable Hilbert space.

2. obtain a Koopman representation of groups of symplectomorphisms for Hamiltonian flows.

3. describe the spectrum of Koopman generator and the subspaces of strong continuity of Koopman group.

# A. Weyl theorem

**Theorem**. *If a topological group $G$ is not locally compact then there is no nontrivial $\sigma$-additive $\sigma$-finite locally finite Borel measure on the group $G$ which is left-invariant.*

Hence there is no nontrivial $\sigma$-additive $\sigma$-finite locally finite Borel shift-invariant measure on an infinite dimensional normed linear space.

There are different opportunity to introduce a measure without some property from the Weil theorem.
We consider finitely-additive measures on a Hilbert space such that these measures are invariant with respect to a group of symplectomorphisms.

# Real Hilbert space with the symplectic structure

Let $E$ be a separable real Hilbert space.
Shift-invariant symplectic form $\omega$ on the space $E$ is nondegenered skew-symmetric bilinear form on $E$.
There is an ONB $\mathcal{E} = \{e_k\}$ in the space $E$ (symplectic ONB) such that

$$\omega(e_{2j}, e_n) = \delta_{2j-1,n} \quad \forall \quad j \in \mathbb{N}.$$

$E = P \oplus Q = \oplus_{k=1}^{\infty} E_k, \quad P = Q = l_2; \quad E_k = \mathbb{R}^2.$
$\mathcal{F} = \{f_j\} = \{e_{2j-1}\}$ – ONB in $P$;
$\mathcal{G} = \{g_j\} = \{e_{2j}\}$ – ONB in $Q$.
$\mathbf{J}$ is linear operator in $E$ associated with the symplectic form

$$\omega(x, y) = (x, \mathbf{J}y)_E$$

$$\mathbf{J}(g_j) = -f_j, \ \mathbf{J}(f_j) = g_j.$$

$$\mathbf{J}^2 = -\mathbf{I}, \ \mathbf{J}^* = -\mathbf{J}.$$

# Reification of complex Hilbert space

Let $H$ be a complex Hilbert space.
Let $E$ be a real Hilbert space with symplectic operator $\mathbf{J}$.

Bijective mapping $\mathbf{R}: \; H \to E$ is called reification of the space $H$ if
1) There is an ONB $\mathcal{H} = \{h_k\}$ in the space $H$ such that
$\mathbf{R}(u) = p + q, \quad u \in H, \;\; E = P \oplus Q,$
where $p = \sum_{j=1}^{\infty} f_j \mathrm{Re}(h_j, u) \in P$ and $q = \sum_{j=1}^{\infty} g_j \mathrm{Im}(h_j, u) \in Q$.
2) $\|u\|_H = \|p + q\|_E$.

The inverse mapping $\mathbf{C} = (\mathbf{R})^{-1}: \; E \to H$ is called complexification of real Hilbert space $E$.

$$\mathbf{R}(iu) = \mathbf{J}\mathbf{R}(u) \;\forall\; u \in H,$$

$$(\mathbf{R}(u_1), \mathbf{R}(u_2))_E = \mathrm{Re}(u_1, u_2)_H \;\forall\; u_1, u_2 \in H.$$

$$\omega(\mathbf{R}(u_1), \mathbf{R}(u_2)) = \mathrm{Im}(u_1, u_2)_H \;\forall\; u_1, u_2 \in H.$$

# Symplectic rectangles

**Definition 1.** A set $\Pi \subset E$ is called measurable symplectic rectangle in the space $E$ if there are the decomposition $E = Q \oplus P$ and ON Bases $\{f_j\}$, $\{g_k\}$ in the spaces $Q$, $P$ such that

$$\Pi = \{z \in E: \ ((z, f_i), (z, g_i)) \in B_i, \ i \in \mathbb{N}\} = B_1 \times B_2 \times \dots, \quad (4)$$

where $B_i$ are Lebesgue-measurable sets in the plane $\mathbb{R}^2$ such that

$$\sum_{j=1}^{\infty} \max\{\ln(\lambda_2(B_j)), 0\} < +\infty$$

(here $\lambda_2$ is Lebesgue measure on the plane $\mathbb{R}^2$).

# Measure on symplectic rectangles

Let $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ be the set of measurable symplectic rectangles which have the form (4) for given pair of ONB $\{f_j\}$, $\{g_k\}$ in the subspaces $Q, P$.

Let the function $\lambda_{\mathcal{K}_{\mathcal{F},\mathcal{G}}}: \ \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0,+\infty)$ be defined by the equality

$$\lambda_{\mathcal{K}_{\mathcal{F},\mathcal{G}}}(\Pi) = \prod_{j=1}^{\infty} \lambda_2(B_j) = \exp(\sum_{j=1}^{\infty} \ln(\lambda_2(B_j))), \ \Pi \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E), \quad (5)$$

in the case $\Pi \neq \oslash$;

$\lambda_{\mathcal{K}_{\mathcal{F},\mathcal{G}}}(\Pi) = 0$ in the case $\Pi = \oslash$.

# Properties of the symplectic-invariant measure

**Lemma 1.** *The function of a set* $\lambda:\ \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)\to[0,+\infty)$ *is additive and shift-invariant.*

Let $r_{\mathcal{F},\mathcal{G}}$ be a ring generated by the collection of sets $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$.

**Lemma 2.** *Additive function of a set* $\lambda:\ \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)\to[0,+\infty)$ *has the unique additive extension on the ring* $r_{\mathcal{F},\mathcal{G}}$.

Let $\mathcal{E}=\mathcal{F}\bigcup\mathcal{G}$ be a symplectic ONB in $E=Q\oplus P$. The outer and interior measure of a set $A\subset E$ are given by equalities

$$\bar{\lambda}_{\mathcal{F},\mathcal{G}}(A)=\inf_{B\in r_{\mathcal{F},\mathcal{G}}:\ B\supset A}\lambda_{\mathcal{F},\mathcal{G}}(B),\ \underline{\lambda}_{\mathcal{F},\mathcal{G}}(A)=\sup_{B\in r_{\mathcal{F},\mathcal{G}}:\ B\subset A}\lambda_{\mathcal{F},\mathcal{G}}(B).$$

$$\mathcal{R}_{\mathcal{F},\mathcal{G}}=\{A\in E:\ \bar{\lambda}_{\mathcal{F},\mathcal{G}}(A)=\underline{\lambda}_{\mathcal{F},\mathcal{G}}(A)\in[0,+\infty)\}$$

**Lemma 3**. *Let* $B_R$ *be a ball in the space* $E$ *of radius* $R$. *Then* $\underline{\lambda}_{\mathcal{F},\mathcal{G}}(B_R)=0$. *If* $R<\frac{1}{\sqrt{\pi}}$, *then* $\lambda_{\mathcal{F},\mathcal{G}}(B_R)=0$. *There in an open set* $S\subset H$ *such that* $\bar{\lambda}_{\mathcal{F},\mathcal{G}}(S)=+\infty$ *и* $\underline{\lambda}_{\mathcal{F},\mathcal{G}}(S)=0$.

# Properties of the symplectic-invariant measure

**Theorem 1**. *A family of sets $\mathcal{R}_{\mathcal{F},\mathcal{G}}$ is the ring.*
*The completion of the measure $\lambda: \ r_{\mathcal{F},\mathcal{G}} \to [0,+\infty)$ is the measure $\lambda_{\mathcal{F},\mathcal{G}}: \ \mathcal{R}_{\mathcal{F},\mathcal{G}} \to [0,+\infty)$ which has following properties*
*1) complete, locally finite and $\sigma$-finite;*
*2) is not $\sigma$-additive;*
*3) its continuation by Lebesgue-Caratheodory scheme vanishes on the class $K(\mathcal{F},\mathcal{G})$;*
*4a) is invariant with respect to a shift on any vector of the space $E$;*
*4b) is invariant with respect to a smooth symplectorphism $\Phi: \ E \to E$ preserving the decomposition $E = \otimes_{k=1}^{\infty} E_k$ onto two-dimensional invariant symplectic subspaces where $E_k = \mathrm{span}(f_k, g_k), \ k \in \mathbb{N}$.*

# Countable family of noninteracting Hamiltonian systems

$$\tilde{\mathbb{H}}(p,q) = \sum_{k\in\mathbb{N}} f_k(p_k,q_k), \quad (p,q) \in E,$$

where $\{f_k\}$ is the sequence of bounded smooth functions such that

$$\sum_{k=1}^{\infty} M_k < \infty,$$

$$M_k = \sup_{(p,q)\in\mathbb{R}^2} (|f_k(p,q)|^2 + |\tfrac{\partial}{\partial p_k} f_k(p,q)|^2 + |\tfrac{\partial}{\partial q_k} f_k(p,q)|^2).$$

The Hamiltonian phase flow preserves the measure $\lambda_{\mathcal{F},\mathcal{G}}$.

# Hilbert space $\mathcal{H}_{\mathcal{F},\mathcal{G}} = L_2(E, \mathcal{R}_{\mathcal{F},\mathcal{G}}, \lambda_{\mathcal{F},\mathcal{G}}, \mathbb{C})$

Linear space of linear combinations of indicator functions of $\lambda_{\mathcal{F},\mathcal{G}}$-measurable sets is endowed by Hermite sesquilinear form.

$$u = \sum_{j=1}^{N} c_j \chi_{B_j} \in \mathrm{span}\{\chi_A,\ A \in \mathcal{R}_{\mathcal{F},G}\} \ \Rightarrow\ (u,u) = \sum_{j=1}^{N} |c_j|^2 \lambda_{\mathcal{F},\mathcal{G}}(B_j).$$

Factor-space of this space by the null-subspace of quadratic form is pre-Hilbert space.

$$N_{\mathcal{F},\mathcal{G}} = \{u \in \mathrm{span}\{\chi_A,\ A \in \mathcal{R}_{\mathcal{F},G}\} :\ (u,u) = 0\}$$

The completion of pre-Hilbert space is Hilbert space $\mathcal{H}_{\mathcal{F},\mathcal{G}} = L_2(E, \mathcal{R}_{\mathcal{F},\mathcal{G}}, \lambda_{\mathcal{F},\mathcal{G}}, \mathbb{C})$.

**Lemma 4.** Hilbert space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ is not separable.
$\mathrm{span}\{\chi_A,\ A \in \mathcal{R}_{\mathcal{F},G}\}/N_{\mathcal{F},\mathcal{G}}$ is dense linear manifold in $\mathcal{H}_{\mathcal{F},\mathcal{G}}$.

# Koopman representation of a Hamiltonian flow

Let $\tilde{\mathbb{H}} = \sum\limits_{k \in \mathbb{N}} f_k(p_k, q_k)$ be a densely defined Hamilton function

$$\tilde{\mathbb{H}} : \; E \supset D_1 \to \mathbb{R}.$$

generating the phase flow $\tilde{\Phi}_t$, $t \in \mathbb{R}$, in the space $E$.
Here $\{f_k\}$ is the sequence of continuously differentiable functions $f_k : \; E_k \to \mathbb{R}$.
Then the measure $\lambda_{\mathcal{F},\mathcal{G}}$ is invariant with respect to the flow $\tilde{\Phi}$.

$$\mathbf{U}_{\tilde{\Phi}}(t)u(x) = u(\tilde{\Phi}_{-t}x), \; t \in \mathbb{R}, \; x \in E, \; u \in \mathcal{H}_{\mathcal{F},\mathcal{G}}.$$

For example, $f_k(p_k, q_k) = \lambda_k(p_k^2 + q_k^2), \; \lambda_k \to +\infty.$

# Koopman group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ and its generator

Let

$$\mathbb{H}(q,p) = \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k(p_k^2 + q_k^2),\ (q,p) \in E_1 = D(\mathbb{H}).$$

The Hamiltonian flow $\Phi$ preserve the 2-dimensional symplectic subspaces $E_k$, $k \in \mathbb{N}$ of the space $E$. Moreover, it preserves the measure $\lambda_{\mathcal{F},\mathcal{G}}$.

**Lemma 5**. The Koopman group $\mathbf{U}_{\mathbf{\Phi}}$ is the unitary group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ *which is strongly continuous* iff the sequence $\{\lambda_k\}$ is finite.

**Remark**. Let $u = \chi_{\Pi_{[-\frac{1}{2},\frac{1}{2}]}}$. Then the function $(\mathbf{U}_{\mathbf{\Phi}}(t)u,u)_{\mathcal{H}_{\mathcal{F},\mathcal{G}}},\ t \in \mathbb{R}$, is continuous iff $\{\lambda_k\} \in l_1$.

Generally unitary group $\mathbf{U}_{\mathbf{\Phi}}$ is not strongly continuous.

# Koopman group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ and its generator

**Theorem 2**. *The Koopman group* $\mathbf{U}_{\mathbf{\Phi}}$ *has the invariant subspace* $\mathcal{H}_{\Phi}$ *of strong continuity in the space* $\mathcal{H}_{\mathcal{F},\mathcal{G}}$. *The generator* $\mathbf{L}_{\Phi}$ *of the* $C_0$*-semigroup* $\mathbf{U}_{\mathbf{\Phi}}|_{\mathcal{H}_{\Phi}}$ *has the countable family of eigenvalues* $\lambda_{m_1,...,m_N} = m_1\lambda_1 + ... + m_N\lambda_N, \quad N \in \mathbb{N}, \; m_1, ..., m_N \in \mathbb{Z}$.

$$\mathrm{Ker}(\mathbf{L}_{\Phi} - \lambda_{m_1,...,m_N}\mathbf{I}) \equiv \mathcal{H}_{\vec{m}} = \mathrm{span}(\prod_{k=1}^{\infty} v_{j_k}(r_k)e^{i\lambda_k m_k \varphi_k}),$$

*where* $\vec{m} \in (\mathbb{N} \to \mathbb{Z})_0$, $\{v_j\}$ *is an ONB in the space* $L_{2,r}([0, +\infty))$, $\{j_k\} : \; \mathbb{N} \to \mathbb{N}$.
*The Hilbert space* $\mathcal{H}_{\Phi} = \oplus_{\vec{m}}\mathcal{H}_{\vec{m}} \subset \mathcal{H}_{\mathcal{F},\mathcal{G}}$ *is the invariant subspace of strong continuity of the Koopman group* $\mathbf{U}_{\mathbf{\Phi}}$.

**Remark**. If $\lambda_k \in \mathbb{N} \; \forall \; k \in \mathbb{N}$ then $\lambda_{\vec{m}} \in \mathbb{Z} \; \forall \; \vec{m} \in (\mathbb{N} \to \mathbb{Z})_0$.

## Shifts operators in space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$

If the Hamiltonian $\mathbb{H}$ of the flow $\Phi$ is a linear functional $\mathbb{H}(q,p) = (a,q)_Q + (b,p)_P$ on the space $E$ then the Koopman group is the group $\mathbf{U}_{a,b}$ of shifts along the vector $h = \mathbf{J}(a,b) = (-b,a) \in E$, where $a \oplus b \in E$.

Let $\mathcal{E} = \mathcal{F} \bigcup \mathcal{G}$ be a symplectic ONB. Let $K(\mathcal{E})$ be a collection of orthogonal measurable rectangles with edges collinear to vectors of ONB $\mathcal{E}$. Let $r_{\mathcal{E}}$ be a ring generated by $K(\mathcal{E})$. Then $r_{\mathcal{E}}$ is the subring of $r_{\mathcal{F},\mathcal{G}}$ and $\mathcal{H}_{\mathcal{E}} = L_2(E, r_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ is the subspace of $\mathcal{H}_{\mathcal{F},\mathcal{G}}$.

Let $h \in E$. Then the linear operator $\mathbf{S}_h$: $\mathbf{S}_h u(x) = u(x+h)$, $x \in E$, $\forall\ u \in \mathcal{H}_{\mathcal{F},\mathcal{G}}$, is unitary.

The family of operators $\mathbf{S}_{th}$, $t \in R$, is the unitary group in $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ and $\mathcal{H}_{\mathcal{E}}$ is its invariant subspace.

Let $\mathcal{E}$ be an ONB in the space $E$. Let $L_1(\mathcal{E}) = \{x \in E : \ \{(x, e_k)\} \in l_1\}$.

# Koopman group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ and its generator

**Lemma 6**. *Let $h \in E$.*

*The group of unitary operators $\mathbf{S}_{th}$, $t \in R$, is strong continuous group in the space $\mathcal{H}_{\mathcal{E}}$ iff $h \in L_1(\mathcal{E})$.*

*The group of unitary operators $\mathbf{S}_{th}$, $t \in R$, is strong continuous group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ iff $\{(h, e_k),\ k \in \mathbf{N}\} \in c_0$.*

Sets $A, B \in \mathcal{R}_{\mathcal{E}}$ are called equivalent if $A = B - h$ for some $h \in L_1(\mathcal{E})$.

Let $K_z(\mathcal{E})$ be a collection of orthogonal measurable rectangles $\Pi \in K(\mathcal{E})$ with geometric center $z \in E/L_1(\mathcal{E})$.

**Theorem 3**. Let $\mathcal{H}_{\mathcal{E},z} = \overline{\mathrm{span}(\mathbf{S}_h K_z(\mathcal{E}),\ h \in L_1(\mathcal{E}))}$. Then
$\mathcal{H}_{\mathcal{E}} = \oplus_{z \in E/L_1(\mathcal{E})} \mathcal{H}_{\mathcal{E},z}$;
$\mathcal{H}_{\mathcal{E},z}$ is invariant under the group $\mathbf{S}_{th}$, $t \in \mathbb{R}$, $\Leftrightarrow h \in L_1(\mathcal{E}) \Leftrightarrow$ $S_{th}$, $t \in \mathbb{R}$, is strong continuous group in the space $\mathcal{H}_{\mathcal{E}}$.

Thank you!