Extensions of the Starobinsky R^2 inflationary model

Sergey Yu. Vernov

Skobeltsyn Institute of Nuclear Physics,
Lomonosov Moscow State University

based on

V.R. Ivanov, S.V. Ketov, E.O. Pozdeeva, S.Yu. Vernov,
JCAP **2203** (2022) 058 [arXiv:2111.09058]
S.V. Ketov, E.O. Pozdeeva, S.Yu. Vernov,
arXiv:2211.01546

New Trends in Mathematical Physics,
MIAN, MOSCOW, 09.11.2022

The Starobinsky R^2 inflationary model

Starobinsky model of inflation, whose action is given by

$$S_{\rm Star.}[g^J_{\mu\nu}] = \frac{M^2_{\rm Pl}}{2}\int d^4x\sqrt{-g_J}\left(R_J + \frac{1}{6m^2}R_J^2\right)\ , \tag{1}$$

includes the inflaton mass m, fixed by CMB measurements,

$$m = 1.3\left(\frac{55}{N}\right)10^{-5}M_{\rm Pl},$$

where N is the number of e-foldings before the end of inflation.
A.A. Starobinsky, *Phys. Lett. B* **91** (1980) 99.
The action (1) is dual to the quintessence (or scalar-tensor gravity) action

$$S_{\rm quint.}[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[\frac{M^2_{Pl}}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_{\rm Star.}(\phi)\right] \tag{2}$$

in terms of the canonical scalar ϕ and another metric $g_{\mu\nu}$ in the Einstein frame, related to $g^J_{\mu\nu}$ (in the Jordan frame) by a Weyl transformation.
The induced scalar potential is given by

$$V_{\rm Star.}(\phi) = \frac{3}{4}M^2_{Pl}m^2\left[1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right)\right]^2\ . \tag{3}$$

The main cosmological parameters of inflation are given by the scalar tilt n_s and the tensor-to-scalar ratio r, whose values are constrained by the combined Planck, WMAP and BICEP/Keck observations of CMB as

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\mathrm{CL}) \qquad \text{and} \qquad r < 0.036 \quad (95\%\mathrm{CL})\ .$$

The Starobinsky model is known as the excellent model of large-field slow-roll cosmological inflation with very good agreement to the observation data.

ϕ_i/M_{Pl}	5.2262	5.4971
n_s	0.961	0.969
r	0.0043	0.0027
N_e	49.258	62.335

The values of the inflationary parameters are sensitive to the duration of inflation and the initial value of the inflaton field, ϕ_i.

The $F(R)$ gravity action can be rewritten as

$$S_J[g^J_{\mu\nu},\sigma] = \int d^4x\sqrt{-g^J}\left[F_{,\sigma}(R_J-\sigma)+F\right] \ , \qquad (4)$$

where the new scalar field σ has been introduced, and $F_{,\sigma}(\sigma) = \frac{dF(\sigma)}{d\sigma}$.
After the Weyl transformation of the metric

$$g_{\mu\nu} = \frac{2F_{,\sigma}(\sigma)}{M^2_{Pl}}g^J_{\mu\nu} \qquad (5)$$

one gets the following action in the Einstein frame:

$$S_E[g_{\mu\nu},\sigma] = \int d^4x\sqrt{-g}\left[\frac{M^2_{Pl}}{2}R - \frac{h(\sigma)}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V\right], \qquad (6)$$

where we have introduced the functions

$$h(\sigma) = \frac{3M^2_{Pl}}{2F^2_{,\sigma}}F^2_{,\sigma\sigma} \quad \text{and} \quad V(\sigma) = M^4_{Pl}\frac{F_{,\sigma}\sigma - F}{4F^2_{,\sigma}} \ . \qquad (7)$$

Introducing the canonical scalar field ϕ instead of σ as

$$\phi = \sqrt{\frac{3}{2}} M_{Pl} \ln \left[\frac{2}{M_{Pl}^2} F_{,\sigma} \right] \tag{8}$$

allows one to rewrite the action S_E to the standard (quintessence or scalar-tensor) form:

$$S_E[g_{\mu\nu}, \phi] = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \tag{9}$$

The inverse transformation reads as follows:

$$R_J = \left[\frac{\sqrt{6}}{M_{Pl}} V_{,\phi} + \frac{4V}{M_{Pl}^2} \right] \exp\left(\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}} \right) , \tag{10}$$

$$F = \frac{M_{Pl}^2}{2} \left[\frac{\sqrt{6}}{M_{Pl}} V_{,\phi} + \frac{2V}{M_{Pl}^2} \right] \exp\left(2\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}} \right) , \tag{11}$$

where $V_{,\phi} = \frac{dV}{d\phi}$, defining the function $F(R_J)$ in the parametric form with the parameter ϕ.

Being motivated by the potential (3), we find useful to introduce the non-canonical dimensionless field

$$y \equiv \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right) = \frac{M_{Pl}^2}{2F_{,\sigma}} > 0 \tag{12}$$

because it is (physically) *small* during slow-roll inflation. Defining $\tilde{V}(y) = V(\phi)$, we simplify Eqs. (10) and (11) as follows:

$$R_J = \frac{2}{M_{Pl}^2}\left(2\frac{\tilde{V}}{y} - \tilde{V}_{,y}\right), \tag{13}$$

$$F = \frac{\tilde{V}}{y^2} - \frac{\tilde{V}_{,y}}{y}, \tag{14}$$

respectively.
In the Starobinsky model, we have

$$\tilde{V}_{\mathrm{Star.}}(y) = V_0(1-y)^2\ , \quad \text{where} \quad V_0 = \frac{3}{4}m^2 M_{Pl}^2\ . \tag{15}$$

In the spatially flat FLRW universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right),$$

the action (9) leads to the standard system of evolution equations:

$$6M_{Pl}^2 H^2 = \dot{\phi}^2 + 2V, \quad (16)$$

$$2M_{Pl}^2 \dot{H} = -\dot{\phi}^2, \quad (17)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (18)$$

where $H = \dot{a}/a$ is the Hubble parameter,
$a(t)$ is the scale factor,
and the dots denote the derivatives with respect to the cosmic time t.

In the inflationary model building, the e-foldings number

$$N_e = \ln\left(\frac{a_{\rm end}}{a}\right) , \qquad (19)$$

where $a_{\rm end}$ is the value of a at the end of inflation, is considered instead of the time variable. Using the relation $d/dt = -H\,d/dN_e$, one can rewrite Eq. (16) as follows:

$$Q = \frac{2V}{6M_{Pl}^2 - \chi^2} , \qquad (20)$$

where $Q \equiv H^2$ and $\chi = \phi' = -\dot{\phi}/H$, and the primes denote the derivatives with respect to N_e.
Equations (17) and (18) yield the dynamical system of equations:

$$Q' = \frac{1}{M_{Pl}^2} Q\chi^2 , \quad \phi' = \chi , \quad \chi' = 3\chi - \frac{1}{2M_{Pl}^2}\chi^3 - \frac{1}{Q}\frac{dV}{d\phi} . \qquad (21)$$

We rewrite the last equation as

$$\chi' = 3\chi - \frac{1}{2M_{Pl}^2}\chi^3 - \frac{6M_{Pl}^2 - \chi^2}{2V}\frac{dV}{d\phi} . \qquad (22)$$

In the Einstein frame, the slow-roll parameters are

$$\epsilon = \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = \frac{y^2}{3}\left(\frac{\tilde{V}_{,y}}{\tilde{V}}\right)^2 ,$$

$$\eta = M_{Pl}^2\left(\frac{V_{,\phi\phi}}{V}\right) = \frac{2y}{3\tilde{V}}\left(\tilde{V}_{,y} + y\tilde{V}_{,yy}\right) .$$

The scalar spectral index n_s and the tensor-to-scalar ratio r in terms of the slow-roll parameters are given by

$$n_s = 1 - 6\epsilon + 2\eta, \qquad r = 16\epsilon . \tag{23}$$

In the slow-roll approximation, the function $\phi(N_e)$ can be found as a solution of

$$\chi \equiv \phi' \simeq \frac{M_{Pl}^2}{V} V_{,\phi} \tag{24}$$

when demanding that $\epsilon = 1$ corresponds to the end of inflation with $a = a_{\rm end}$.

Equation (24) is equivalent to

$$y' = \frac{2y^2\tilde{V}_{,y}}{3\tilde{V}} . \tag{25}$$

The $(R + R^2 + R^3)$ gravity models of inflation

To the best of our knowledge, adding the higher-order terms in R was first proposed in
J.D. Barrow and S. Cotsakis, *Phys. Lett. B* **214** (1988) 515.
A generic $(R + R^2 + R^3)$ gravity action is given by

$$S_{3-\mathrm{gen.}} = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g_J}\left[(1+\delta_1)R_J + \frac{(1+\delta_2)}{6m^2}R_J^2 + \frac{\delta_3}{36m^4}R_J^3\right] \; ,$$

where we have introduced the three dimensionless parameters δ_i.
The corresponding inflaton scalar potential (7) is given by

$$V(\sigma) = \frac{16V_0\tilde{\sigma}^2\left[3(1+\delta_2)+\delta_3\tilde{\sigma}\right]}{3\left[12(1+\delta_1)+4(1+\delta_2)\tilde{\sigma}+\delta_3\tilde{\sigma}^2\right]^2} \; ,$$

where the dimensionless variable $\tilde{\sigma} = \sigma/m^2$ has been introduced.
$V(0) = 0$, $V(\tilde{\sigma}) > 0$ at $\tilde{\sigma} > 0$, and V tends to zero at $\tilde{\sigma} \to +\infty$, while the potential has a maximum at some positive value of $\tilde{\sigma}$. The equation $V' = 0$ has only one positive root given by $\tilde{\sigma}_{\mathrm{max.}} = 6\sqrt{\frac{1+\delta_1}{\delta_3}}$.

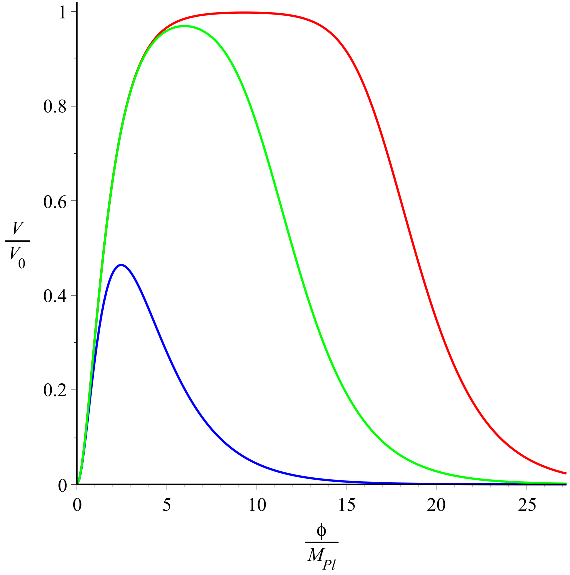

Figure: The normalized potential $V(\phi)/V_0$ with $\delta_1 = \delta_2 = 0$ for $\delta_3 = 0.000001$ (red), $\delta_3 = 0.000247$ (blue), and $\delta_3 = 1/3$ (green).

To study the impact of the R^3-term on inflation in more detail, let us consider the simplest non-trivial case with $\delta_1 = \delta_2 = 0$, in which Eq. (12) implies

$$\frac{1}{y} = 1 + \frac{1}{3}\tilde{\sigma} + \frac{\delta_3}{12}\tilde{\sigma}^2 \,. \tag{26}$$

Equation (26) is a quadratic equation on $\tilde{\sigma}$ as a function of y. The only positive root of this equation is given by

$$\tilde{\sigma} = \frac{2}{\delta_3}\left[\sqrt{1+3\delta_3\left(y^{-1}-1\right)}-1\right] = \frac{2}{\delta_3}\left[\sqrt{1+3\delta_3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right)}-1\right] .$$

Using Eqs. (7) and (12), we find the scalar potential in terms of y or the inflaton field ϕ as follows:

$$\tilde{V}(y) = \frac{4V_0}{27\delta_3^2 y}\left[y + 2\sqrt{y(y+3\delta_3(1-y))}\right]\left(y - \sqrt{y(y+3\delta_3(1-y))}\right)^2 .$$

It is worth noticing that $\tilde{V}_{\mathrm{Star.}}(y)$ is reproduced in the limit $\delta_3 \to 0$.

The condition $\phi_i < \phi_{\rm max.}$ yields the additional restriction on the possible initial values of ϕ, being represented by the blue curve on the left-hand-side of Fig. 2.

The upper bound on the parameter δ_3 can be estimated by assuming the observable value of n_s to be calculated at the maximum of the potential. Then we find

$$n_s(\phi_{\rm max.}) = 1 - \frac{8\sqrt{\delta_3}\left(1 + 4\sqrt{\delta_3} + 4\delta_3\right)}{3\left(3\sqrt{\delta_3}+1\right)\left(2\sqrt{\delta_3}+1\right)^2} \quad . \tag{27}$$

Since observations require $n_s > 0.960$, we get $\delta_3 < 0.0002467$. The dependence of n_s upon δ_3 is given on the right-hand-side of Fig. 2. Therefore, the domain of allowed values of δ_3 and ϕ is highly restricted.

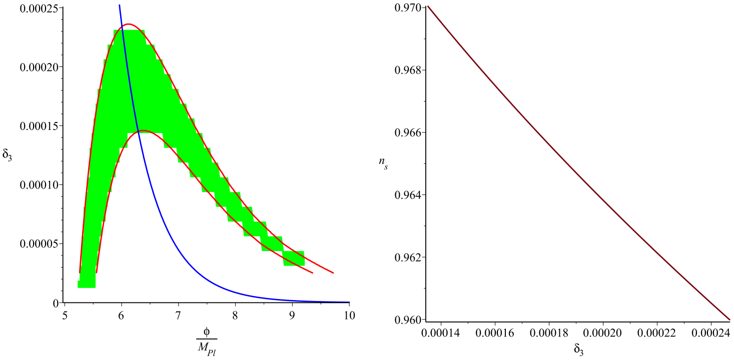

Figure: The allowed range of δ_3 and ϕ from the observational constraints (Planck): $0.961 < n_s < 0.969$ (left), and the dependence of n_s upon δ_3 (right), under the assumption that inflation started at the maximum of the potential.

Let

$$F(R_J) = \frac{M_{Pl}^2}{2}\left[R_J + \frac{1}{6m^2}R_J^2 + \frac{\delta}{m}R_J^{3/2}\right] , \qquad (28)$$

where we have introduced the dimensionless parameter δ.

The $R^{3/2}$ term appears in the (chiral) modified supergravity

S.V. Ketov and A.A. Starobinsky, Phys. Rev. D 83 (2011) 063512 [1011.0240].

S.V. Ketov and S. Tsujikawa, Phys. Rev. D 86 (2012) 023529 [1205.2918].

The $R^{3/2}$-term in $F(R)$ gravity arises in an approximate description of the Higgs field with a small cubic term in its scalar potential and a large non-minimal coupling to R

J.S. Martins, O.F. Piattella, I.L. Shapiro and A.A. Starobinsky, 2010.14639.

Given $\tilde{\sigma} > 0$, we find

$$F_{,R_J} = \frac{M_{Pl}^2}{6}\left(\sqrt{R_J} + \frac{9\delta}{4}\right)^2 - \frac{3}{16}\left(27\delta^2 - 16\right) > 0 \qquad (29)$$

when $\delta > -4\sqrt{3}/9$, and

$$F_{,R_J R_J} = \frac{M_{Pl}^2}{24m^2}\left(4 + \frac{9\delta}{\sqrt{R_J}}\right) > 0 \qquad (30)$$

only when $\delta > 0$.
Hence, the condition $\delta > 0$ is necessary to get a stable $F(R)$ gravity model for all $R_J > 0$.

The corresponding scalar potential (7) is given by

$$V = \frac{4V_0\tilde{\sigma}(3\delta\sqrt{\tilde{\sigma}} + \tilde{\sigma})}{(6 + 9\delta\sqrt{\tilde{\sigma}} + 2\tilde{\sigma})^2} \quad . \tag{31}$$

Equation (12) in this case is a quadratic equation on $\sqrt{\tilde{\sigma}}$, and its only real solution is

$$\tilde{\sigma} = \frac{3(1-y)}{y} + \frac{9\delta}{8y}\left[9\delta\, y - \sqrt{3y(27\delta^2 y - 16y + 16)}\right] \quad . \tag{32}$$

The potential can be rewritten as

$$\begin{aligned}\tilde{V} &= \frac{V_0}{2304y^2}\,(s + 3\delta\, y)\,(s - 9\delta\, y)^3 \\ &= \frac{243\,V_0\delta^4 y^2}{256}\left(3\sqrt{1 + \frac{16(1-y)}{27\delta^2 y}} + 1\right)\left(\sqrt{1 + \frac{16(1-y)}{27\delta^2 y}} - 1\right)^3 \quad ,\end{aligned}$$

where we have introduced $s = \sqrt{3y(27\delta^2 y - 16y + 16)}$.

When $\delta = 4\sqrt{3}/9$, the $F_{,\sigma}$ function is a perfect square, and the potential simplifies as

$$\tilde{V}_{\rm special}(y) = \frac{V_0}{3}\left(3+\sqrt{y}\right)\left(1-\sqrt{y}\right)^3 \,, \tag{33}$$

or

$$V_{\rm special}(\phi) = \frac{V_0}{3}\left({\rm e}^{\phi/(\sqrt{6}M_{Pl})}-1\right)^3\left(1+3{\rm e}^{\phi/(\sqrt{6}M_{Pl})}\right){\rm e}^{-2\sqrt{2/3}\,\phi/M_{Pl}} \,.$$

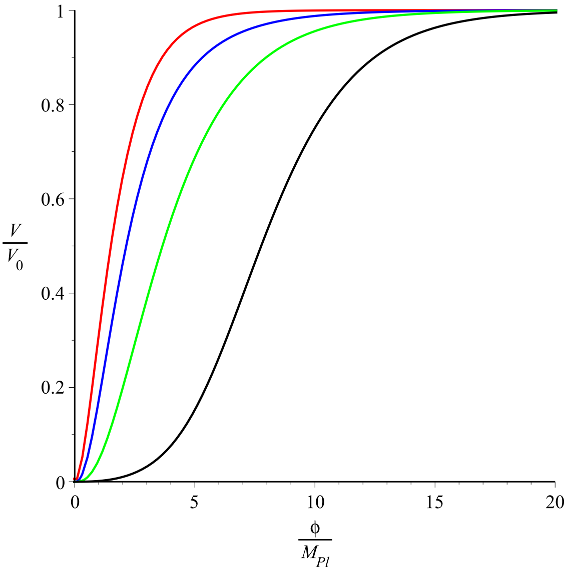

Figure: The potential $V(\phi)$ for $\delta = 0$ (red), $\delta = 1/5$ (blue), $\delta = 4\sqrt{3}/9$ (green), and $\delta = 5$ (black).

The inflationary parameters

The inflationary parameters are given by

$$n_s = 1 + \frac{8y\left(3s(3\delta(9\delta^2-16)s+720\delta^2-256)y^2 - s^3\delta(39\delta y+s)\right)}{\left(-9\,\delta\,y+s\right)^2\left(3\,\delta\,y+s\right)^2 s} - \frac{8y\left[72\delta(4-9\delta^2)(27\delta^2-16)y^4 + [(768-1215\delta^4-432\delta^2)s - 144\delta(45\delta^2-16)]y^3\right]}{\left(-9\,\delta\,y+s\right)^2\left(3\,\delta\,y+s\right)^2 s} \tag{34}$$

and

$$r = \frac{768\,y^2\left(-9\,\delta^2 y + s\delta + 8\,y\right)^2}{\left(-9\,\delta\,y+s\right)^2\left(3\,\delta\,y+s\right)^2} \quad . \tag{35}$$

The amplitude of scalar perturbations is given by

$$A_s = \frac{\left(-9\,\delta\,y+s\right)^5\left(3\,\delta\,y+s\right)^3 m^2}{3538944 y^4 \pi^2\left(-9\,\delta^2 y + s\delta + 8\,y\right)^2} \quad . \tag{36}$$

The observed value of A_s determines the value of the parameter m.

The slow-roll evolution equation allows us to relate N_e with y at the end of inflation,

$$N_e = \left(\frac{9}{8} - \delta^{-2}\right) \ln\left[9\delta^3\left(9\delta\, y - s\right) + 24\left(1-4y\right)\delta^2 + 8\left(\delta s + 4y\right)\right] + \left(\delta^{-2} - \frac{3}{8}\right) \ln y + \frac{s}{4\delta\, y} - N_0 \ ,$$

where the integration constant N_0 is fixed by the condition $N_e(y_{end}) = 0$. The analytic formula for $N_0(\delta)$ is obtained by substituting $N_e = 0$ and $y = y_{end}$.
The condition $\epsilon = 1$ gives

$$y_{end} = \frac{3(4-3\delta^2+\sqrt{3}\delta^2) - \sqrt{9(4-3\delta^2+\sqrt{3}\delta^2)^2 - 72(2-3\delta^2)}}{2(2-3\delta^2)(3+2\sqrt{3})} \ .$$

It is worth noticing that this solution has no singularity at $\delta = \sqrt{2/3}$, while $y_{end}(\delta)$ is a smooth monotonically decreasing function.

Table: The values of y, N_e and r corresponding to $n_s = 0.961$ and $n_s = 0.969$, respectively, and the values of y_{end} for some values of the parameter δ.

δ	y_{end}	$y_{in,\,n_s=0.961}$	$y_{in,\,n_s=0.969}$	$N_{e,0.961}$	$N_{e,0.969}$	$r_{n_s=0.961}$	$r_{n_s=0.969}$
0	0.464	0.0140	0.0112	49.3	62.3	0.0043	0.0027
0.2	0.395	0.00682	0.00505	45.0	56.8	0.0096	0.0065
$\frac{4\sqrt{3}}{9}$	0.299	0.00146	0.000968	48.1	60.9	0.0152	0.0099
1	0.279	0.000939	0.000616	49.4	62.4	0.0157	0.0102
5	0.205	$4.32 \cdot 10^{-5}$	$2.75 \cdot 10^{-5}$	56.3	69.7	0.0168	0.0108
10	0.199	$1.08 \cdot 10^{-5}$	$6.91 \cdot 10^{-6}$	58.7	72.0	0.0168	0.0108
25	0.197	$1.74 \cdot 10^{-6}$	$1.11 \cdot 10^{-6}$	61.4	74.8	0.0169	0.0108
50	0.196	$4.34 \cdot 10^{-7}$	$2.77 \cdot 10^{-7}$	63.5	76.9	0.0169	0.0108
100	0.196	$1.09 \cdot 10^{-7}$	$6.92 \cdot 10^{-8}$	65.5	79.1	0.0169	0.0108

Superstring theory is the theory of quantum gravity that can provide the UV-completion to viable inflation models.
We modify the Starobinsky inflation model by adding the Bel-Robinson (BR) tensor $T^{\mu\nu\lambda\rho}$ squared term proposed as the leading quantum correction inspired by superstring theory.
In 4 spacetime dimensions is defined by the BR tensor squared can be rewritten in terms of the Euler and Pontryagin densities squared by using the identities

$$T^{\mu\nu\lambda\rho}T_{\mu\nu\lambda\rho} = \frac{1}{4}\left(P_4^2 - E_4^2\right), \tag{37}$$

where the Euler and Pontryagin (topological) densities are

$$E_4 = \mathcal{G} = {}^*R_{\mu\nu\lambda\rho}{}^*R^{\mu\nu\lambda\rho} \quad \text{and} \quad P_4 = {}^*R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}\ , \tag{38}$$

respectively.
The Euler density coincides with the Gauss-Bonnet (GB) term, $E_4 = \mathcal{G}$.

Therefore,

$$S_{\mathrm{SBR}}[g_{\mu\nu}] = \frac{M_{\mathrm{Pl}}^2}{2} \int d^4x \sqrt{-g} \left[R + \frac{1}{6m^2} R^2 + \frac{\beta}{32m^6} \left(\mathcal{G}^2 - P_4^2 \right) \right] , \quad (39)$$

where we have introduced the dimensionless coupling constant $\beta > 0$.
In the context of superstrings theory, the constant β must be positive.
S.V. Ketov, *Universe* **8** (2022) 351
The positive sign of β is consistent with the physical requirement in the $F(\mathcal{G})$ modified theories of gravity, demanding the second derivative of the F-function to be positive.
A. De Felice and S. Tsujikawa, *Phys. Lett. B* **675** (2009) 1

In the spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2\left(dx_1^2 + dx_2^2 + dx_3^2\right) \tag{40}$$

with the cosmic scale factor $a(t)$, the P_4 term in the SBR action does *not* contribute to the equations of motion.
In the spatially flat FLRW metric, we get only one independent equation:

$$2\left(m^4+3\beta H^4\right)H\ddot{H}-\left(m^4-9\beta H^4\right)\dot{H}^2+6\left(m^4+3\beta H^4\right)H^2\dot{H}-3\beta H^8+m^6H^2=0\,. \tag{41}$$

The parameter m can be removed from Eq. (41) by using the dimensionless variables $h = H/m$ and $\tau = mt$, which yields

$$2h\left(1+3\beta h^4\right)\frac{d^2h}{d\tau^2}-\left(1-9\beta h^4\right)\left(\frac{dh}{d\tau}\right)^2+6h^2\left(1+3\beta h^4\right)\frac{dh}{d\tau}+h^2\left(1-3\beta h^6\right)=0\,.$$

or, equivalently,

$$\begin{aligned}\frac{dR_h}{d\tau} &= \frac{1}{h(1+3\beta h^4)}\left[\frac{1}{12}R_h^2-R_hh^2-3h^2-\frac{3}{4}\beta\left(R_h-10h^2\right)\left(R_h-18h^2\right)h^4\right],\\ \frac{dh}{d\tau} &= \frac{R_h}{6}-2h^2,\end{aligned}$$

with $R_h = R/m^2$.

INFLATION

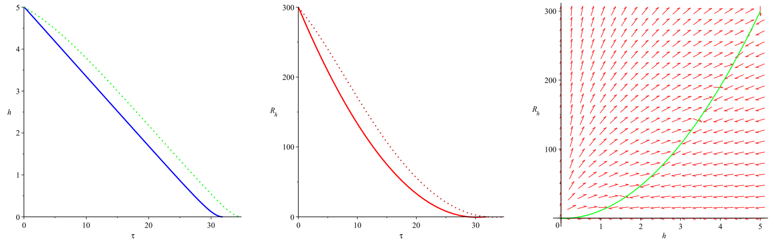

Figure: The functions $h(\tau)$ (on the left), $R_h(\tau)$ (in the center) for $\beta = 0$ (blue and red curves) and $\beta = 10^{-5}$ (green and brown curves), and the phase portrait (R_h, h) for $\beta = 10^{-5}$ (on the right).

SLOW-ROLL SOLUTIONS FOR THE STAROBINSKY INFLATION

In the slow-roll approximation defined by the conditions

$$\left|\ddot{H}\right| \ll \left|H\dot{H}\right| \quad \text{and} \quad \left|\dot{H}\right| \ll H^2 \ , \tag{42}$$

Eq. (41) with $\beta = 0$ is greatly simplified to

$$6\dot{H} + m^2 \approx 0 \ , \tag{43}$$

and has the well-known slow-roll solution of the Starobinsky inflation

$$H(t) \approx \frac{m^2}{6}(t_0 - t) \ , \tag{44}$$

where t_0 is the integration constant that apparently corresponds to the end of inflation. The slow roll conditions (42) are valid provided that $m(t_0 - t) \gg 1$.

SLOW-ROLL SOLUTIONS FOR $\beta \ll 1$

We search for a perturbative solution to Eq. (41) in the first order with respect to the parameter β, in the dimensionless form

$$H(t) = \frac{1}{6}(t_0 - t) - \beta H_1(t), \tag{45}$$

where we have kept only the leading term in the solution to the Hubble function of the Starobinsky inflation during the slow-roll regime, with the unknown function $h_1(\tau)$.
Taking into account the slow-roll conditions (42) allows us to simplify Eq. (41) to the non-linear ordinary differential equation

$$6\left(m^4 + 3\beta H^4\right)\dot{H} - 3\beta H^6 + m^6 = 0\ . \tag{46}$$

When searching for a perturbative solution to this equation in the first order with respect to β, we find

$$H(t) \approx \frac{m^2(t_0 - t)}{6} - \beta\left(\frac{m}{6}\right)^6 (t_0 - t)^5 \left[\frac{m^2}{14}(t_0 - t)^2 + \frac{18}{5}\right]\ . \tag{47}$$

INFLATIONARY PARAMETERS

Various scenarios of inflation in the neighborhood of the Starobinsky model modified by the higher-order curvature terms, depending upon the unknown effective function $F(H^2)$ entering the equations of motion in the FLRW universe have been studied in

P.A. Cano, K. Fransen and T. Hertog, *Phys. Rev. D* **103** (2021) 103531

Though our modification of the Starobinsky model is outside their modified (higher-derivative) gravity theories because Eq. (41) includes $\ddot{H}$, we can apply their results in the slow-roll approximation.

The effective $F(H^2)$ function in our case is given by

$$F(H^2) = H^2 - \frac{3\beta}{m^4}\left(H^2\right)^3 - \frac{3\beta}{m^6}\left(H^2\right)^4 \ . \tag{48}$$

Upper bound on β can be obtained from CMB measurements. The results of the Cano Fransen Hertog paper for the inflationary parameters gives in our case,

$$n_s = 1 - \frac{2}{N} - \frac{8\beta N}{9} - \frac{\beta N^2}{2} \tag{49}$$

and

$$r = \frac{12}{N^2} - \frac{16}{3}\beta - 2\beta N \ , \tag{50}$$

According to the CMB observation data, we have

$$n_s = 0.9649 \pm 0.0042, \quad r < 0.036 \quad , \tag{51}$$

for the tilt n_s of scalar (curvature) perturbations and the tensor-to-scalar ratio r.
In order to be consistent with the observed value of the spectral index n_s, we should demand

$$\beta \leqslant 3.9 \cdot 10^{-6} \quad . \tag{52}$$

The tensor-to-scalar-ratio r is under the upper bound of Eq. (51) for these values of β.
For the amplitude A_s of scalar perturbations, the β-correction is very small

$$A_s \approx 2.1 \cdot 10^{-9} + 5.5 \cdot 10^{-11} N^3 \beta \quad . \tag{53}$$

For instance, when $N = 65$ and $\beta = 10^{-6}$, we get the β-correction of the order $\mathcal{O}(10^{-3})\bar{A}_s$.

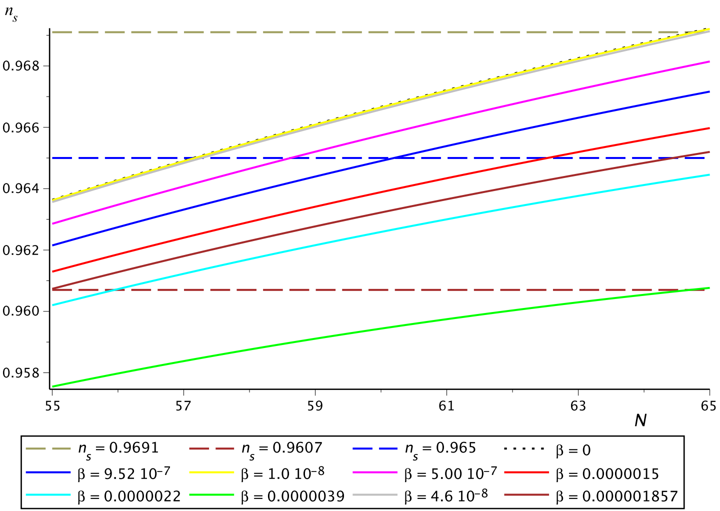

Figure: The spectral index n_s for $0 \leqslant \beta \leqslant 3.9 \cdot 10^{-6}$ with the e-foldings $55 \leqslant N \leqslant 65$. The dotted lines are the boundaries for the observed value of n_s set by the CMB data.

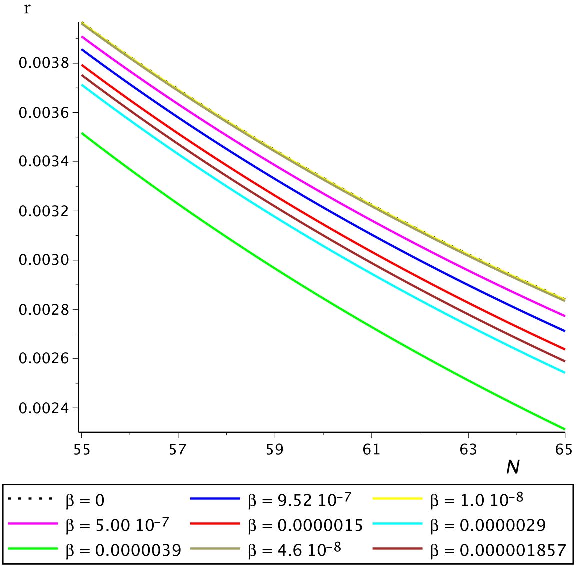

Figure: The tensor-to-scalar ratio r for $0 \leqslant \beta \leqslant 1.857 \cdot 10^{-6}$ with the e-folding number $55 \leqslant N \leqslant 65$.

CONCLUSIONS

• We studied several extensions of the Starobinsky inflation model of the $(R + R^2)$ gravity in the context of $F(R)$ gravity and string-inspired models. By deforming the scalar potential of the Starobinsky model, and derived the corresponding F-function in the analytic form in a new model with a single parameter, and found the lower and upper bounds on the values of the parameters.

• The modification of the Starobinsky model by the $R^{3/2}$ term does not lead to significant constraints on its coefficient in slow-roll inflation, at least for $0 < \delta < 100$.

The $R^{3/2}$ term has a significant impact on the value of the tensor-to-scalar ratio r.

• In the $(R + \frac{1}{6m^2}R^2 - \frac{\beta}{8m^6}T^2)$ model, find the physical bounds on the value of the parameter β by demanding the absence of ghosts and consistency of the derived inflationary observables with the measurements of the cosmic microwave background radiation.

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Thank for your attention