Dynamics of open quantum systems in symplectic tomography

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Contents

Motivation

I. Fundamental: An alternative picture of quantum dynamics — the probability representation of quantum mechanics

- A quantum state characterized by a set of probability distributions
- The way to link the formalism of quantum mechanics and classical statistical physics

II. Practical: Controlling the state of a quantum system

- The quantum tomography is used to recover the density matrix of the system using experimental data
- Step-wise control of quantum state in designing quantum devices
- It is more efficient to predict the state of an evolving system directly in tomographic picture (including consideration of interaction with environment)

I. Introduction

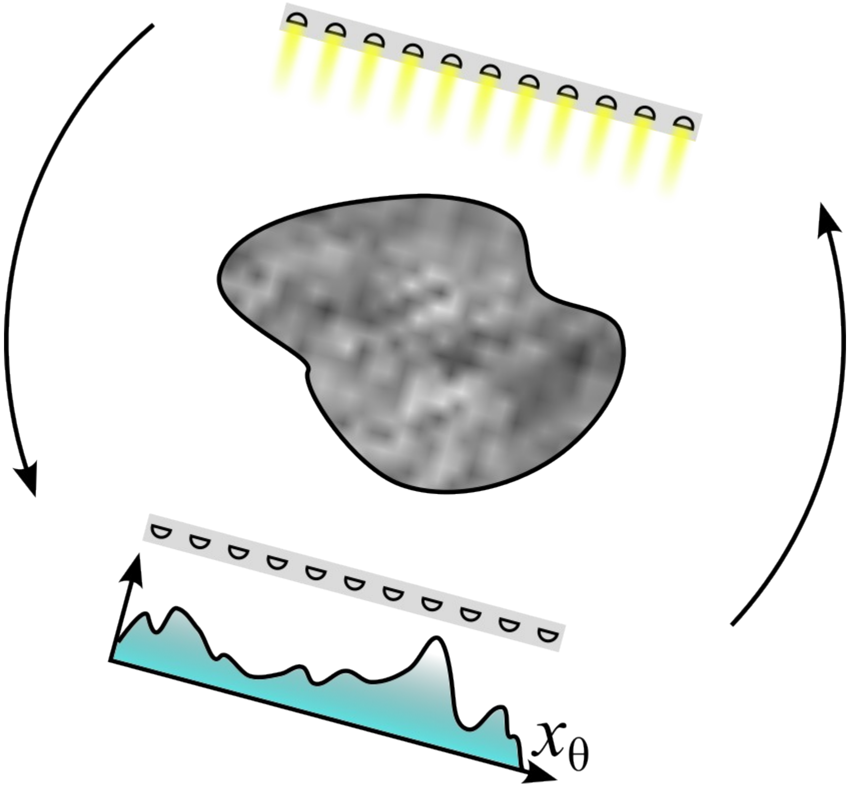
I. The most famous alternative representations of a quantum state

		s-parameter
Wigner quasiprobability distribution	$W(q,p) = \frac{1}{\pi}\int \langle q+y\|\hat{\rho}\|q-y\rangle e^{-2ipy}dy$	$s=0$
Husimi Q representation	$Q(\alpha) = \frac{1}{\pi}\langle\alpha\|\hat{\rho}\|\alpha\rangle$	$s=-1$
Glauber–Sudarshan P representation	$\hat{\rho} = \int P(\alpha)\|\alpha\rangle\langle\alpha\|d^2\alpha$	$s=1$

A new representation:

Mancini, S., Man'ko, V. I., Tombesi, P. (1996). Symplectic tomography as classical approach to quantum systems. Physics Letters A, 213(1-2), 1-6.

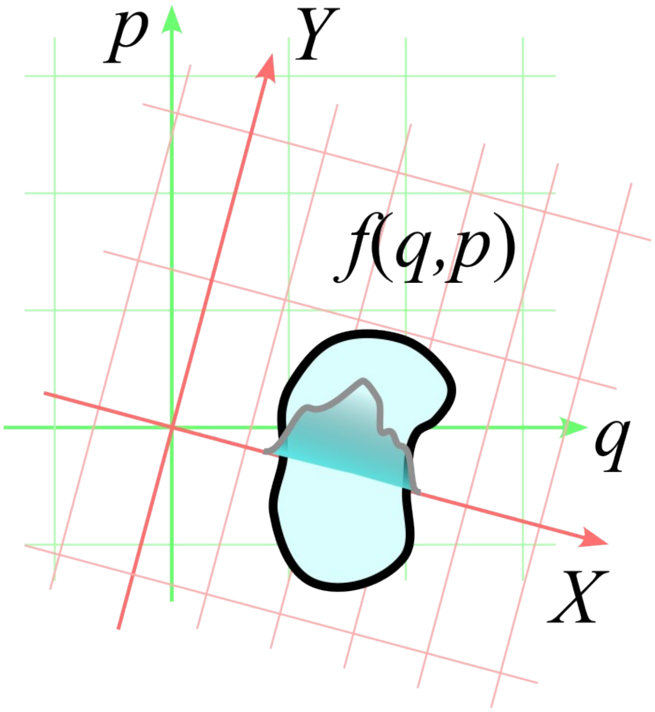
I. The idea of tomography



Tomography
(ancient Greek τόμος — "slice, section")

is imaging by sections or sectioning
that uses any kind of penetrating wave.

I. Symplectic tomography in classical physics



Probability density $f(q,p) \longleftrightarrow \mathcal{T}(X,\mu,\nu)$ Tomogram

Symplectic transformation of the phase space, preserving its symplectic structure

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \varkappa & \chi \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad \begin{pmatrix} \mu & \nu \\ \varkappa & \chi \end{pmatrix} = \Lambda \in Sp(2,\mathbb{R})$$

Meaning of the parameters: $\mu = e^{\lambda}\cos\theta, \nu = e^{-\lambda}\sin\theta$

Probability distribution for X:

$$\mathcal{T}(X,\mu,\nu) = \int f(q,p)\delta(X-\mu q-\nu p)dqdp$$

Recovery of initial probability distribution:

$$f(q,p) = \frac{1}{4\pi^2}\int \mathcal{T}(X,\mu,\nu)e^{-i(\mu q+\nu p-X)}dXd\mu d\nu$$

I. Symplectic tomography in quantum physics

Let $\hat{X} = \mu\hat{q} + \nu\hat{p}$ which eigenstates are $|X\rangle$: $\quad \hat{X}|X\rangle = X|X\rangle$

$$\varphi_X(q) = \langle q|X\rangle = (2\pi|\nu|)^{-1} e^{i\frac{X}{\nu}q - \frac{i}{2}\frac{\mu}{\nu}q^2}$$

Define the *symplectic tomogram* to be $\mathcal{T}(X, \mu, \nu) = \langle X|\hat{\rho}|X\rangle$

Connection with the Wigner function has the form as for probability density in the classical phase space:

$$\mathcal{T}(X, \mu, \nu) = \int W(q,p)\delta(X - \mu q - \nu p)dqdp \qquad W(q,p) = \frac{1}{4\pi^2}\int \mathcal{T}(X, \mu, \nu)e^{-i(\mu q + \nu p - X)}dXd\mu d\nu$$

where the Wigner function is $W(q,p) = \frac{1}{\pi}\int \langle q + y|\hat{\rho}|q - y\rangle e^{-2ipy}dy$

Mancini, S., Man'ko, V. I., Tombesi, P. (1996). Symplectic tomography as classical approach to quantum systems. Physics Letters A, 213(1-2), 1-6.

II. Formalism of operator symbols

II. Formalism of operator symbols for symplectic tomography

Let $\vec{x}=(X,\mu,\nu)$ be a vector consisting of three real variables

Define $\hat{\mathcal{U}}(\vec{x})=\delta(X-\mu\hat{q}-\nu\hat{p})=(2\pi)^{-1}\int\exp\{ikX-ik\mu\hat{q}-ik\nu\hat{p}\}dk$ («Dequantizer»)

$\hat{\mathcal{D}}(\vec{x})=(2\pi)^{-1}\exp\{iX-i\mu\hat{q}-i\nu\hat{p}\}$ («Quantizer»)

for which $\mathrm{Tr}\{\hat{\mathcal{U}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\}=\delta(\vec{x}_1-\vec{x}_2)$

operator $\hat{A}\longleftrightarrow f_{\hat{A}}(\vec{x})$ operator symbol

$$f_{\hat{A}}(\vec{x})=\mathrm{Tr}\{\hat{A}\hat{\mathcal{U}}(\vec{x})\}$$

$$\hat{A}=\int f_{\hat{A}}(\vec{x})\hat{\mathcal{D}}(\vec{x})d\vec{x}$$

II. Product of operator symbols

Noncommutativity of operators

Star product

$$f_{\hat{A}\hat{B}}(\vec{x}) \equiv f_{\hat{A}}(\vec{x}) \star f_{\hat{B}}(\vec{x}) = \int f_{\hat{A}}(\vec{x}_1) f_{\hat{B}}(\vec{x}_2) \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} d\vec{x}_1 d\vec{x}_2$$

The kernel

$$\mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} = (2\pi)^{-2}\delta(\mu(\nu_1+\nu_2)-\nu(\mu_1+\mu_2))e^{iX_1+iX_2}e^{-i\frac{(\nu_1+\nu_2)}{\nu}X}e^{\frac{i}{2}(\nu_1\mu_2-\nu_2\mu_1)}$$

Scalar product: $$(f_{\hat{A}}, f_{\hat{B}}) = \frac{1}{2\pi}\int f^*_{\hat{A}}(X,\mu,\nu) f_{\hat{B}}(X',\mu,\nu) e^{i(X-X')} dX' dX d\mu d\nu$$

II. Formalism of operator symbols for symplectic tomography

Let $\vec{x} = (X, \mu, \nu)$ be a vector consisting of three real variables

Define $\hat{\mathcal{U}}(\vec{x}) = \delta(X - \mu\hat{q} - \nu\hat{p}) = (2\pi)^{-1}\int \exp\{ikX - ik\mu\hat{q} - ik\nu\hat{p}\}dk$ («Dequantizer»)

$\hat{\mathcal{D}}(\vec{x}) = (2\pi)^{-1}\exp\{iX - i\mu\hat{q} - i\nu\hat{p}\}$ («Quantizer»)

for which $\mathrm{Tr}\{\hat{\mathcal{U}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\} = \delta(\vec{x}_1 - \vec{x}_2)$

Density matrix $\hat{\rho} \longleftrightarrow \mathcal{T}(\vec{x})$ tomogram

$$\mathcal{T}(\vec{x}) \equiv f_{\hat{\rho}}(\vec{x}) = \mathrm{Tr}\{\hat{\rho}\hat{\mathcal{U}}(\vec{x})\}$$

$$\hat{\rho} = \int \mathcal{T}(\vec{x})\hat{\mathcal{D}}(\vec{x})d\vec{x}$$

Space of quantum states: Hilbert space → space of probability densities

II. Formalism of operator symbols for symplectic tomography

Properties of symplectic tomograms

It is a probability density

$$\mathcal{T}(\vec{x}) > 0, \quad \int \mathcal{T}(\vec{x}) dX = 1$$

$$\mathcal{T}(X, 1, 0) = \langle q|\hat{\rho}|q\rangle$$

$$\mathcal{T}(X, 0, 1) = \langle p|\hat{\rho}|p\rangle$$

The mean value

$$\langle \hat{A} \rangle = \mathrm{Tr}\{\hat{\rho}\hat{A}\}$$

$$= \frac{1}{2\pi} \int \mathcal{T}(X, \mu, \nu) f_{\hat{A}}(X' - X, \mu, \nu) dX dX' d\mu d\nu$$

The probability of transition from state 1 to 2

$$P_{1\to 2} = (\mathcal{T}_1(\vec{x}), \mathcal{T}_2(\vec{x}))$$

Indicator of the mixed state

$$(\mathcal{T}(\vec{x}), \mathcal{T}(\vec{x})) < 1$$

Reparametrization:

$$\mu = e^{\lambda} \cos\theta, \nu = e^{-\lambda} \sin\theta$$

$$\lambda = 1 : \mu = \cos\theta, \nu = \sin\theta$$

$$\mathcal{T}(X, \mu, \nu) \longrightarrow w(X, \theta)$$ optical tomogram

II. Unitary evolution of the tomograms

Classical quantum mechanics

Von Neumann equation

$$\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}]$$

Through the evolution operator

$$\hat{\rho}(t) = \hat{U}_t \hat{\rho}(0) \hat{U}_t^\dagger$$

Simplectic tomography

$$\frac{\partial \mathcal{T}}{\partial t} = -i \int h(\vec{x}_1)\mathcal{T}(\vec{x}_2) \cdot \mathrm{Tr}\left\{[\hat{\mathcal{D}}(\vec{x}_1), \hat{\mathcal{D}}(\vec{x}_2)]\hat{\mathcal{U}}(\vec{x})\right\} d\vec{x}_1 d\vec{x}_2 \equiv -i[h, \mathcal{T}]_\star$$

(Fokker-Plank-style equation)

$$\mathcal{T}(\vec{x}', t) = \int \mathcal{T}(\vec{x}', 0)\Pi_t(\vec{x}', \vec{x}) d\vec{x}'$$

$$\Pi_T(\vec{x}', \vec{x}) \equiv \mathrm{Tr}\left\{\hat{\mathcal{D}}(\vec{x}')(\hat{U}_T)^\dagger \hat{\mathcal{U}}(\vec{x})\hat{U}_T\right\}$$

(«classical» propagator)

$$\frac{\partial \Pi_t}{\partial t} = -i[f_{\hat{H}}, \Pi_t]_\star, \quad \Pi_0 = \delta(\vec{\bar{x}} - \vec{x})$$

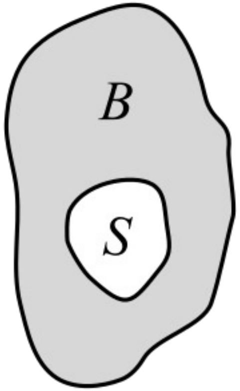
Example: free particle $\hat{H} = \hat{p}^2/(2m)$

$$\frac{\partial \mathcal{T}}{\partial t} - \frac{\mu}{m}\frac{\partial \mathcal{T}}{\partial \nu} = 0$$

$$\Pi_T(\vec{x}', \vec{x}) = \delta\left(X' - X\right)\delta\left(\mu' - \mu\right)\delta\left(\nu' - \nu - \mu\frac{T}{m}\right)$$

III. Quantum processes

III. Operator-sum representation of quantum processes



Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{int}$

Evolution operator $\hat{U}_t = \int_0^t e^{-i\hat{H}s} ds$

State transformation $\hat{\rho}(t) = \hat{U}_t \hat{\rho}(0) \hat{U}_t^\dagger$

Initial state $\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0)$

$$\hat{\rho}_B(0) = \sum_m p_m |m\rangle\langle m|$$

Operator-sum representation

$$\hat{\rho}_S(t) = \sum_{m,n} \hat{K}_{mn} \hat{\rho}_S(0) \hat{K}_{mn}^\dagger$$

$$\hat{K}_{mn} = \sqrt{p_n} \langle m | \hat{U}(t) | n \rangle$$

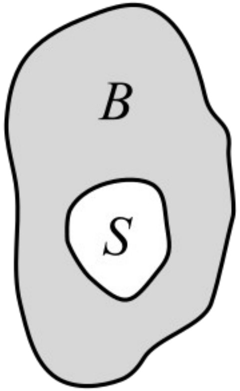
$$\sum_{m,n} \hat{K}_{mn}^\dagger \hat{K}_{mn} = \hat{I}$$

Quantum process

$\Phi : \hat{\rho}(0) \to \hat{\rho}(t)$

1. Trace preserving: $\mathrm{Tr}\{\Phi(\hat{\rho}(0))\} = \mathrm{Tr}\{\hat{\rho}(0)\} = 1$
2. Linearity: $\Phi(a\hat{\rho}_1 + b\hat{\rho}_2) = a\Phi(\hat{\rho}_1) + b\Phi(\hat{\rho}_2)$
3. Complete Positivity: $(\Phi \otimes \mathcal{I}_R)(\hat{A}) > 0 \quad \forall \dim R$

III. Quantum process in symplectic tomography



Evolution of S+B: $\hat{U}(t) = \int e^{-is\hat{H}} ds \quad \hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$

The symbol of evolution operator: $f_{\hat{U}}(\vec{x},\vec{y}) = \mathrm{Tr}\{\hat{U}\hat{\mathcal{U}}(\vec{x},\vec{y})\} \qquad \hat{\mathcal{U}}(\vec{x},\vec{y}) = \hat{\mathcal{U}}(\vec{x})\hat{\mathcal{U}}^B(\vec{y})$

The Kraus operator:

$$\hat{K}_{mn} = \sqrt{p_n}\langle m|\hat{U}(t)|n\rangle = \sqrt{p_n}\langle m|\left[\int f_{\hat{U}}(\vec{x},\vec{y})\hat{\mathcal{D}}(\vec{x})\hat{\mathcal{D}}^B(\vec{y})d\vec{x}d\vec{y}\right]|n\rangle =$$

$$= \sqrt{p_n}\left[\int f_{\hat{U}}(\vec{x},\vec{y})\hat{\mathcal{D}}(\vec{x})\langle m|\hat{\mathcal{D}}^B(\vec{y})|n\rangle d\vec{x}d\vec{y}\right] = \int\left[\sqrt{p_n}\int f_{\hat{U}}(\vec{x},\vec{y})\mathcal{D}^B_{mn}(\vec{y})d\vec{y}\right]\hat{\mathcal{D}}(\vec{x})d\vec{x}$$

The symbol of the Kraus operator: $f_{\hat{K}_{mn}} = \sqrt{p_n}\int f_{\hat{U}}(\vec{x},\vec{y})\mathcal{D}^B_{mn}(\vec{y})d\vec{y} \qquad f_{\hat{K}} = \sqrt{p(q)}\int f_{\hat{U}}(\vec{x},\vec{y})\mathcal{D}^B(Q,q,\vec{y})d\vec{y}$

$$\mathcal{D}^B_{mn}(\vec{x}) = \langle m|\hat{\mathcal{D}}^B(\vec{x})|n\rangle = \frac{1}{2\pi}e^{iX}e^{i\mu\nu/2}\int \psi^*_m(q)\psi_n(q-\nu)e^{-i\mu q}dq$$

$$\mathcal{D}^B(Q,q,\vec{x}) = \langle Q|\hat{\mathcal{D}}^B(\vec{x})|q\rangle = \frac{1}{2\pi}e^{iX}e^{-i\mu(Q+q)/2}\delta(Q-q-\nu)$$

III. Quantum process in symplectic tomography

Transformation of a state

$$\mathcal{T}'(\vec{x}) = \sum_m \mathrm{Tr}\{\hat{A}_m \hat{\rho} \hat{A}_m^\dagger \hat{\mathcal{U}}(\vec{x})\} = \sum_m f_{\hat{A}_m} \star \mathcal{T} \star f^*_{\hat{A}_m} = \sum_m \int \mathcal{T}(\vec{\bar{x}}) K_m(\vec{\bar{x}}, \vec{x}) d\vec{\bar{x}}$$

The probability of outcome *m*: $p_m = \int \mathcal{T}'_m(\vec{\bar{x}}) dX = \int \mathcal{T}(\vec{\bar{x}}) \left(\int K_m(\vec{\bar{x}}, \vec{x}) dX \right) d\vec{\bar{x}}$

The kernel of a partial quantum operation

$$K_m(\vec{\bar{x}}, \vec{x}) = \int f_{\hat{A}_m}(\vec{x}_1) f^*_{\hat{A}_m}(\vec{x}_2)\, \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} d\vec{x}_1 d\vec{x}_2$$

$$\mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} = \frac{1}{(2\pi)^3} \exp\left\{ i \left(\bar{X} + X_1 + X_2 - X \frac{\bar{\nu} + \nu_1 + \nu_2}{\nu} \right) \right\} \cdot$$

$$\cdot \exp\left\{ \frac{i}{2} \left(\bar{\mu}(\nu_1 - \nu_2) - (\mu_1 - \mu_2)\bar{\nu} - \mu_1\nu_2 + \mu_2\nu_1 \right) \right\} \delta((\bar{\mu} + \mu_1 + \mu_2)\nu - (\bar{\nu} + \nu_1 + \nu_2)\mu)$$

III. Quantum process in symplectic tomography

Representing a quantum process through the scalar product

$$\mathcal{T}'(\vec{x}) = \sum_m \mathrm{Tr}\{\hat{A}_m \hat{\rho} \hat{A}_m^\dagger \hat{\mathcal{U}}(\vec{x})\} = \sum_m \mathrm{Tr}\{\hat{\rho} \underbrace{\hat{A}_m^\dagger \hat{\mathcal{U}}(\vec{x}) \hat{A}_m}_{\hat{\mathcal{U}}_m}\} = \sum_m (\mathcal{T}(\vec{\bar{x}}), f_{\hat{\mathcal{U}}_m}(\vec{\bar{x}}, \vec{x}))$$

where $f_{\hat{\mathcal{U}}_m} = f^*_{\hat{A}_m} \star f_{\hat{\mathcal{U}}} \star f_{\hat{A}_m}$,

$$f_{\hat{\mathcal{U}}}(\vec{\bar{x}}, \vec{x}) = \mathrm{Tr}\{\hat{\mathcal{U}}(\vec{\bar{x}}) \hat{\mathcal{U}}(\vec{x})\} = \frac{1}{2\pi} \int e^{ik\bar{X} + isX} \delta(k\bar{\mu} + s\mu) \delta(k\bar{\nu} + s\nu) dk ds$$

$$= \frac{1}{2\pi} \delta\left(\bar{\mu}\nu - \mu\bar{\nu}\right) \int d\bar{k} \frac{1}{|k|} e^{i\left(X - \frac{\mu}{\bar{\mu}}\bar{X}\right)k}$$

$$\left(\int f_{\hat{\mathcal{U}}}(\vec{\bar{x}}, \vec{x}) e^{iX} dX = \delta\left(\bar{\mu}\nu - \mu\bar{\nu}\right) e^{i\bar{X}\mu/\bar{\mu}} \right)$$

III. Quantum process in symplectic tomography

Completeness condition for symbol of operators

Transformation of a state

$$\mathcal{T}'(\vec{x}) = \sum_m \mathrm{Tr}\{\hat{A}_m \hat{\rho} \hat{A}_m^\dagger \hat{\mathcal{U}}(\vec{x})\} = \sum_m f_{\hat{A}_m} \star \mathcal{T} \star f^*_{\hat{A}_m} = \sum_m \int \mathcal{T}(\vec{\bar{x}}) K_m(\vec{\bar{x}}, \vec{x}) d\vec{\bar{x}}$$

From the normalization condition $\int \mathcal{T}'(\vec{x}) dX = 1$ it follows that

$$1 = \int \mathcal{T}(\vec{\bar{x}}) \left[\int \sum_m K_m(\vec{\bar{x}}, \vec{x}) dX \right] d\vec{\bar{x}} \Longrightarrow \left[\int \sum_m K_m(\vec{\bar{x}}, \vec{x}) dX \right] = e^{i\bar{X}} \delta(\bar{\mu}) \delta(\bar{\nu})$$

$$\frac{1}{(2\pi)^2} \sum_m \int f_{\hat{A}_m}(X_1, \mu_1, \nu_1) f^*_{\hat{A}_m}(X_2, \mu_1 + \bar{\mu}, \nu_1 + \bar{\nu}) e^{i(X_1 - X_2)} e^{\frac{i}{2}(\bar{\mu}\nu_1 - \mu_1 \bar{\nu})} dX_1 dX_2 d\mu_1 d\nu_1 = \\ = \delta(\bar{\mu}) \delta(\bar{\nu})$$

III. Quantum process in symplectic tomography

Example: qubit phase flipping

Operators: $A_{1,nm} = \sqrt{p}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $A_{2,nm} = \sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

The operation kernels:

$$K_1(\vec{\bar{x}}, \vec{x}) = p\delta(\vec{\bar{x}} - \vec{x})$$

$$K_2(\vec{\bar{x}}, \vec{x}) = (1-p)\bigg(\frac{1}{\pi}\mathcal{T}_{00}(\vec{x})\int \mathcal{T}_{00}(\bar{X}+X', \bar{\mu}, \bar{\nu})e^{-iX'}dX' + \\ + \frac{1}{\pi}\mathcal{T}_{11}(\vec{x})\int \mathcal{T}_{11}(\bar{X}+X', \bar{\mu}, \bar{\nu})e^{-iX'}dX' - \delta(\vec{\bar{x}} - \vec{x})\bigg)$$

III. Example: the von Neumann quantum measurement model

The initial state of the system $$|\psi_0\rangle = \sum_{i=1}^{N} c_i |a_i\rangle$$

The initial state of the pointer $$|\phi_0\rangle = \left(\frac{\kappa}{2\pi}\right)^{\frac{1}{4}} \int e^{-\frac{\kappa}{4}q^2} |q\rangle dq$$

System-pointer interaction $$\hat{H}_{int}(t) = g\delta(t - t_0)\hat{A} \otimes \hat{p}$$

The evolution operator $$\hat{U} = e^{-ig\hat{A}\otimes\hat{p}}$$

The Kraus operators

$$\hat{M}_Q \equiv \langle Q|\hat{U}|\phi_0\rangle = \left(\frac{\kappa}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{\kappa}{4}(Q-g\hat{A})^2}$$

$$\hat{L}_j \equiv \langle a_j|\hat{U}|\psi_0\rangle = c_j e^{-iga_j\hat{p}}$$

System:

$$\mathrm{K}_Q^{(S)}(\vec{\bar{x}}, \vec{x}) = \sqrt{\tfrac{\kappa}{2\pi}} \textstyle\sum_{i,j} e^{-\frac{\kappa}{4}(Q-ga_i)^2 - \frac{\kappa}{4}(Q-ga_j)^2} \cdot$$
$$\cdot \int \mathcal{T}_i(\vec{x}_1)\mathcal{T}_j^*(\vec{x}_2)\,\mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} d\vec{x}_1 d\vec{x}_2$$

Pointer:

$$\mathrm{K}_j^{(P)}(\vec{\bar{x}}, \vec{x}) = \tfrac{|c_j|^2}{2\pi} \exp\left[i\left(\bar{X} - X\tfrac{\bar{\nu}}{\nu} + \bar{\mu}ga_j\right)\right] \cdot$$
$$\cdot \delta(\bar{\mu}\nu - \bar{\nu}\mu)$$

IV. Markovian dynamics

IV. Markovian dynamics

Transformation of the state in tomographic representation

$$\mathcal{T}_{\Delta t}(\vec{x}) = \sum_{\alpha} \int \mathcal{T}_0(\vec{\bar{x}}) K_\alpha(\vec{\bar{x}}, \vec{x}) d\vec{\bar{x}}$$

The partial quantum operation

$$K_\alpha(\vec{\bar{x}}, \vec{x}) = \int f_\alpha(\vec{x}_1) f^*_\alpha(\vec{x}_2) \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} d\vec{x}_1 d\vec{x}_2$$

$$\mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} = \frac{1}{(2\pi)^3} \exp\left[i\left(\bar{X} + X_1 + X_2 - X\frac{\bar{\nu} + \nu_1 + \nu_2}{\nu}\right)\right]$$
$$\exp\left[\frac{i}{2}\left(\bar{\mu}(\nu_1 - \nu_2) - (\mu_1 - \mu_2)\bar{\nu} - \mu_1\nu_2 + \mu_2\nu_1\right)\right] \delta((\bar{\mu} + \mu_1 + \mu_2)\nu - (\bar{\nu} + \nu_1 + \nu_2)\mu)$$

The completeness condition follows from $\int \mathcal{T}(\vec{x}) dX = 1$

$$\frac{1}{(2\pi)^2} \sum_{\alpha} \int f_\alpha(X_1, \mu_1, \nu_1) f^*_\alpha(X_2, \mu_1 + \bar{\mu}, \nu_1 + \bar{\nu}) e^{i(X_1 - X_2)} e^{\frac{i}{2}(\bar{\mu}\nu_1 - \mu_1\bar{\nu})} dX_1 dX_2 d\mu_1 d\nu_1 =$$
$$= \delta(\bar{\mu})\delta(\bar{\nu})$$

IV. Markovian dynamics

Let the symbol of the Kraus operators be

$$f_0(\vec{x}, \Delta t) = 1(\vec{x}) + [a(\vec{x}) - ih(\vec{x})]\Delta t \qquad\qquad a(\vec{x}),\ h(\vec{x}) \in \mathbb{R}$$

$$f_\alpha(\vec{x}, \Delta t) = \sqrt{\gamma_\alpha \Delta t} g_\alpha(\vec{x})$$

The corresponding partial operations are

$$K(\vec{\bar{x}}, \vec{x}, \Delta t) = \delta(\vec{\bar{x}} - \vec{x}) + \Delta t \int \left[1(\vec{x}_1)a(\vec{x}_2) + a(\vec{x}_1)1(\vec{x}_2)\right] \mathcal{K}(\vec{x}_1, \vec{\bar{x}}, \vec{x}_2, \vec{x}) d\vec{x}_1 d\vec{x}_2 + \\ + i\Delta t \int \left[1(\vec{x}_1)h(\vec{x}_2) - h(\vec{x}_1)1(\vec{x}_2)\right] \mathcal{K}(\vec{x}_1, \vec{\bar{x}}, \vec{x}_2, \vec{x}) d\vec{x}_1 d\vec{x}_2 + o(\Delta t^2)$$

$$K_\alpha(\vec{\bar{x}}, \vec{x}, \Delta t) = \gamma_\alpha \Delta t \int g_\alpha(\vec{x}_1) g_\alpha^*(\vec{x}_2) \mathcal{K}(\vec{x}_1, \vec{\bar{x}}, \vec{x}_2, \vec{x}) d\vec{x}_1 d\vec{x}_2$$

$$\mathcal{K}(\vec{x}_1, \vec{\bar{x}}, \vec{x}_2, \vec{x}) = \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\}$$

IV. Markovian dynamics

Using completeness equation, one gets $a(\vec{x}) = -\frac{1}{2}\sum_{\alpha}\gamma_\alpha \int g^*_\alpha(\vec{x}_1) g_\alpha(\vec{x}_2)\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})d\vec{x}_1 d\vec{x}_2$

so $\frac{\mathcal{T}_{\Delta t}(\vec{x}) - \mathcal{T}_0(\vec{x})}{\Delta t} \equiv \frac{\partial \mathcal{T}(\vec{x})}{\partial t}\Big|_{t=0} = \int \mathcal{T}_0(\vec{\bar{x}})\mathcal{L}(\vec{\bar{x}}, \vec{x})d\vec{\bar{x}}$

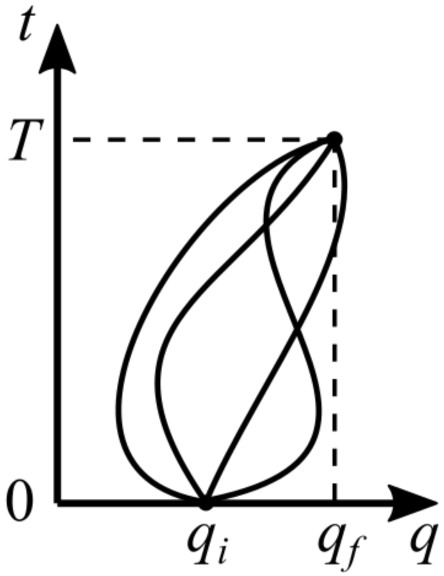
$$\mathcal{L}(\vec{\bar{x}}, \vec{x}) = \int \Big[- ih(\vec{x}')1(\vec{x}'')\Big(\mathcal{K}(\vec{x}', \vec{\bar{x}}, \vec{x}'', \vec{x}) - \mathcal{K}(\vec{x}'', \vec{\bar{x}}, \vec{x}', \vec{x})\Big) + \sum \gamma_\alpha g_\alpha(\vec{x}')g^*_\alpha(\vec{x}'')\left(\mathcal{K}(\vec{x}', \vec{\bar{x}}, \vec{x}'', \vec{x}) - \frac{1}{2}\mathcal{K}(\vec{\bar{x}}, \vec{x}'', \vec{x}', \vec{x}) - \frac{1}{2}\mathcal{K}(\vec{x}'', \vec{x}', \vec{\bar{x}}, \vec{x})\right)\Big] d\vec{x}' d\vec{x}''$$

The Markovity assumption: $\frac{\partial \mathcal{T}(\vec{x})}{\partial t}\Big|_{t=0} \longrightarrow \frac{\partial \mathcal{T}(\vec{x})}{\partial t}\ \forall t$

$$\mathcal{K}(\vec{x}_1, \vec{\bar{x}}, \vec{x}_2, \vec{x}) = \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{\bar{x}})\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\}$$

V. Non-Markovian dynamics

V. Transition amplitude based on the path integral



Configuration space

Action $$S(q) = \int_0^T L(q, \dot{q}, t) dt$$

Paths in the configuration space $$[q] = \{q(t) : 0 \leq t \leq t, q(0) = q_i, q(T) = q_f\}$$

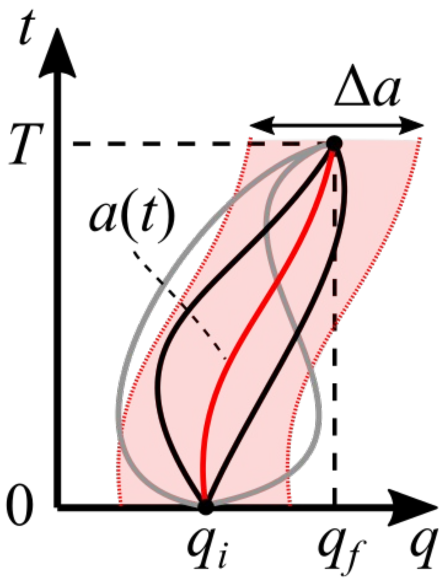
Amplitude of going the particular path $q(t)$ $$e^{iS[q]}$$

Amplitude of the transition $q_i \to q_f$ $$U_T(q_f, q_i) \equiv \langle q_f | \hat{U}_T | q_i \rangle = \int_{q_i,0}^{q_f,T} d[q] e^{iS[q]}$$

Transformation of the state $$\hat{\rho}(T) = \hat{U}_T \hat{\rho}(0) \hat{U}_T^\dagger$$

V. Restricted path integral

Based on the information obtained during the measurement, a weight functional is introduced for the paths, $w_a,\ 0 \le w_a[q] \le 1$, which characterizes how far the path $q(t)$ is from the result $a(t)$



Configuration space

Amplitude of going the particular path $q(t)$ $\quad w_a[q]e^{iS[q]}$

Amplitude of the transition $q_i \to q_f$ $\quad U_{T,a} \equiv \langle q_f|\hat{U}_T|q_i\rangle = \int_{q_i,0}^{q_f,T} d[q]w_a[q]e^{iS[q]}$

Transformation of the state $\quad \hat{\rho}_a(T) = \hat{U}_{T,a}\hat{\rho}(0)\hat{U}^\dagger_{T,a}$

The examples of weight functional

$$w_a[q] = \begin{cases} 1 & |q(t) - a(t)| \le \Delta a, \\ 0 & |q(t) - a(t)| > \Delta a, \end{cases}$$

$$w_a[q] = \exp\left\{-\sum_{n=1}^{\infty}\frac{(q_n - a_n)^2}{\Delta a_n^2}\right\} \quad w_a[q] = \exp\left\{-\frac{2}{T\Delta a^2}\int_0^T (q(t) - a(t))^2 dt\right\}$$

Mensky, M. B. (1979). Quantum restrictions for continuous observation of an oscillator. Physical Review D, 20(2), 384

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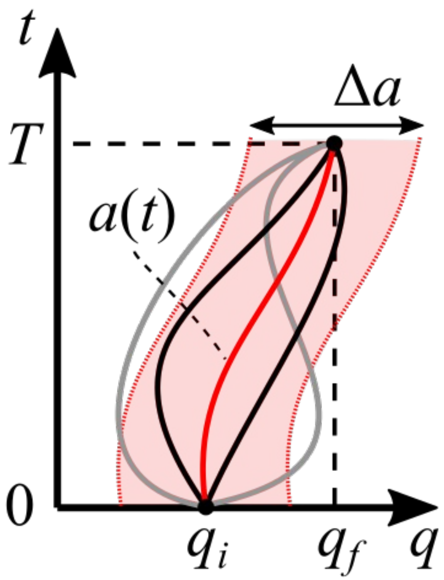

Configuration space

Amplitude of going the particular path $q(t)$	$w_a[q]e^{iS[q]}$
Amplitude of the transition $q_i \to q_f$	$U_{a,T} \equiv \langle q_f \lvert \hat{U}_T \rvert q_i \rangle = \int_{q_i,0}^{q_f,T} d[q] w_a[q] e^{iS[q]}$
Transformation of the state	$\hat{\rho}_a(T) = \hat{U}_{a,T}\hat{\rho}(0)\hat{U}^\dagger_{a,T}$
Propagator property	$\hat{U}^\dagger_{a,T}\hat{U}_{a,T} < 1 \quad \Longleftrightarrow \quad \mathrm{Tr}\{\hat{\rho}_a\} < 1$
The probability of the outcome $a(t)$	$P(a, da) = \mathrm{Tr}\{\hat{\rho}_a(T)\}da$
The state for unknown result	$\hat{\tilde{\rho}}(T) = \int \hat{U}_{a,T}\hat{\rho}(0)\hat{U}^\dagger_{a,T} da$
Normalization	$\mathrm{Tr}\{\hat{\tilde{\rho}}(T)\} = 1 \Leftrightarrow \int \hat{U}^\dagger_{a,T}\hat{U}_{a,T} da = 1$

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V. The tomograpgic propagator based on the rectricted path integral

Selective measurement

The partial tomographic propagator (density)

$$\Pi_{t,a}(\vec{x}', \vec{x}) = \frac{1}{4\pi^2} \int dk dq_f dq_i \cdot e^{ik(X'+X)} \int\limits_{q_i,0}^{q_f+k\nu,T} d[q_1] \int\limits_{q_i+k\nu',0}^{q_f,T} d[q_2]$$

$$\cdot \exp\left\{ i \left[-S[q_1] - i \ln w_a[q_1] - k\frac{\mu' q_i + \mu(q_f + k\nu)}{2} + S[q_2] - i \ln w_a[q_2] - k\frac{\mu'(q_i + k\nu') + \mu q_f}{2} \right] \right\}$$

The tomogram density $\mathcal{T}_a(\vec{x}, T) = \int \mathcal{T}(\vec{\bar{x}}, 0)\Pi_{a,T}(\vec{\bar{x}}, \vec{x})d\vec{\bar{x}}$

Non-selective measurement

The tomogram of the system for the non-selective measurement: $\mathcal{T}(\vec{x}, T) = \int \mathcal{T}_a(\vec{x}, T)da = \int \mathcal{T}(\vec{\bar{x}}, 0)\tilde{\Pi}_T(\vec{\bar{x}}, \vec{x})d\vec{\bar{x}}$

The tomographic propagator for the non-selective measurement: $\tilde{\Pi}_T(\vec{\bar{x}}, \vec{x}) = \int \Pi_{a,T}(\vec{\bar{x}}, \vec{x})da$

V. Example: particle scattering

Continuous measuring of particle position using the restricted path integral

The Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m}$

Gaussian-type weight functional $w_a[q] = \exp\left\{ -\frac{2}{T\Delta a^2} \int\limits_0^T (q(t) - a(t))^2 dt \right\}$

Effective Lagrangian $L_a(q, \dot{q}, t) = \frac{m\dot{q}^2}{2} + i\frac{2}{T\Delta a^2} \int\limits_0^T (q(t) - a(t))^2 dt$

The partial propagator

$$U(q_{f,1}q_{i,1}|q_{f,2}q_{i,2}) = N \exp\left\{ \frac{im}{2T}[(q_{f,1} - q_{i,1})^2 - (q_{f,2} - q_{i,2})^2] \right\}$$
$$\cdot \exp\left\{ -\frac{1}{3\Delta a^2}[(q_{i,1} - q_{i,2})^2 + (q_{f,1} - q_{f,2})^2 + (q_{i,1} - q_{i,2})(q_{f,1} - q_{f,2})] \right\}$$

V. Example: particle scattering

Continuous measuring of particle position using the restricted path integral

Without measurement

$$\Pi_T(\vec{x}', \vec{x}) = \delta\left(X' - X\right)\delta\left(\mu' - \mu\right)\delta\left(\nu' - \nu - \mu\frac{T}{m}\right)$$

With measurement

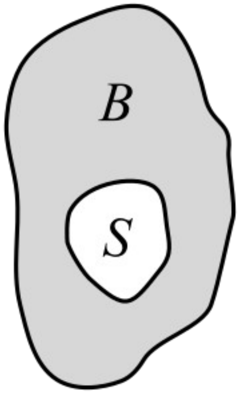
$$\tilde{\Pi}_T(\vec{x}', \vec{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{(X'-X-\bar{X})^2}{2\sigma^2}}\delta\left(\mu' - \mu\right)\delta\left(\nu' - \nu - \mu\frac{T}{m}\right)$$

$$\sigma^2 = \frac{2}{3\Delta a^2}\left(3\nu^2 + 3\nu\mu\frac{T}{m} + \mu^2\frac{T^2}{m^2}\right)$$

The partial propagator indeed satisfies the master equation

$$\frac{\partial\tilde{\Pi}_t}{\partial t} - \frac{\mu}{m}\frac{\partial\tilde{\Pi}_t}{\partial\nu} - k\nu^2\frac{\partial^2\tilde{\Pi}_t}{\partial X^2} = 0$$

V. Nakajima-Zwanzig formalism



The starting point:

Total Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{int}$

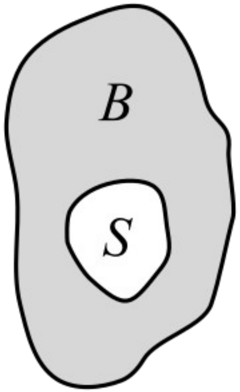
Total (unitary) state evolution $\partial_t \hat{\rho} = -i\alpha[\hat{H}(t), \hat{\rho}(t)] \equiv \alpha\mathcal{L}\hat{\rho}(t)$

Projection operators $\mathcal{P}\hat{\rho} = \mathrm{Tr}_B\{\hat{\rho}\} \otimes \hat{\rho}_B$

$$\mathcal{Q} = 1 - \mathcal{P}$$

$$\mathcal{P}^2 = \mathcal{P}, \quad \mathcal{Q}^2 = \mathcal{Q}, \quad \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$$

V. Nakajima-Zwanzig formalism in symplectic tomography



Total Hamiltonian $$h(\vec{x}) = h_S(\vec{x}) + h_B(\vec{x}) + h_{int}(\vec{x})$$

$$h(\vec{x}) \in \mathbb{R} \quad (f_{\hat{H}^\dagger}(\vec{x}) = f^*_{\hat{H}}(\vec{x}))$$

Total tomogram evolution

$$\partial_t \mathcal{T} = -i\alpha \int h(\vec{x}_1)\mathcal{T}(\vec{x}_2) C_{\vec{x}_1 \vec{x}_2}(\vec{x}) d\vec{x}_1 d\vec{x}_2 \equiv \alpha W \mathcal{T}$$

The kernel of evolution equation:

$$C_{\vec{x}_1 \vec{x}_2}(\vec{x}) = \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} - \mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{U}}(\vec{x})\}$$

$$\mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x}_1)\hat{\mathcal{D}}(\vec{x}_2)\hat{\mathcal{U}}(\vec{x})\} = (2\pi)^{-2}\delta(\mu(\nu_1+\nu_2) - \nu(\mu_1+\mu_2)) e^{iX_1+iX_2} e^{-i\frac{(\nu_1+\nu_2)}{\nu}X} e^{\frac{i}{2}(\nu_1\mu_2 - \nu_2\mu_1)}$$

V. Nakajima-Zwanzig formalism in symplectic tomography

The density matrix $$\hat{\rho} = \int \mathcal{T}(\vec{x}, \vec{x}')\hat{\mathcal{D}}_S(\vec{x})\hat{\mathcal{D}}_B(\vec{x}')d\vec{x}d\vec{x}'$$

The tomograms of the system $$\mathcal{T}_S(\vec{x}) = \int \mathcal{T}(\vec{x}, \vec{x}')\,\mathrm{Tr}_B\{\hat{\mathcal{D}}_B(\vec{x}')\}d\vec{x}'$$

Projection operators for tomograms

$$\mathcal{P}[\mathcal{T}](\vec{x}, \vec{x}') = \mathcal{T}_B(\vec{x}') \int \mathcal{T}(\vec{x}, \vec{x}'')\,\mathrm{Tr}_B\{\hat{\mathcal{D}}_B(\vec{x}'')\}d\vec{x}''$$

$$\mathcal{Q}[\mathcal{T}](\vec{x}, \vec{x}') = \mathcal{T}_B(\vec{x}') \int \mathcal{T}(\vec{x}, \vec{x}'')\Big[\delta(\vec{x}'' - \vec{x}') - \mathrm{Tr}_B\{\hat{\mathcal{D}}_B(\vec{x}'')\}\Big]d\vec{x}''$$

$$\mathcal{P}^2 = \mathcal{P}, \quad \mathcal{Q}^2 = \mathcal{Q}, \quad \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$$

$$\mathrm{Tr}\{\hat{\mathcal{D}}(\vec{x})\} = e^{iX}\delta(\mu)\delta(\nu)$$

V. Nakajima-Zwanzig formalism in symplectic tomography

$$\partial_t \mathcal{P}\mathcal{T} = \alpha \mathcal{P} W \mathcal{P}\mathcal{T} + \alpha \mathcal{P} W \mathcal{Q}\mathcal{T}$$
$$\partial_t \mathcal{Q}\mathcal{T} = \alpha \mathcal{Q} W \mathcal{P}\mathcal{T} + \alpha \mathcal{Q} W \mathcal{Q}\mathcal{T}$$

⬇

$$\mathcal{Q}\mathcal{T}(t) = \mathcal{G}(t,t_0)\mathcal{Q}\mathcal{T}(t_0) + \alpha \int_{t_0}^{t} ds \mathcal{G}(t,s)\mathcal{Q}W(s)\mathcal{P}\mathcal{T}(s)$$

⬇

Integral form $$\mathcal{P}\mathcal{T}(t) = \alpha \mathcal{P}W(t)\mathcal{P}\mathcal{T}(t) + \alpha \int_{t_0}^{t} ds \mathcal{P}W(t)\mathcal{G}(t,s)\mathcal{Q}W(s)\mathcal{P}\mathcal{T}(s)$$

Time-convolutionless form $$\partial_t \mathcal{P}\mathcal{T}(t) = \alpha \mathcal{P}W(t)(1-\Sigma(t))^{-1}\mathcal{G}(t,t_0)\mathcal{Q}\mathcal{T}(t_0) + \alpha \mathcal{P}W(1-\Sigma(t))^{-1}\mathcal{P}\mathcal{T}(t)$$

$$\mathcal{G}(t,s) = \overleftarrow{T} \exp\left\{\alpha \int_{s}^{t} ds' \mathcal{Q}W(s')\right\}$$
$$\mathcal{R}(t,s) = \overrightarrow{T} \exp\left\{-\alpha \int_{s}^{t} ds' W(s')\right\}$$

$$\Sigma(t) = \alpha \int_{t_0}^{t} ds \mathcal{G}(t,s)\mathcal{Q}W(s)\mathcal{P}\mathcal{R}(t,s)$$

Thank you!