Morse matching method for conformal cohomologies

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General definition

- (Co-)chain complex $(C^\bullet, \Delta_\bullet)$ is a sequence

$$\ldots \longrightarrow C^{n-1} \xrightarrow{\Delta_{n-1}} C^n \xrightarrow{\Delta_n} C^{n+1} \longrightarrow \ldots$$

$$\Delta_n \Delta_{n-1} = 0$$

- (Co)cycles

$$Z^n := Ker \Delta_n$$

- (Co)boundaries

$$\beta^n := Im \Delta_{n-1}$$

- The n^{th} (co)homology group

$$H^n = Z^n / \beta^n$$

General definition

- Chain complex $(B_\bullet, d_\bullet)$ is a sequence

$$\ldots \longleftarrow B_{n-1} \xleftarrow{d_{n-1}} B_n \xleftarrow{d_n} B_{n+1} \longleftarrow \ldots$$

$$d_{n-1} d_n = 0$$

- We can get (co)chain from chain complex by replacing

$$Hom_k(B_n, \mathbb{k}) = C^n$$

$$d_n^* = \Delta_n$$

Hochschild cohomology

We assume that Λ — associative $\mathbb{k}$-algebra with unity and $\mathrm{char}(\mathbb{k}) = 0$. Let Λ has an *augmentation*, i.e., a $\mathbb{k}$-algebra map $\varepsilon : \Lambda \to \mathbb{k}$.

- Hochschild complex (bar resolution)

$$B_n = \Lambda \otimes (\Lambda/\mathbb{k})^{\otimes n} \otimes \Lambda$$

- Notation:

$$[a_1|a_2|\ldots|a_n] := a_1 \otimes a_2 \otimes \ldots \otimes a_n$$

Hochschild cohomology

- The Hochschild differential:

$$d_n[a_1|a_2|\dots|a_{n+1}] = a_1[a_2|a_3|\dots|a_{n+1}] + \sum_{i=1}^{n}(-1)^i[a_1|\dots|a_i a_{i+1}|\dots|a_{n+1}] + (-1)^{n+1}[a_1|a_2|\dots|a_n]a_{n+1}.$$

  Let M be a Λ-bimodule (= left $\Lambda \otimes \Lambda^{op}$-module).

- $Hom_{\Lambda\otimes\Lambda^{op}}(B_n, M) = C^n(\Lambda, M)$ then $H^n(\Lambda, M)$ is the Hochschild cohomology group.

Cartan-Eilenberg cohomology

We assume that $\mathfrak{g}$ — $\mathbb{k}$- Lie algebra. Let $U(\mathfrak{g})$ is the universal enveloping algebra.

- Chevalley–Eilenberg complex:

$$B_n = U(\mathfrak{g}) \otimes \Lambda^{\otimes n}\mathfrak{g}$$

- Chevalley–Eilenberg differential:

$$d_n(a_1, \dots, a_{n+1}) = \sum_{i=1}^{n} (-1)^i a_i(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) + \sum_{i<j} (-1)^{i+j}([a_i, a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{n+1}).$$

Let M be a $\mathfrak{g}$-module ($= U(\mathfrak{g})$-module).

- $Hom_{U(\mathfrak{g})}(B_n, M) = C^n(\mathfrak{g}, M)$ then $H^n(\mathfrak{g}, M)$ is the Cartan-Eilenberg cohomology group.

Why do we need cohomology?

- Derivations of algebras;
- Extensions of modules and algebras;
- Deformations of algebras;
- Combinatorial invariants (Poincaré series, Hilbert series).

Every conformal algebra C is a left module over the polynomial algebra $H = \Bbbk[\partial]$. The structure of a conformal algebra on an H-module C may be expressed by means of a single polynomial-valued map called λ-product:

- $(\cdot\,{}_{(\lambda)}\,\cdot) : C \otimes C \longrightarrow C[\lambda]$, $(a\,{}_{(\lambda)}\,b) = \sum_{n=0}^{N(a,b)-1} \frac{\lambda^n}{n!}(a\,{}_{(n)}\,b)$;
- 3/2-linearity: $(a\,{}_{(\lambda)}\,\partial b) = (\lambda + \partial)(a\,{}_{(\lambda)}\,b)$;
- $(\partial a\,{}_{(\lambda)}\,b) = -\lambda(a\,{}_{(\lambda)}\,b)$;
- (Associativity): $(a\,{}_{(\lambda)}\,b)\,{}_{(\mu)}\,c = a\,{}_{(\lambda)}\,(b\,{}_{(-\lambda+\mu)}\,c)$;
- (Anti-commutativity): $(a\,{}_{(\lambda)}\,b) = -(b\,{}_{(-\lambda-\partial)}\,a)$;
- (Jacobi): $a\,{}_{(\lambda)}\,(b\,{}_{(\mu)}\,c) - b\,{}_{(\mu)}\,(a\,{}_{(\lambda)}\,c) = (a\,{}_{(\lambda)}\,b)\,{}_{(\lambda+\mu)}\,c$

Examples of conformal algebras

(1) $C = \Bbbk[\partial]v$

$$(v \,{}_{(\lambda)}\, v) = (\partial + 2\lambda)v$$

— Virasoro (Lie) conformal algebra, Vir

(2) $C = \Bbbk[\partial, v]v$

$$(v^n \,{}_{(\lambda)}\, v^m) = v^n(v+\lambda)^m$$

— Weyl (associative) conformal algebra, $U(2)$

(3) An associative conformal algebra C turns into a Lie one, $C^{(-)}$:

$$[a \,{}_{(\lambda)}\, b] = (a \,{}_{(\lambda)}\, b) - (b \,{}_{(-\lambda-\partial)}\, a)$$

e.g., $\mathrm{Vir} \subset U(2)^{(-)}$

Problem

It is well-known that

$$\mathrm{H}^n(\mathfrak{g}, M) = \mathrm{H}^n(U(\mathfrak{g}), M)$$

Chevalley–Eilengerg = Hochschild of the univ. envelope

(?) What happens to conformal algebras?

Given a Lie conformal algebra L there exists a series $U(L; N)$ of universal enveloping associative conformal algebras [Roitman, Sel.Math., 2000] e.g.,

$$U(2) = U(\mathrm{Vir}; N = 2)$$

but

$$\mathrm{H}^2(\mathrm{Vir}, \Bbbk) \neq 0, \quad \mathrm{H}^2(U(2), \Bbbk) = 0$$

Problem

Consider

$$U(3) = U(\mathrm{Vir}; N = 3)$$

— associative conformal algebra generated by v relative to

$$\deg_\lambda(v \,_{(\lambda)}\, v) < N = 3,$$

$$(v \,_{(\lambda)}\, v) - (v \,_{(-\partial-\lambda)}\, v) = (\partial + 2\lambda)v$$

[Cheng, Kac, 1997] $\Rightarrow$

Every irreducible Vir-module is a conformal module over $U(3)$.

$$\mathrm{H}^n(U(3), \Bbbk) = (?)$$

[AlHussein, K., J.Math.Phys., 2021]: $\dim \mathrm{H}^2(U(3), \Bbbk) = 1$

Coefficient algebra

Let C is a conformal algebra. The coefficient algebra $A = Coeff(C)$ is linear space spanned over $\Bbbk$ by the set $\{a(n) | a \in C, n \geq 0\}$ subject to linear relations $(\alpha a)(n) = \alpha a(n), \alpha \in \Bbbk$, $(a+b)(n) = a(n) + b(n)$ and $(\partial a)(n) = na(n-1)$. The product in A is defined by

$$a(m)b(n) = \sum_{s \geq 0} \binom{m}{s} (a_{\,(s)}\, b)(m+n-s)$$

[Bakalov, Kac, Voronov, 1999]$\Rightarrow$

$$H^n(C, \Bbbk) = H^n(C(A_+, \Bbbk)/\partial C(A_+, \Bbbk))$$

Cohomology coefficient algebra

- $C^n(A_+, \mathbb{k}) = Hom_{\mathbb{k}}(B_n, \mathbb{k}); \quad B_n = A_+^{\otimes n}$
- $d_n : B_n \to B_{n-1}$
- $\partial_n : B_n \to B_n$
- $\partial_n[\lambda_1|\dots|\lambda_n] = \sum_{i=1}^n [\lambda_1|\dots|\partial\lambda_i|\dots|\lambda_n]$
- $d_{n+1}^* : C^n \to C^{n+1}$
- $\partial_n^* : C^n \to C^n$
- $C^n(A_+, \mathbb{k})/\partial C^n(A_+, \mathbb{k}) = (ker\partial_n)^*$

Morse matching method

[Forman, Adv.Math., 1998];
[Sköldberg, Trans.Amer.Math.Soc., 2006];
[Jöllenbeck, Welker, Mem.Amer.Math.Soc., 2009]

R is an augmented associative algebra;

Chain complex $(B_\bullet, \delta_\bullet)$ of free R-modules $\Rightarrow$ Oriented graph $\Gamma(B_\bullet)$
Vertices $V = R$-bases of all B_ns
Weighted edges $E = [u \stackrel{\lambda}{\to} v]$ for $\delta_n(u) = \dots + \lambda v + \dots$, $0 \neq \lambda \in R$

Morse matching method

A subset $M \subseteq E$ of the set of edges is called an acyclic matching, if it satisfies the following three conditions:

(1) Each vertex $v \in V$ lies in at most one edge $e \in M$;

(2) For all edges $[u \xrightarrow{\lambda} v]$ in M the weight λ lies in the center of R and is a unit in R;

(3) The graph $\Gamma^M(B_\bullet) = (V, E^M)$ has no directed cycles, where

$$E^M = (E \setminus M) \cup \{[v \xrightarrow{-\lambda^{-1}} u] \mid [u \xrightarrow{\lambda} v] \in M\}.$$

Critical cells $V^M =$ all vertices that do not belong to edges from M

Anick resolution via Morse mathching method

Λ augmented assoc. algebra, V is a Λ-bimodule
$R = \Lambda \otimes \Lambda^{op}$
$B_n(\Lambda, \Lambda) = \Lambda \otimes (\Lambda/\Bbbk)^{\otimes n} \otimes \Lambda$ bar-resolution
$(\mathrm{C}^n(\Lambda, V) = \mathrm{Hom}_R(B_n(\Lambda, \Lambda), V))$

Then there exists a matching M such that $B_\bullet^M = A_\bullet(\Lambda, \Lambda)$
is the Anick (bimodule) resolution [Anick, 1986]

A_n is generated by critical cells = Anick chains
[Jöllenbeck, Welker, 2009]

Hochschild cohomology via Morse matching

Given an associative conformal algebra C,
how to find $\mathrm{H}^n(C,\Bbbk)$:

1. Find the Gröbner–Shirshov basis of the annihilation algebra $\mathcal{A}_+(C)$ [Kac, 1998]
   — this is a derivation algebra spanned by $a(n)$, $a \in C$, $n \geq 0$, such that $(\partial a)(n) = -na(n-1)$;
2. Construct the Anick resolution $(A_\bullet(\Lambda,\Lambda),\tilde{\delta}_\bullet)$ for $\Lambda = \mathcal{A}_+(C) \oplus \Bbbk 1$;
3. Derive $A_\bullet = A_\bullet(\Lambda,\Lambda) \otimes_R \Bbbk$, $R = \Lambda \otimes \Lambda^{op}$;
4. Find the homotopy image $\tilde{\partial}_n : A_n \to A_n$ of $\partial_n : B_n \to B_n$
   — this is a morphism of complexes;
5. Evaluate the cohomologies of the sub-complex $\ker \tilde{\partial}_\bullet$ in $(A_\bullet,\tilde{\delta}_\bullet)$

[Bakalov, Kac, Voronov, 1999] $\Rightarrow$

$$\mathrm{H}^n(C,\Bbbk) = \left(\ker \tilde{\delta}_n|_{\ker \tilde{\partial}_n} / \mathrm{Im}\, \tilde{\delta}_{n+1}|_{\ker \tilde{\partial}_{n+1}} \right)^*$$

Application (ArXiv 2204.10837)

$C = U(3)$
The annihilation algebra $\mathcal{A}_+(U(3))$ is generated by $v(n)$, $n \geq 0$,
GSB:

$$v(1)v(0) = v(0)v(1) + v(0),$$
$$v(n)v(m) = \frac{nm}{n+m-1}v(1)v(n+m-1) - \frac{(n-1)(m-1)}{n+m-1}v(0)v(n+m) + \frac{n(n-1)}{n+m-1}v(n+m-1), \quad n \geq 2.$$

The Anick chains are of the following form:

$$[v(m_1)|v(m_2)|\dots|v(m_{n-1})|v(m_n)],$$

where $m_1, \dots, m_{n-2} \geq 2$ and either $m_{n-1} \geq 2$ or $(m_{n-1}, m_n) = (1, 0)$
— this is a basis of A_n

Application

To find the differentials in $A_\bullet$, apply the Morse matching method: e.g.,

$$
\begin{tikzcd}[column sep=small]
 & & {[v(i_1)|\cdots|v(i_{n-1})|v(1)|v(0)]} \arrow[dll] \arrow[d] \arrow[drr, "(-1)^n"] \arrow[ddl, "(-1)^j"', "j=1,\dots,n-2"] \arrow[ddrr, "(-1)^{n-1}"] & & \\
\cdots & & \cdots & & {[v(i_1)|\cdots|v(i_{n-1})|v(1)]} \\
 & {[v(i_1)|\cdots|v(i_j)v(i_{j+1})|\cdots|v(i_{n-1})|v(1)|v(0)]} \arrow[dl, "\frac{i_j i_{j+1}}{i_j+i_{j+1}-1}"'] \arrow[drr, "\frac{i_j(i_j-1)}{i_j+i_{j+1}-1}"] \arrow[ddr, "-\frac{(i_j-1)(i_{j+1}-1)}{i_j+i_{j+1}-1}"] & & & {[v(i_1)|\cdots|v(i_{n-1})v(1)|v(0)]} \\
{[v(i_1)|\cdots|v(1)v(i_j+i_{j+1}-1)|\cdots|v(i_{n-1})|v(0)]} & & & {[v(i_1)|\cdots|v(i_j+i_{j+1}-1)|\cdots|v(i_{n-1})|v(1)|v(0)]} & \\
 & & {[v(i_1)|\cdots|v(0)v(i_j+i_{j+1})|\cdots|v(i_{n-1})|v(1)|v(0)]} & &
\end{tikzcd}
$$

Application

$$
\begin{tikzcd}[column sep=small]
 & {[v(i_1)|\cdots|v(i_{t-1})|v(0)v(p+1)|\cdots|v(i_{n+1})]} \arrow[d, shift right=2ex, "(-1)^{t+1}"'] & \\
\begin{array}{c}\text{no}\\ \text{Anick}\\ \text{chains}\end{array} & {[v(i_1)|\cdots|v(i_{t-1})|v(0)|v(p+1)|\cdots|v(i_{n+1})]} \arrow[u, dashed, shift right=2ex, "(-1)^{t}"'] \arrow[l, shift right=1ex] \arrow[l, shift left=1ex, "\vdots"'] \arrow[d, "(-1)^{t-1}"'] & \\
 & {[v(i_1)|\cdots|v(i_{t-1})v(0)|v(p+1)|\cdots|v(i_{n+1})]} \arrow[dl, "1"'] \arrow[dr, "i_{t-1}"] & \\
{[v(i_1)|\cdots|v(0)v(i_{t-1})|v(p+1)|\cdots|v(i_{n+1})]} & & {[v(i_1)|\cdots|v(i_{t-1}-1)|v(p+1)|\cdots|v(i_{n+1})]}
\end{tikzcd}
$$

Application

$$
\begin{tikzcd}
 & & \color{blue}{[v(i_1)|\cdots|v(i_{t-1})|v(1)v(p)|\cdots|v(i_{n+1})]} \arrow[d, shift right, "(-1)^{t+1}"'] & \\
\text{no Anick chains} & & {[v(i_1)|\cdots|v(i_{t-1})|v(1)|v(p)|\cdots|v(i_{n+1})} \arrow[u, dashed, shift right, "(-1)^{t}"'] \arrow[ll, shift right, "\vdots"'] \arrow[ll, shift left] \arrow[d, "(-1)^{t-1}"'] & \\
 & & {[v(i_1)|\cdots|v(i_{t-1})v(1)|v(p)|\cdots|v(i_{n+1})]} \arrow[dl, "1"'] \arrow[dr, "(i_{t-1}-1)"] & \\
 & \color{blue}{[v(i_1)|\cdots|v(1)v(i_{t-1})|v(p)|\cdots|v(i_{n+1})]} & & {[v(i_1)|\cdots|v(i_{t-1})|v(p)|\cdots|v(i_{n+1})]}
\end{tikzcd}
$$

Application

The action of $\tilde{\partial}_n$:

$$\partial(v(n)) = nv(n-1)$$

expand to Anick chains by the Leibiz rule

$$[v(m_1)|\dots|v(m_n)] \mapsto \sum_{s=1}^{n} m_s[v(m_1)|\dots|v(m_s-1)|\dots|v(m_n)]$$

and remove all non-Anick chains,
e.g.,

$$\tilde{\partial}_3[v(2)|v(2)|v(1)] = [v(2)|v(2)|v(0)]$$

$$\tilde{\partial}_3[v(3)|v(2)|v(0)] = 3[v(2)|v(2)|v(0)] + 2[v(3)|v(1)|v(0)]$$

etc.

Application

Theorem

For the associative conformal algebra $C = U(3) = U(\mathrm{Vir}; N = 3)$ we have

$$\dim \mathrm{H}^n(C, \Bbbk) = 1, \quad n = 0, 2, 3,$$

and $\mathrm{H}^n(C, \Bbbk) = 0$ otherwise.

Application

Theorem

For the associative conformal algebra $C = U(3) = U(\mathrm{Vir}; N = 3)$ we have

$$\dim \mathrm{H}^n(C, \Bbbk) = 1, \quad n = 0, 2, 3,$$

and $\mathrm{H}^n(C, \Bbbk) = 0$ otherwise.

We can see $\mathrm{H}^n(U(3), \Bbbk) = \mathrm{H}^n(\mathrm{Vir}, \Bbbk)$, as for ordinary algebras.

Remark

This is no longer the case for non-scalar modules:
there exists an irreducible Vir-module M such that
$\mathrm{H}^1(\mathrm{Vir}, M) \neq \mathrm{H}^1(U(3), M)$.

Thank you!