Ramified chaotic attractors of smooth geometrically integrable self-maps of a cylinder

November, 09, 2022

L.S. Efremova,
Nizhni Novgorod State University,
Moscow Institute of Physics and Technology, Russia

Content

1. The concept of the geometric integrability of discrete dynamical systems on a cylinder: geometric and analytic criteria.

2. Geometric properties of maps under consideration.

2. Example of the family of integrable maps with global ramified attractors.

Introduction

There is a vast literature devoted to different integrability aspects of dynamical systems both with continuous and discrete time (see, e.g., V.E. Adler, A.I. Bobenko, S.V. Bolotin, V.V. Kozlov, A.M. Stepin, Yu.B. Suris, D.V. Treschev, A.P. Veselov, I.V. Volovich).
Originally, the concept of integrability of dynamical systems with discrete time was introduced for systems obtained by digitization of known differential equations. But there are discrete dynamical systems that do not belong to this class. We consider here precisely this case.

Recall the following Birkhoff's thought: "If we try to formulate the exact definition of integrability then we see that many definitions are possible, and every of them is of the specific theoretical interest".

Some comments

First works contained deep results on integrable maps are works by G.Julia, P.Fatou and J.Ritt (although the term "integrability"is not using in these works). The problem considered in the above works is the description of pairs of commuting rational (in particular, polynomial) maps (in the projective plane)

$$H(G(x)) = G(H(x)).$$

The important conclusion follows from the above papers: the existence of the commuting maps with suitable properties implies integrability of the corresponding dynamical systems (in the sense of the explicit description of dynamics) (see A.P. Veselov, "Integrable maps", Russian Math. Surveys, **48**: 5((281) (1991), 3-42).

One can consider another function ψ instead G in the right part of the above equality. Then we obtain the integrability definition (for rational and polynomial maps) by R.I. Grigorchuk.

The definition of geometric integrability

Let S^1 be a circle presented as the interval $[0, 1]^*$ with identified ends, $I_2 = [a_2, b_2]$ be a segment, $M = S^1 \times I_2$ be a cylinder.

Definition 1. A map $F : M \to M$ is said to be *geometrically integrable on a nonempty F-invariant set $A \subseteq M$* if there exist a self-map ψ of an arc $J \subseteq S^1$ and ψ-invariant set $B(\psi) \subseteq J$ such that the restriction $F_{|A(F)}$ is semiconjugate with the restriction $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$, i.e. the following equality holds:

$$H \circ F_{|A(F)} = \psi_{|B(\psi)} \circ H. \tag{1}$$

The map $\psi_{|B(\psi)}$ is said to be *the quotient* of $F_{|A(F)}$.

Remark 1. In the framework of our approach the concept of geometric integrability is introduced for some multifunctions (see L.S. Efremova, The Trace Map and Integrability of the Multifunctions, J. Phys.: Conf. Ser., **990** (2018), 012003).

The geometric criterion of integrability on an invariant set

We use further first pr_1 and second pr_2 natural projections.

Theorem 2. *Let $F \in C^0(M)$, $A(F)$ be a nonempty closed F-invariant subset of the cylinder M satisfying*

$$pr_2(A(F)) = I_2. \tag{2}$$

Let $J \subseteq S^1$ be an arc, ψ be a self-map of J, $B(\psi) \subseteq J$ be a closed ψ-invariant set. Then $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in I_2$ the map H is an injection on x, if and only if $A(F)$ is the support of a continuous invariant lamination for $A(F) \neq M$ (of a continuous invariant foliation for $A(F) = M$) with fibres $\{\gamma_{x'}\}_{x' \in B(\psi)}$ that are pairwise disjoint graphs of continuous functions $x = x_{x'}(y)$ for every $y \in I_2$. Moreover, the inclusion $F(\gamma_{x'}) \subseteq \gamma_{\psi(x')}$ holds.

D.V. Anosov, E.V. Zhuzhoma, *Nonlocal asymptotic behavior of curves and leaves of laminations on universal coverings*, Proc. Steklov Inst. Math., 2(249) (2005), p.p. 1-219.

Skew products and the analytic criterion of integrability

The dynamical system $\Phi : M \to M$ is said to be *a skew product* if

$$\Phi(x, y) = (f(x), g_x(y)), \text{ for all } (x; y) \in M. \qquad (3)$$

Theorem 3. *Let $F \in C^0(M)$, $A(F)$ be a nonempty closed F-invariant subset of M satisfying the equality (2). Let $J, \psi : J \to J$, $B(\psi) \subseteq J$ be as in Theorem 2.*
Then $F_{|A(F)}$ is geometrically integrable with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in I_2$ the map H is an injection on x, if and only if there is a homeomorphism $\widetilde{H}$ that maps the set $A(F)$ on the set $B(\psi) \times I_2$ and reduces the restriction $F_{|A(F)}$ to the skew product $\Phi_{|B(\psi)\times I_2}$ satisfying the equality

$$\Phi_{|B(\psi)\times I_2}(u, v) = (\psi_{|B(\psi)}(u), g_{x'}(v)), \ \ x' = pr_1 \circ \widetilde{H}^{-1}(u, v). \qquad (4)$$

Here $\widetilde{H}^{-1} : B(\psi) \times J \to A(F)$ is the inverse homeomorphism for $\widetilde{H}$,

$$\textit{and } \ \widetilde{H}(x, y) = (H(x, y), y), \ \textit{for all } (x, y) \in A(F). \qquad (5)$$

Maps under consideration, I

We consider a C^1-smooth map $F : M \to M$ so that for every point $(x,\, y) \in M$ the equality holds:

$$F(x,\, y) = (f(x) + \mu(x,\, y),\ g_x(y)), \ \text{ where } \ g_x(y) = g(x,\, y). \quad (6)$$

Important representatives of general class of maps (6), are Hénon and Lozi maps (on the plane) and Belykh map on the cylinder.

Definition 2. A map $\varphi \in C^1(S^1)$ is said to be *structurally stable* (in the C^1-norm) if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for a map $\psi \in B^1_{1,\,\varepsilon}(\varphi)$ one can find δ-close in the C^0-norm (of the uniform convergence) to the identity map homeomorphism $h : S^1 \to S^1$ satisfying the equality

$$h \circ \varphi = \psi \circ h.$$

Maps under consideration, II

Denote by $\Omega(\cdot)$ the nonwandering set of a map.

Definition 3. A map $\varphi \in C^1(S^1)$ is said to be Ω-*stable* (in the C^1-norm) if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for a map $\psi \in B^1_{1,\varepsilon}(\varphi)$ one can find δ-close in the C^0-norm to the identity map homeomorphism $h : \Omega(\varphi) \to \Omega(\psi)$ satisfying the equality

$$h \circ \varphi_{|\Omega(\varphi)} = \psi_{|\Omega(\psi)} \circ h.$$

Remark 2. The above definitions shows that the Ω-stable map $\varphi \in C^1(S^1)$ satisfying $\Omega(\varphi) = S^1$, is C^1-structurally stable.

D.V. Anosov, *Structurally stable systems*, Tr. Mat. Inst. Steklova, [*English translation*], Proc. Steklov Inst. Math., 169 (1986), p.p. 61-95.

Maps under consideration, III

Recall the main result of the paper
M.V. Jakobson, *Smooth mappings of the circle into itself*, Mat. Sb. (N.S.), [*English translation*], Mathematics of the USSR – Sbornik **14** (2) (1971), pp. 161-185.

Theorem 1. *In the space $C^1(S^1)$ there is an open everywhere dense set* $\mathbf{L}$ *of maps that equals the union of two subsets* $\mathbf{L_1}$ *and* $\mathbf{L_2}$*, where* $\mathbf{L_1}$ *is the set of* Ω*-stable maps with a completely disconnected nonwandering set, and* $\mathbf{L_2}$ *is the set of structurally stable maps such that every* $\varphi \in \mathbf{L_2}$ *is topologically conjugate with an expanding map of the same degree* $\deg \varphi$*, where* $|\deg \varphi| > 1$*. (A map* $\varphi \in C^1(S^1)$ *is said to be expanding if for every* $x \in S^1$ *the inequality holds* $|f'(x)| > 1$*).*

Maps under consideration, IV

We consider the class $C^1_L(M)$ of C^1-smooth maps (6), where $f \in \mathbf{L}$, and μ satisfies the condition:
(i_μ) $\mu(0,\, y) = \mu(1,\, y) = \mu(x,\, a) = \mu(x,\, b) = 0$ for $x \in [0,\, 1]^*$, $y \in [a_2,\, b_2]$.
For every $y \in I_2$ we have:

$$||\mu||_{C^1(S^1)} \leqslant ||\mu||_{C^1(M,\, S^1)}. \tag{7}$$

Let $\delta > 0$. Since $f \in \mathbf{L}$ then we find $\varepsilon > 0$, using Definitions 2, 3 and Theorem 1. We suppose that $\mu \in C^1(M,\, S^1)$ satisfies the "condition of smallness"in the C^1-norm. It means that the following inequality is valid:
$(ii_{\mu,\,\varepsilon})$ $||\mu||_{C^1(M,\, S^1)} < \varepsilon$.
Then by (7) and $(ii_{\mu,\,\varepsilon})$ we have for every $y \in I_2$:

$$(f + \mu) \in B^1_{1,\,\varepsilon}(f). \tag{8}$$

Existence of C^1-smooth local lamination

Denote by $T^1_L(M)$ the space of C^1-smooth skew products on a cylinder with quotients from the set $\mathbf{L}$.

Theorem 4. *Let $\Phi \in T^1_L(M)$ be a map of the form (3). Let $\delta > 0$. Then there is an ε-neighborhood $B^1_\varepsilon(\Phi)$ of the map Φ in the space $C^1_L(M)$ such that every map $F \in B^1_\varepsilon(\Phi)$ obtained from Φ by means of the C^1-smooth perturbation $\mu = \mu(x, y)$, where μ satisfies the condition $(ii_{\mu,\varepsilon})$, has an invariant C^1-smooth local lamination $L^1_{loc}(F)$, which is a lamination for $f \in \mathbf{L_1}$, and a foliation for $f \in \mathbf{L_2}$. Fibres of this local lamination start from the points of the set $\Omega(f) \times \{a_2\}$ and are pairwise disjoint graphs of C^1-smooth functions $x = x(y)$ on the interval I_2. Moreover, every curvilinear fibre is ε'-close in the C^1-norm (for some $\varepsilon' > 0$, $\varepsilon' = \varepsilon'(\delta)$) to the vertical closed interval that starts from the same initial point of the set $\Omega(f) \times \{a_2\}$ just as the curvilinear fibre.*

Important corollary of Theorem 4

Corollary 1. *Let* $\Phi \in T^1_L(M)$ *be a map of the form (3). Let* $\delta > 0$.

Then there is an ε*-neighborhood* $B^1_\varepsilon(\Phi)$ *of the map* Φ *in the space* $C^1_L(M)$ *such that every map* $F \in B^1_\varepsilon(\Phi)$ *obtained from* Φ *by means of the* C^1*-smooth perturbation* $\mu = \mu(x, y)$*, where* μ *satisfies the condition* $(ii_{\mu,\varepsilon})$*, has the following property: the curvilinear projection* H *(defined by the equality* $H(\gamma_{x'}) = x'$*, where* $\gamma_{x'}$ *is the fibre of* $L^1_{loc}(F)$ *that starts from a point* (x', a_2)*,* $x' \in \Omega(f)$*) of the support of the local lamination* $L^1_{loc}(F)$ *on* $\Omega(f)$ *is* C^1*-smooth injection on* x *for every* $y \in I_2$*, moreover,*

$$\frac{\partial}{\partial x} H(x, y) \neq 0 \text{ on the support of } L^1_{loc}(F). \tag{9}$$

Integrability of maps under consideration

Theorem 5. *Let* $\Phi \in T^1_L(M)$ *be a map of the form (3). Let* $\delta > 0$.

Then there is an ε*-neighborhood* $B^1_\varepsilon(\Phi)$ *of the map* Φ *in the space* $C^1_L(M)$ *such that every map* $F \in B^1_\varepsilon(\Phi)$ *obtained from* Φ *by means of the* C^1*-smooth perturbation* $\mu = \mu(x, y)$*, where* μ *satisfies the condition* $(ii_{\mu,\varepsilon})$*, is geometrically integrable on the support of the local lamination* $L^1_{loc}(F)$ *with the quotient* $f_{|\Omega(f)}$ *by means of a* C^1*-smooth curvilinear projection* H *such that for every* $y \in I_2$ *the map* H *is an injection on* x *satisfying the inequality (9).*

Example of maps with one-dimensional ramified attractors, I

Consider the family of C^1-smooth maps $F_k : M \to M$ ($k > 1$, $k \in \mathbb{N}$, $I_2 = [0, 1]$) satisfying

$$F_k(x, y) = (f_k(x) + \mu(x, y), g(x, y)) \text{ for } (x, y) \in M, \tag{10}$$

where

$$f_k(x) = \begin{cases} kx - i, & \text{if} \quad x \in \left[\frac{i}{k}, \frac{i+1}{k}\right), \; i = 0, 1, \ldots, k-1, \\ 1, & \text{if} \quad x = 1, \end{cases} \tag{11}$$

Construct maps $g_x(y)$ for the cylinder map (10) satisfying (11). Using positive $\delta < 1/2$, we choose $\varepsilon' > 0$ so small that it satisfies the inequality $\varepsilon' < 1/2$ (see Theorem 4). Then the straight line $y = 1 - x$ intersects every fibre $\gamma_{x'}$ in the unique point without tangency, and the equality $y = 1 - x_{x'}(y)$ is equivalent to $y = y(x')$, where $x' \in [0, 1]^*$; moreover, the function $y = y(x')$ is continuous on the circle.

Example of maps with one-dimensional ramified attractors, II

We use two connected sets

$$D' = \{(x,\, y) \in \bigcup_{x' \in [0,\,1]^*} \gamma_{x'} : 0 \leqslant y \leqslant 1 - x\};$$
$$D'' = \{(x,\, y) \in \bigcup_{x' \in [0,\,1]^*} \gamma_{x'} : 1 - x < y \leqslant 1\}.$$

Define C^1-smooth functions $g_x(y)$ for maps (10) satisfying (11).

$$g_x(y) = \begin{cases} y, & \text{if} \quad (x,\, y) \in D', \\ 1 - x + \sin(y - 1 + x), & \text{if} \quad (x,\, y) \in D''. \end{cases} \tag{12}$$

Every map (10) satisfying (11) and (12), is C^1-smoothly conjugate (under C^1-smooth diffeomorphism $\widetilde{H}$) to the skew product

$$\widehat{\Phi}_k(x',\, y) = (f_k(x'),\, g_x(y)), \text{ for } x = pr_1 \circ \widetilde{H}^{-1}(x',\, y), \tag{13}$$

so that finally $\widehat{\Phi}_k(x',\, y) = (f_k(x'),\, \widehat{g}_{x'}(y))$ $(k \geqslant 2)$.

On the global attractor, I

Theorem 6. *Let $\Phi_k \in T^1_L(M)$ be a map of the form*

$$\Phi_k(x,\, y) = (f_k(x),\;\, g_x(y))$$

satisfying (11) and (12). Let δ be a positive number, $\delta < 1/2$.

Then there is an ε-neighborhood $B^1_\varepsilon(\Phi_k)$ of the map Φ_k in the space $C^1_L(M)$ such that every map $F_k \in B^1_\varepsilon(\Phi_k)$ obtained from Φ_k by means of the C^1-smooth perturbation $\mu = \mu(x,\, y)$, where μ satisfies the condition $(ii_{\mu,\,\varepsilon})$, has one-dimensional global attractor $A(F_k)$, which is a ramified continuum such that the cardinality of the set of its ramification points equals continuum; the order of every such point equals 3.

Moreover, the attractor $A(F_k)$ possesses the properties:

On the global attractor, II

(i) $A(F_k) = \overline{Per(F_k)}$, and all F_k-periodic points are not hyperbolic;

(ii) $A(F_k)$ consists of two types of C^1-smooth arcs: on the circle $S^1 \times \{0\}$ the map F_k is mixing, and on the each nondegenerate arc of the second type (the set of these arcs has continuum cardinality) the map F_k is not mixing; moreover, among nondegenerate arcs of the second type there is a countable number of arcs over f_k-periodic points satisfying: for every natural number $m \geqslant 1$ there is an arc such that m-th iteration of F_k on this arc is the identity map.

[1] L.S. Efremova, *Ramified continua as global attractors of C^1-smooth self-maps of a cylinder close to skew products*, J. Difference Equ. Appl., Special issue "Lozi, Hénon and other chaotic attractors. Theory and applications", **28** (10) (2022), 22 p.

[2] L.S. Efremova, *Geometrically integrable maps in the plane and their periodic orbits*, Lobachevskii J. Math., **42** (10) (2021), p.p. 2315–2324.