Some notes on the Weyl geometry, particle production and induced gravity

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A little bit of mathematics

Differential geometry

Metric tensor $g_{\mu\nu}$ $\Rightarrow$ interval

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \qquad (g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu)$$

Connections $\Gamma^\lambda_{\mu\nu}$ $\Rightarrow$ covariant derivative

$$\nabla_\mu l^\nu = l^\nu_{,\mu} + \Gamma^\nu_{\sigma\mu} l^\sigma, \ \ldots$$

Curvature tensor $R^\mu{}_{\nu\lambda\sigma}$

$$R^\mu{}_{\nu\lambda\sigma} = \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\lambda} - \frac{\partial \Gamma^\mu_{\nu\lambda}}{\partial x^\sigma} + \Gamma^\mu_{\varkappa\lambda}\Gamma^\varkappa_{\nu\sigma} - \Gamma^\mu_{\varkappa\sigma}\Gamma^\varkappa_{\nu\lambda}$$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Curvature scalar $R = g^{\mu\lambda}R_{\mu\lambda}$

$$g_{\mu\nu}(x), \quad \Gamma^\lambda_{\mu\nu}(x)$$

A little bit of mathematics

Differential geometry II

Three tensors	$g_{\mu\nu}(x),\ S^{\lambda}_{\mu\nu},\ Q_{\lambda\mu\nu}$
Torsion	$S^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$
Nonmetricity	$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu}$
Connections	$\Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}$

$$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\varkappa}(g_{\varkappa\mu,\nu} + g_{\varkappa\nu,\mu} - g_{\mu\nu,\varkappa})$$

$$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} \quad \Rightarrow \quad Q_{\lambda\mu\nu} = Q_{\lambda\nu\mu}$$

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(S^{\lambda}{}_{\mu\nu} - S_{\mu}{}^{\lambda}{}_{\nu} - S_{\nu}{}^{\lambda}{}_{\mu})$$

$$L^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu})$$

A little bit of mathematics
Differential geometry III

Riemann geometry

$$S^{\lambda}_{\mu\nu} = 0, \quad Q_{\lambda\mu\nu} = 0$$

$$S^{\lambda}_{\mu\nu} = 0, \quad Q_{\lambda\mu\nu} = 0, \quad \Rightarrow \quad \Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu}$$

Weyl geometry

$$S^{\lambda}_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$$

$$Q_{\lambda\mu\nu} = A_{\lambda}(x) g_{\mu\nu}(x)$$

$$\Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu} + W^{\lambda}_{\mu\nu}$$

$$W^{\lambda}_{\mu\nu} = -\frac{1}{2}(A_{\mu}\delta^{\lambda}_{\nu} + A_{\nu}\delta^{\lambda}_{\mu} - A^{\lambda} g_{\mu\nu})$$

$A_{\lambda}(x)$ — "Weyl vector"

Local conformal transformation

$$ds^2 = \Omega^2(x) d\hat{s}^2 = \Omega^2 \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{\Omega^2} \hat{g}^{\mu\nu}$$

$$\sqrt{-g} = \Omega^4(x) \sqrt{-\hat{g}}$$

$$C^\lambda_{\mu\nu} = \hat{C}^\lambda_{\mu\nu} + \left(\frac{\Omega_{,\nu}}{\Omega} \delta^\lambda_\mu + \frac{\Omega_{,\mu}}{\Omega} \delta^\lambda_\nu - g^{\lambda\varkappa} \frac{\Omega_{,\varkappa}}{\Omega} \hat{g}_{\mu\nu} \right)$$

If $A_\mu =$ gauge field

$$A_\mu = \hat{A}_\mu + 2 \frac{\Omega_{,\mu}}{\Omega} \quad \Rightarrow$$

$$\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} \quad \Rightarrow$$

$$R^\mu{}_{\nu\lambda\sigma} = \hat{R}^\mu{}_{\nu\lambda\sigma}, \quad R_{\mu\nu} = \hat{R}_{\mu\nu}$$

Strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}$$

Weyl gravity

$$\mathcal{L}_{\rm W} = \alpha_1 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu}F^{\mu\nu}$$

$$S_{\rm W} = \int \mathcal{L}_{\rm W}\sqrt{-g}\, d^4x, \quad \frac{\delta S_{\rm W}}{\delta \Omega} = 0$$

Dynamical variables: $g_{\mu\nu}(x)$, $A_\mu x$

Matter fields $S_{\rm tot} = S_{\rm W} + S_{\rm m}$, $S_{\rm m} = \int \mathcal{L}_{\rm m}\sqrt{-g}\, d^4x$

$S_{\rm m}$ is not necessary conformal invariant

$\delta S_{\rm m}$ does!

$$\delta S_{\rm m} \stackrel{\rm def}{=} -\frac{1}{2}\int T^{\mu\nu}(\delta g_{\mu\nu})\sqrt{-g}\, d^4x - \int G^\mu(\delta A_\mu)\sqrt{-g}\, d^4x$$

$$+ \int \frac{\delta \mathcal{L}_{\rm m}}{\delta \Psi}(\delta\Psi)\sqrt{-g}\, d^4x = 0$$

Ψ — collective dynamical variable for matter fields, $\delta\mathcal{L}_{\rm m}/\delta\Psi = 0$

$T^{\mu\nu}$ — energy-momentum tensor, G^μ — "Weyl current"

Weyl gravity II

$$\delta g_{\mu\nu} = \frac{2}{\Omega} g_{\mu\nu}(\delta\Omega) + \Omega^2(\delta\hat{g}_{\mu\nu})$$

$$\delta A_\mu = (\delta A_\mu) + 2(\delta(\log\Omega)_{,\mu}$$

$$\delta\hat{g}_{\mu\nu} : \quad \frac{\delta S_{\rm m}}{\delta\hat{g}_{\mu\nu}} \stackrel{\rm def}{=} -\frac{1}{2}\hat{T}_{\mu\nu} \quad \Rightarrow \quad \hat{T}_{\mu\nu} = \Omega^2 T_{\mu\nu}$$

$$(\hat{T}_\mu^\nu = \Omega^4 T_\mu^\nu, \ \hat{T}^{\mu\nu} = \Omega^6 T_{\mu\nu})$$

$$\delta\hat{A}_\mu : \quad \frac{\delta S_{\rm m}}{\delta\hat{A}_\mu} \stackrel{\rm def}{=} -\hat{G}^\mu \quad \Rightarrow \quad \hat{G}^\mu = \Omega^4 G^\mu$$

Self-consistency condition

$$\delta\Omega : \quad 2G^\mu_{;\mu} = Trace[T^{\mu\nu}]$$

";" — covariant derivative with $C^\lambda_{\mu\nu}$

Cosmology

Cosmology $=$ homogeneity and isotropy

Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right), \quad k = 0, \pm 1$$

$$A_\mu = (A_0(t), 0, 0, 0) \quad \Rightarrow$$

$$\boxed{F_{\mu\nu} \equiv 0}$$

$$T^\mu_\nu = (T^0_0(t), T^1_1(t) = T^2_2 = T^3_3)$$

Special gauge

$$A_0(t) = 0$$

Field equations

$$\delta A_\mu : \quad -6\gamma \dot{R} = G^0, \quad \gamma = \frac{1}{3}(\alpha_1 + \alpha_2 + 3\alpha_3)$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$$

$$\delta g_{\mu\nu} : \quad -12\gamma\left\{\frac{\dot{a}}{a}\dot{R} + R\left(\frac{R}{12} + \frac{\dot{a}^2 + k}{a^2}\right)\right\} = T^0_0$$

$$-4\gamma\left\{\ddot{R} + 2\frac{\dot{a}}{a}\dot{R} - R\left(\frac{R}{12} + \frac{\dot{a}^2 + k}{a^2}\right)\right\} = T^1_1$$

Self-consistency condition:

$$2\frac{(G^0 a^3)^{\dot{}}}{a^3} = T^0_0 + 3\,T^1_1$$

Perfect fluid, Riemann geometry I

J.R. Ray (1972):

$$S_{\mathrm{m}} = -\int \varepsilon(X,n)\sqrt{-g}\,d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\,d^4x + \int \lambda_1 (n u^\mu)_{;\mu}\sqrt{-g}\,d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g}\,d^4x$$

$\varepsilon(X,n)$ — energy density
$u^\mu(x)$ — four-velocity
$n(x)$ — particle number density,
$X(x)$ — auxiliary variable,
$\lambda_i(x)$ — Lagrange multipliers

Constraints

$u^\mu u_\mu = 1$ — normalization
$(n u^\mu)_{;\mu} = 0$ — particle number conservation
$X_{,\mu} u^\mu = 0$ — numbering of the trajectories

Perfect fluid, Riemann geometry II

Energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - p g^{\mu\nu}$$

Hydrodynamical pressure

$$p = n \frac{\partial \varepsilon}{\partial n} - \varepsilon$$

V.A.Berezin (1987):
Phenomenological description of particle creation

$$(n u^{\mu})_{;\mu} = 0 \quad \Rightarrow \quad (n u^{\mu})_{;\mu} - \Phi(inv) = 0$$

Perfect fluid, Weyl geometry: G^{μ} — ? How to incorporate A_{μ}?

$$\Downarrow$$

New possibilities

Single particle u^μ

Riemann geometry

$$S_{\rm part} = -m\int ds$$

Weyl geometry — new invariant $B = A_\mu u^\mu$

$$S_{\rm part} = \int f_1(B)ds + \int f_2(B)d\tau = \int\{f_1(B)\sqrt{g_{\mu\nu}u^\mu u^\nu} + f_2(B)\}d\tau$$

Equations of motion

$$\begin{aligned} f_1(B)u_{\lambda;\mu}u^\mu &= \left((f_1^{'}(B) + f_2^{''}(B))A_\lambda - f_1^{'}(B)u_\lambda\right) B_{,\mu}u^\mu \\ &+ (f_1^{'}(B) + f_2^{'}(B))F_{\lambda\mu}u^\mu \end{aligned}$$

Since $F_{\lambda\mu}u^\lambda u^\mu \equiv 0$ and $u_{\lambda;\sigma}u^\lambda \equiv 0$

Either $(f_1^{'} + f_2^{''})B = f_1^{'}$ or $B_{,\mu}u^\mu = 0$

The invariant $B = A_\mu u^\mu$ is closely tied to the number density n. Hence,

$$\varepsilon(X, n) \;\Rightarrow\; \varepsilon(X, \varphi(B) n)$$

And how about the constraint $(nu^\mu)_{;\mu} = \Phi(inv)$?
Conformal transformation

$$(nu^\mu)_{;\mu}\sqrt{-g} = (nu^\mu\sqrt{-g})_{,\mu}$$

$$n = \frac{\hat{n}}{\Omega^3}, \quad u^\mu = \frac{\hat{u}^\mu}{\Omega}, \quad \sqrt{-g} = \Omega^4\sqrt{-\hat{g}} \;\Rightarrow$$

$$(nu^\mu\sqrt{-g})_{,\mu} = (\hat{n}\hat{u}^\mu\sqrt{-\hat{g}})_{,\mu} \quad \Rightarrow$$

$\Phi\sqrt{-g}$ — conformal invariant $\Rightarrow$
In the absence of the classical fields, i.e., in the case of the particle creation by the vacuum fluctuations

$$\Phi = \alpha_1' R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2' R_{\mu\nu} R^{\mu\nu} + \alpha_3' R^2 + \alpha_4' F_{\mu\nu} F^{\mu\nu}$$

Riemann geometry — the only possibility

$$\Phi \propto C^2$$

$C^2 = C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$ — square of the Weyl tensor.

Y.B.Zel'dovich and A.A.Starobinsky (1977):

Particle creation by the vacuum fluctuation of the massless scalar field on the homogeneous slightly anisotropic cosmological background.
Now it becomes fundamental!

Gravitational cosmological equations

Vector:

$$-6\gamma\dot{R} = G^0$$

Tensor:

$$-\gamma\left(12\frac{\dot{a}}{a}\dot{R} + R(4R_0^0 - R)\right) = T_0^0$$

$$-12\gamma\left(\ddot{R} + 3\frac{\dot{a}}{a}R\right) = T$$

Self-consistency condition

$$2\frac{(G^0a^3)^{\dot{}}}{a^3} = T_0^0 + 3T_1^1 = T$$

It is quite clear that the self-consistency condition is just the consequence of the vector and trace equations. At last, let us write down the expressions for the $\binom{0}{0}$-component of Ricci tensor and the scalar curvature in terms of the scale factor $a(t)$

$$R^0_0 = -3\frac{\ddot{a}}{a}$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right), \quad k = 0, \pm 1$$

Note that in cosmology we need to know only $T^{00} = T^0_0$ and $T = \mathrm{Trace}\, T^{\mu\nu}$, since $T^{0i} = 0$ and $T^{ij} = T^1_1 g^{ij}$, $T^1_1 = (1/3)(T - T^0_0)$. Thus,

$$T = T[\mathrm{part}] + T[\mathrm{cr}]$$

$$T[\mathrm{part}] = \varepsilon - 3p$$

$$T[\mathrm{cr}] = \ddot{\lambda}_1(8\beta' R^0_0 - 4\beta' R - 12\gamma' R)$$

$$-4\dot{\lambda}_1\left(\beta'\frac{\dot{a}}{a}(R + 2R^0_0) + 6\gamma'\dot{R} + 9\gamma'\frac{\dot{a}}{a}R\right) - 12\lambda_1\gamma'(\ddot{R} + 3\frac{\dot{a}}{a}R)$$

$$T^0_0 = T^0_0[\mathrm{part}] + T^0_0[\mathrm{cr}], \quad T^0_0[\mathrm{part}] = \varepsilon$$

$$T_0^0[\mathrm{cr}] = 8\gamma'\dot{\lambda}_1\frac{\dot{a}}{a}R_0^0 - 4(\beta' + 3\gamma')\dot{\lambda}_1\frac{\dot{a}}{a}R$$

$$-\gamma'\lambda_1\left(12\frac{\dot{a}}{a}\dot{R} + R(4R_0^0 - R)\right)$$

How about the equations of motion for the cosmological perfect fluid? We are left with only one equation plus the law of the particle creation, namely

$$\begin{cases} \dot{\lambda}_1 = -\frac{\varepsilon+p}{n} \\ \frac{(na^3)^{\cdot}}{a^3} = \Phi(\mathrm{inv}) \end{cases}$$

$$\Phi(\mathrm{inv}) = -\frac{4}{3}\beta' R_0^0(2R_0^0 - R) + \gamma' R^2.$$

Creation of the universe from nothing

Empty from the very beginning
Vacuum is not absolute, but physical
May or may not it persists?
"Pregnant" vacuum

$$\Phi(\mathrm{inv}) = 0, \quad |\beta'| + |\gamma'| \neq 0$$

$$\frac{4}{3}\beta' R_0^0 (2R_0^0 - R) = \gamma' R^2$$

$$G^0[\mathrm{part}] = T_0^0[\mathrm{part}] = T[\mathrm{part}] = 0$$

$$n = 0$$

$$\dot{\lambda}_1 = -\frac{\varepsilon + p}{n} \quad \Rightarrow$$

1. Non-dust matter
   $\dot{\lambda}_1 = 0 \quad \Rightarrow \quad \lambda_1 = const$
2. Dust matter
   $\dot{\lambda}_1 = -\phi(0) = const \quad \Rightarrow \quad \lambda_1 = -\phi(0)(t - t_0)$

Non-dust pregnancy I

General case: $\beta', \gamma' \neq 0$

$$\lambda_1 = const$$

$$R = \xi R_0^0 \quad \Rightarrow$$

$$(3\gamma'\xi^2 + 4\beta'(\xi - 2))R_0^0 = 0$$

$$-6(\gamma - \gamma'\lambda_1)\dot{R} = 0$$

$$-12(\gamma - \gamma'\lambda_1)\frac{(\dot{R}a^3)^{\cdot}}{a^3} = 0$$

$$-(\gamma - \gamma'\lambda_1)\left\{12\frac{\dot{a}}{a}\dot{R} + R(R - 4R_0^0)\right\} = 0$$

Let, first, $\gamma \neq \gamma'\lambda_1 \quad \Rightarrow \quad \dot{R} = 0$
Either $R = 0 \quad \Rightarrow \quad R_0^0 = 0 \quad \Rightarrow$ Milne universe
Or $\xi = 4 \quad \Rightarrow$ de Sitter for $\beta + 6\gamma' = 0$

Non-dust pregnancy II

General case: $\beta', \gamma' \neq 0$

$$\gamma = \gamma' \lambda_1$$

It is not a special condition, but the solution λ_1

$$\dot{a}^2 + k = C_0 a^{\frac{4}{\xi - 2}}$$

Instability and so on...

Dust pregnancy

$$\lambda_1 = -\phi(0)(t - t_0) \quad \Rightarrow \quad \dot{\lambda}_1 = -\phi < 0, \quad \ddot{\lambda}_1 = 0$$

Surely, there exists the solution with $R = R_0^0 = 0$, i. e., Milne universe.

It can be shown that the only other solution is just the de Sitter universe, with $\xi = 4$ ($R,\ R_0^0 = const$) for $\beta' + 6\gamma' = 0$.

If it is not so, the universe emerged from the vacuum foam, like Aphrodite, immediately starts to produce dust particles !!!

The End