Restrictions imposed by the wave function on the results of measurements of a particle momentum

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Born's interpretation of the wave function

For simplicity we consider a quantum particle moving in the one-dimensional configuration space (OCS). Let its state be described by the wave function

$$\psi(x,t) = \sqrt{w(x,t)}\, e^{i\frac{S(x,t)}{\hbar}}. \qquad (1)$$

By Born, only its amplitude has a physical meaning: $w(x,t)dx$ is the probability of finding a particle in the interval $[x, x+dx]$.

Most physicists believe that this interpretation exhaustively reflects the properties of the quantum mechanical state, and therefore it is on this basis that the interpretation of QM itself should be developed. For example, to emphasize its importance, John Bell begins his famous book "Speakable and unspeakable in quantum mechanics" with the statement "To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements."

On the Bohmian mechanics

But Born's interpretation is not commonly accepted. For example, according to Bohm, the x-derivative of the function $S(x,t)$ in the phase of the wave function also has a physical meaning. In his approach the equations

$$\frac{\partial w}{\partial t} + \frac{1}{m}\frac{\partial}{\partial x}\left(w\frac{\partial S}{\partial x}\right) = 0 : \ continuity \quad eq. \tag{2}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + K_B + V = 0 : \ Hamilton - Jacobi\ eq. \tag{3}$$

describe the motion of a particle in the external potential $V(x)$ and the "quantum mechanical potential" $K_B(x,t)$. He assumes that the $\partial S/\partial x$ is the momentum of the particle, and the probability current density lines in the OCS are its trajectories;

$$K_B = K_w + U_w; \quad K_w = \frac{\hbar^2}{8mw^2}\left(\frac{\partial w}{\partial x}\right)^2, \quad U_w = -\frac{\hbar^2}{4mw}\frac{\partial^2 w}{\partial x^2}$$

Born's view on the standard expression for expectation values of observables

We agree that Born's interpretation is incomplete. But to complete it, no new assumptions should be made. All the necessary information is already contained in the standard expression $\langle A\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle$ for the average value of an observable A. In the x-representation

$$\langle A\rangle_\psi = \int_{-\infty}^{\infty} \psi^*(x,t)\hat{A}\psi(x,t)dx. \tag{4}$$

By Born, if $A = f(x,t)$, then Exp. (4) takes on the form

$$\langle A\rangle_\psi = \int_{-\infty}^{\infty} f(x,t)\, w(x,t)dx, \tag{5}$$

which is standard in classical probability theory, but if $[\hat{A}, \hat{x}] \neq 0$ then Exp. (4) cannot be reduced to this form.

New look at the standard expression of the average value of observables

Our aim is to show that this Born restriction does not follow from QM itself: in fact QM suggests that exp. (4) can be reduced to Exp. (5) without additional assumptions:

$$\langle A\rangle_\psi = \int_{-\infty}^{\infty}\psi^*(x,t)\hat{A}\psi(x,t)dx \quad\Rightarrow\quad \int_{-\infty}^{\infty}A(x,t)w(x,t)dx$$

As $\langle A\rangle_\psi$ is the first initial moment of the random variable A, the function $A(x,t)$ will be called the field of the first initial moment of the observable A or, in short, the field of the operator $\hat{A}$.

We consider only three one-particle observables – the momentum of a particle, its kinetic energy, and its total energy – that is, only those quantities that play a key role in classical mechanics in describing one-particle dynamics.

Momentum operator field

It is easy to show that

$$\langle p\rangle_\psi = -i\hbar\int_{-\infty}^{\infty}\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x}dx = \int_{-\infty}^{\infty}\frac{\partial S(x,t)}{\partial x}w(x,t)dx.$$

That is, $\langle p\rangle_\psi$ is equal to the average value of the function

$$p(x,t) = \frac{\partial S(x,t)}{\partial x} \tag{6}$$

which appears in the Bohmian mechanics as the momentum of a particle; but, as will be seen from the following, this interpretation is erroneous. In our approach, the function $p(x,t)$ will be called the field of the momentum operator.

Kinetic energy operator field

Same way, for the operator $\hat{K} = \hat{p}^2/2m$ we have

$$\langle K \rangle_\psi = \int_{-\infty}^{\infty} \psi^*(x,t)\hat{K}\psi(x,t)dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{[p(x,t)]^2}{2m} + K_B(x,t) \right) w(x,t)dx.$$

The function

$$K(x,t) = \frac{1}{2m}\left[p(x,t)\right]^2 + K_B(x,t) \neq \frac{1}{2m}\left[p(x,t)\right]^2 \qquad (7)$$

is the field of the kinetic energy operator. As is seen, $p(x,t)$ cannot be interpreted as the momentum of the particle at the point x at the time t.

Hamilton operator field

And finally, for the operator $\hat{H} = \hat{K} + V(x)$ we have

$$\langle H \rangle_\psi = \int_{-\infty}^{\infty} \psi^*(x,t) \hat{H} \psi(x,t) dx = \int_{-\infty}^{\infty} E(x,t) w(x,t) dx$$

where, as expected, the field $E(x,t)$ of the operator $\hat{H} = \hat{K} + V(x)$ is

$$E(x,t) \equiv K(x,t) + V(x). \tag{8}$$

In addition to the three fields associated with single-particle observables, we also introduce two fields to characterize the wave function.

Wave-packet's analogs of the de Broglie and Einstein relations

Similar to the (constant) wave number k and frequency ω of the wave $e^{i(px-Et)/\hbar}$, let us define the wave number field $k(x,t)$ and the frequency field $\omega(x,t)$ for the wave packet $\psi(x,t)$:

$$k(x,t) = \frac{1}{\hbar}\frac{\partial S(x,t)}{\partial x}, \qquad \omega(x,t) = -\frac{1}{\hbar}\frac{\partial S(x,t)}{\partial t}. \tag{9}$$

With these two fields, Def. (6) of $p(x,t)$ can now be rewritten in the form of the de Broglie relation

$$p(x,t) \equiv \hbar k(x,t), \tag{10}$$

and Eq. (3) takes now the form of the Planck-Einstein relation:

$$E(x,t) \equiv K(x,t) + V(x) = \hbar\omega(x,t). \tag{11}$$

Note, the physical meaning of the quantities in these relations changes in the transition from the de Broglie wave to a wave packet of a general form. To reveal this meaning, let us turn to König's theorem on the kinetic energy of a system of particles.

König's theorem (mechanics)

We apply it to a 1D system of N identical particles with momenta p_i $(i = 1, ..., N)$ located at the moment t at the point x. By Koenig's theorem, for its kinetic energy we have

$$\sum_i^N \frac{p_i^2}{2m} = N\frac{\bar{p}^2}{2m} + \sum_i^N \frac{(p_i - \bar{p})^2}{2m} \equiv N \cdot \bar{\mathcal{K}}; \qquad \bar{p} = \frac{1}{N}\sum_i^N p_i.$$

- The first term is the kinetic energy associated to the movement of the center of mass (CM) of particles;
- the second is the kinetic energy associated to the movement of the particles relative to the CM;

$\bar{\mathcal{K}}$ is the average kinetic energy of particles in the system:

$$\bar{\mathcal{K}} = \frac{\bar{p}^2}{2m} + \frac{1}{N}\sum_i^N \frac{(p_i - \bar{p})^2}{2m} \Leftrightarrow K(x,t) = \frac{[p(x,t)]^2}{2m} + K_B(x,t)$$

On the physical meaning of $p(x,t)$ and $K(x,t)$

From the comparison of these two expressions we see that the field $p(x,t)$ plays the role of the average value $\bar{p}(x,t)$ and the field $K(x,t)$ plays the role of the average kinetic energy $\bar{\mathcal{K}}(x,t)$. As a consequence, we obtain two equations on the momenta $p_1(x,t)$,..., $p_N(x,t)$:

$$\bar{p}(x,t) \equiv \frac{1}{N}\sum_i^N p_i(x,t) = p(x,t),$$

$$\bar{\mathcal{K}}(x,t) \equiv \frac{1}{N}\sum_i^N \frac{[p_i(x,t)]^2}{2m} = K(x,t). \tag{12}$$

On the two momentum fields in the OCS

Since there are no other physically motivated restrictions on these momenta, $N = 2$. Otherwise, Eqs. (12) would contain $N - 2$ arbitrary momenta independent of x, what is unacceptable in QM. This result we treat as follows: a one-particle quantum ensemble consists of pairs of (non-interacting) one-particle systems in which particles at the moment t have the same coordinate x, but two different (equiprobable) values of the momentum – $p_1(x,t)$ and $p_2(x,t)$:

$$\frac{1}{2}(p_1 + p_2) = p, \quad \frac{1}{2}\left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m}\right) = \frac{p^2}{2m} + K_B :$$

$$p_1 = p - \sqrt{2mK_B}, \quad p_2 = p + \sqrt{2mK_B}. \tag{13}$$

Inequality for the variances of the fields $p_1(x)$ and $p_2(x)$

$$D_p = \int_{-\infty}^{\infty} \left[p_{1,2}(x,t) - p(x,t)\right]^2 w(x,t)\, dx,$$

$$D_x = \int_{-\infty}^{\infty} (x- <x>)^2 \, w(x,t)\, dx; \quad <x>= \int_{-\infty}^{\infty} x\, w(x,t) dx.$$

$$D_p = \frac{\hbar^2}{4} \int_{-\infty}^{\infty} \left[\left(\frac{1}{w} \frac{\partial w}{\partial x} \right)^2 - \frac{2}{w} \frac{\partial^2 w}{\partial x^2} \right] w\, dx = \frac{\hbar^2}{4} \int_{-\infty}^{\infty} \left(\frac{1}{w} \frac{\partial w}{\partial x} \right)^2 w\, dx$$

$w(x,t)$ with all its x-derivatives tend to zero when $|x| \to \infty$.
Using the Cauchy-Bunyakovsky theorem, we obtain

$$D_p D_x = \left\{ \frac{\hbar^2}{4} \int_{-\infty}^{\infty} \left(\frac{1}{w} \frac{\partial w}{\partial x} \right)^2 w\, dx \right\} \left\{ \int_{-\infty}^{\infty} (x- <x>)^2 \, w\, dx \right\}$$

$$\geq \frac{\hbar^2}{4} \left[\int_{-\infty}^{\infty} \left(\frac{1}{w} \frac{\partial w}{\partial x} \right) (x- <x>)\, w\, dx \right]^2 = \frac{\hbar^2}{4}$$

Equations for $p_1(x,t)$, $p_2(x,t)$ and $w(x,t)$

Note that Eq. (2) as well as Eq. (3) (after differentiating with respect to x) can now be rewritten as follows:

$$\frac{\partial w}{\partial t} + \frac{1}{m}\frac{\partial (wp)}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + \frac{p}{m}\frac{\partial p}{\partial x} = -\frac{\partial K_B}{\partial x} - \frac{\partial V}{\partial x};$$

$$p = \frac{\partial S}{\partial x} = \frac{p_2 + p_1}{2},$$

$$K_B = \frac{\hbar^2}{8mw^2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{\hbar^2}{4mw}\frac{\partial^2 w}{\partial x^2} = \frac{1}{2m}\left(\frac{p_2 - p_1}{2}\right)^2.$$

What follows from these equations

- The probability current density lines in the OCS are the trajectories of the CM of pairs of particles with two equally probable momenta, and not the trajectories of a particle with a certain momentum.
- The “Bohmian potential” K_B actually represents the kinetic energy (per one particle) associated with the movement of a pair of particles located at the point x at the time t relative to their CM.

What follows from these equations

- These CMs are pushed out of the regions of the OCS, where the kinetic energy $K_B(x,t)$ is greater, into those regions where it is less. This mechanism leads to the appearance of probability flows in the OCS, which tends to smooth the function $K_B(x,t)$ and, hence, leads to the spreading of the wave packet. Stationary states arise when this mechanism is balanced by the action of an external field $V(x)$.
- Since the kinetic energy $p^2/2m$ of CMs is always non-negative, it is due to this mechanism that the CMs of paired particles can fall into the classically forbidden regions of the OCS (where K_B is negative).

On the experimental verification of this approach

Our approach implies the following:

- The Heisenberg uncertainty relation imposes restrictions on the variances of the random variables x and p, but not on the errors in their measurement — in particular, it does not prohibit the simultaneous measurement of x and p (or, of x and K).
- What is really forbidden by this principle is the existence of a joint probability distribution $w(x, p, t)$, where x and p play the role of two independent random variables.
- Instead, it implies the existence of a pair of two (joint) distributions – $w(x, t)$ and $w(p, t)$ – associated with the x- and p-representations of the same state vector.

On the experimental verification of this approach

These distributions complement each other, giving a complete information about the quantum mechanical state:

- According to our approach, in the x-representation QM predicts not only the probability field $w(x,t)$, but also the momentum fields $p_1(x,t)$ and $p_2(x,t)$ (and the corresponding kinetic energy fields $K_1(x,t)$ and $K_2(x,t)$). For experimental verification of these fields, it is necessary to collect pairs (x,p) of jointly measured values x and p (or pairs (x,K)); these measurement data must satisfy the condition $\int_{-\infty}^{\infty} w(x,t)dx = 1$.

On the experimental verification of this approach

- In the p-representation QM predicts not only the probability field $w(p,t)$, but also the field $x(p,t)$. Again, for experimental verification of these fields, it is necessary to collect the pairs (x,p) of jointly measured values x and p, but now the measurement data must satisfy the condition $\int_{-\infty}^{\infty} w(p,t)dp = 1$. In the p-representation, the usual relation $K = p^2/2m$ between the particle's kinetic energy and its momentum is satisfied: in this case, there is no need for König's theorem relating to a *system* of particles.

Thank you very much for attention!