Noncompact foliations of mechanical systems in pseudo-Euclidean space

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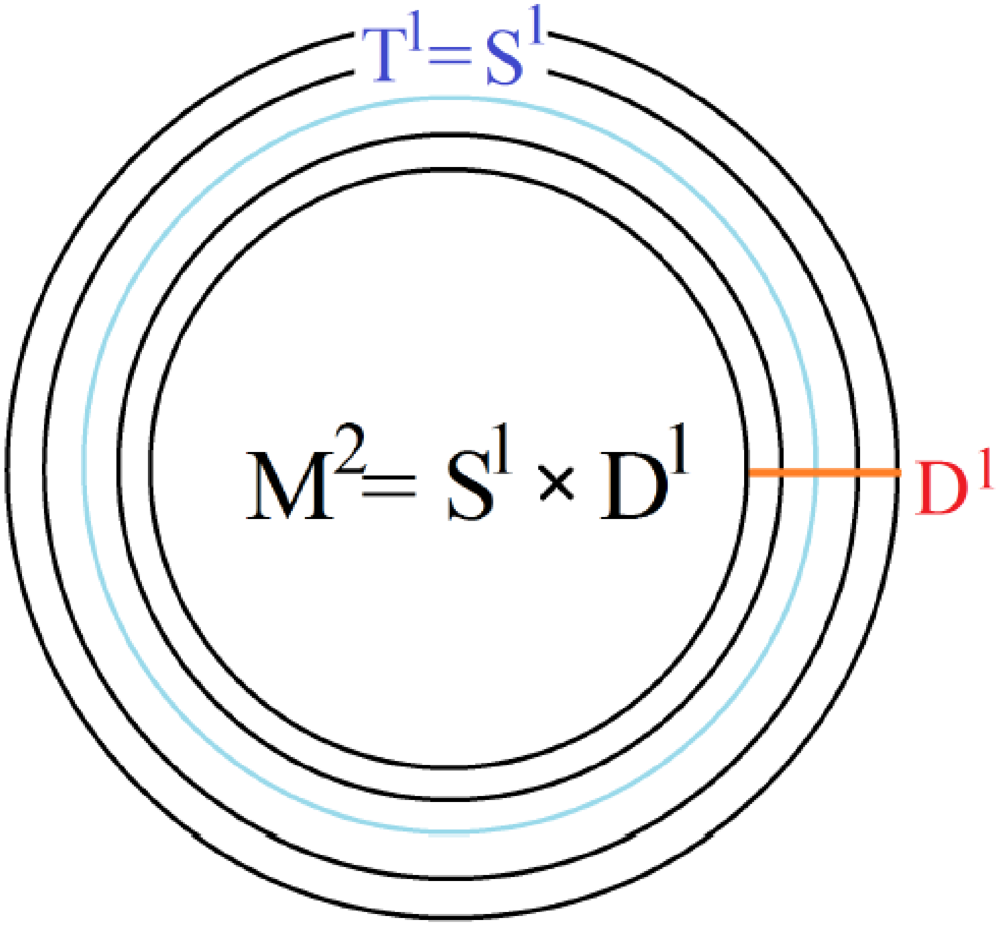
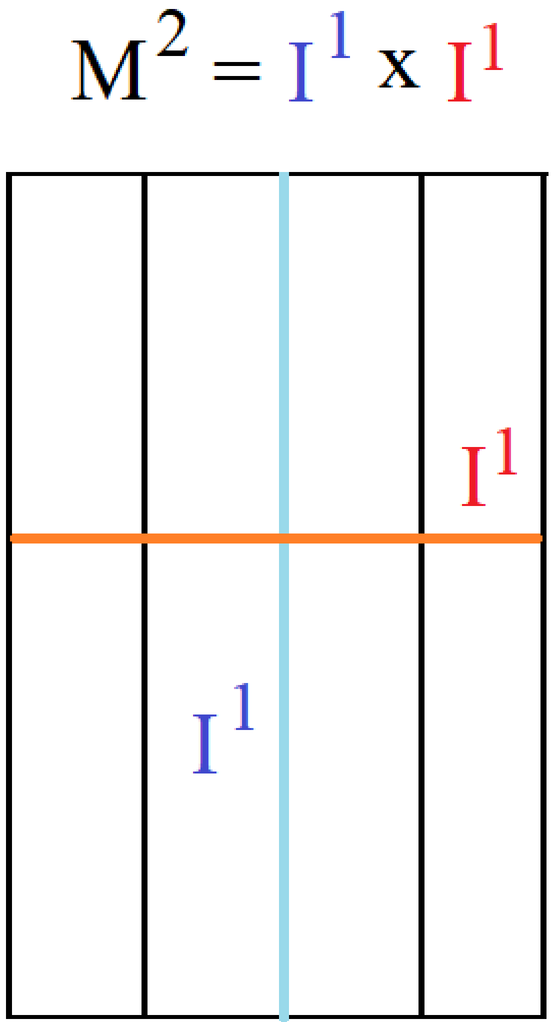
Compact and noncompact foliations

One-dimensional torus T^1 is a circle S^1 and 1-disk D^1 is an interval.

Fibration has structure of a direct product of disk and circle.





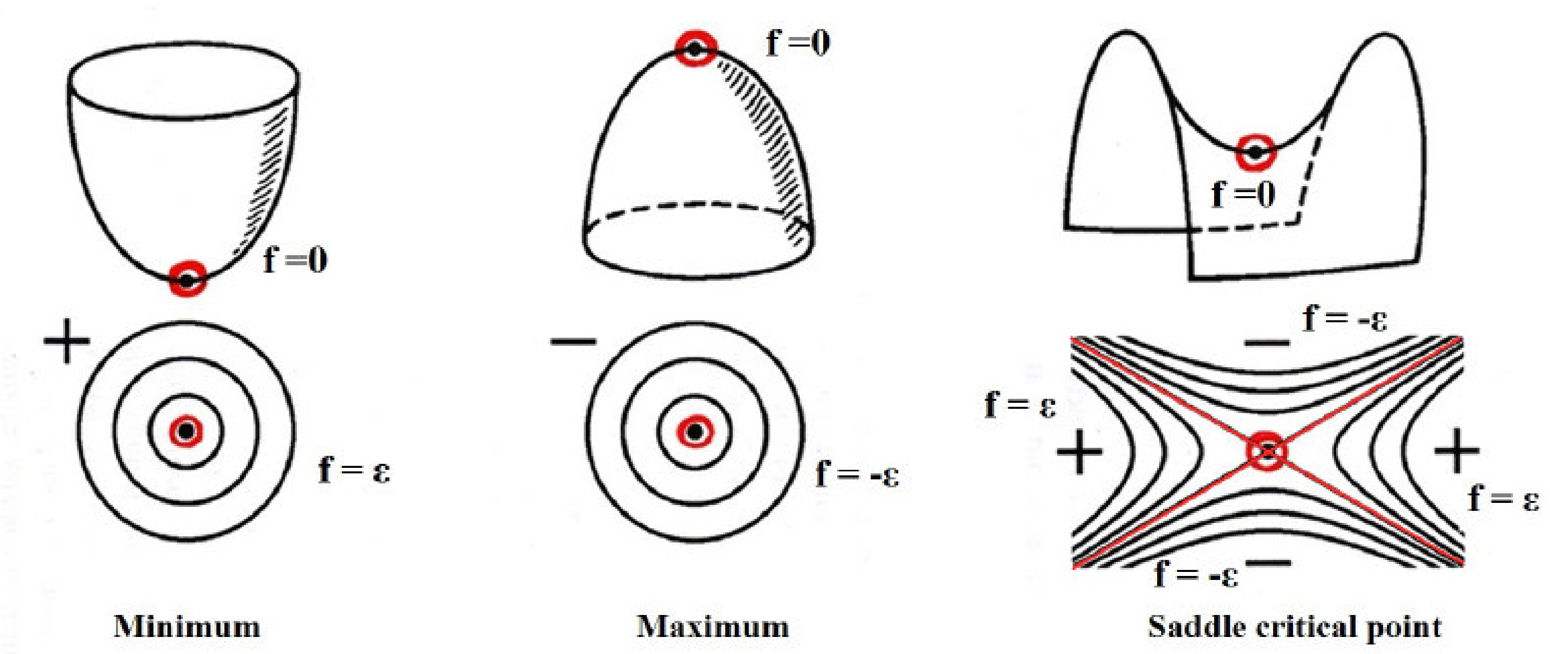
Critical singularities in 1 d.o.f. case

1. Regular point x: $\mathrm{d}H(x) \neq 0$,
Regular fiber is diffeomorphic to union of circles $\coprod_1^k S_i^1$.

2. Singular point x: $\mathrm{d}H(x) = 0$ is of the Morse type if

$$\det d^2 H(x) \neq 0.$$

There are only two types: elliptic $H = x^2 + y^2$ and saddle $H = x^2 - y^2$.

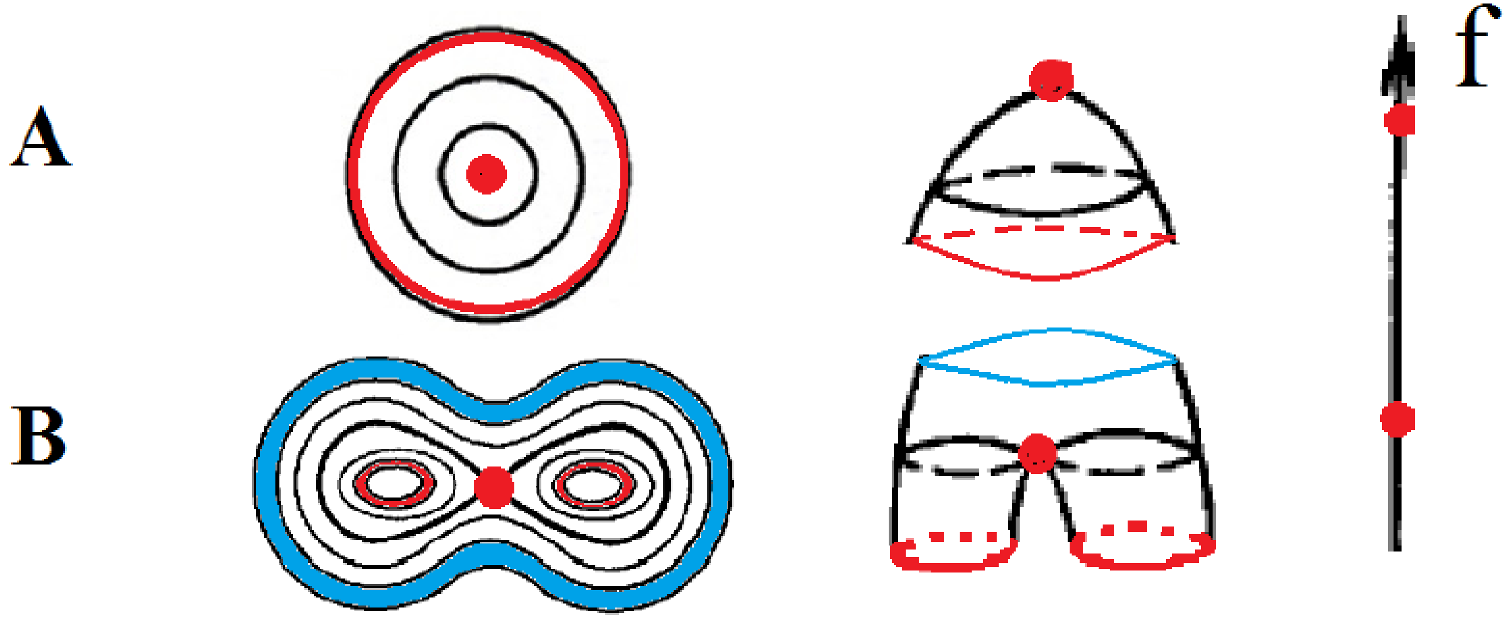


Examples of 2-atoms

- **Singular fiber** is a fiber that contains at least one critical point of H

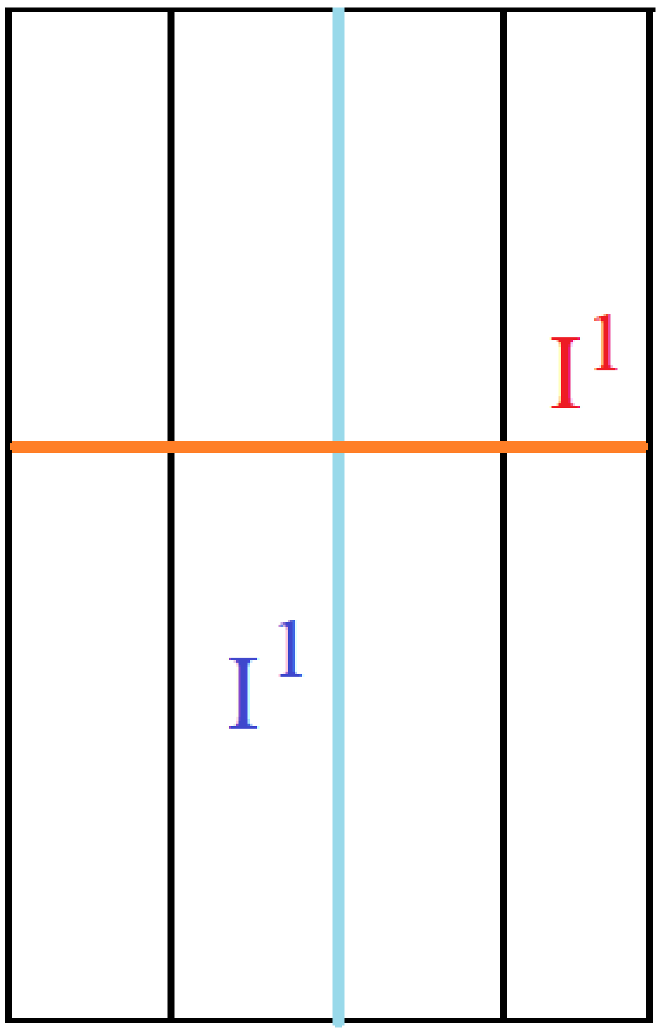
- **2-atom** is a class of fiber-wise diffeomorphic foliateв neighbourhood of a special fiber.

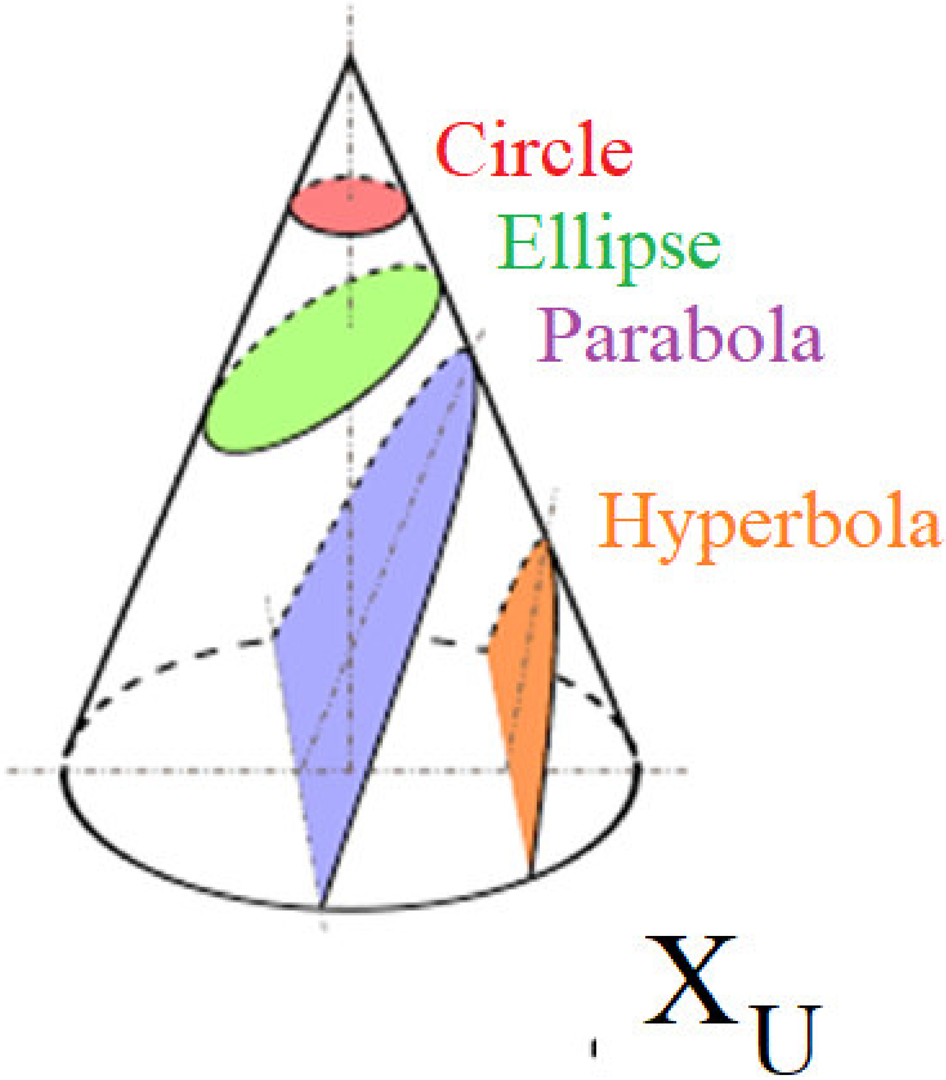
- two different visualisations of hyperbolic atom B and (the unique) elliptic atom A.



Noncompact reg. foliation and noncritical bifurcation

- $M^2 = D^1 \times R^1$



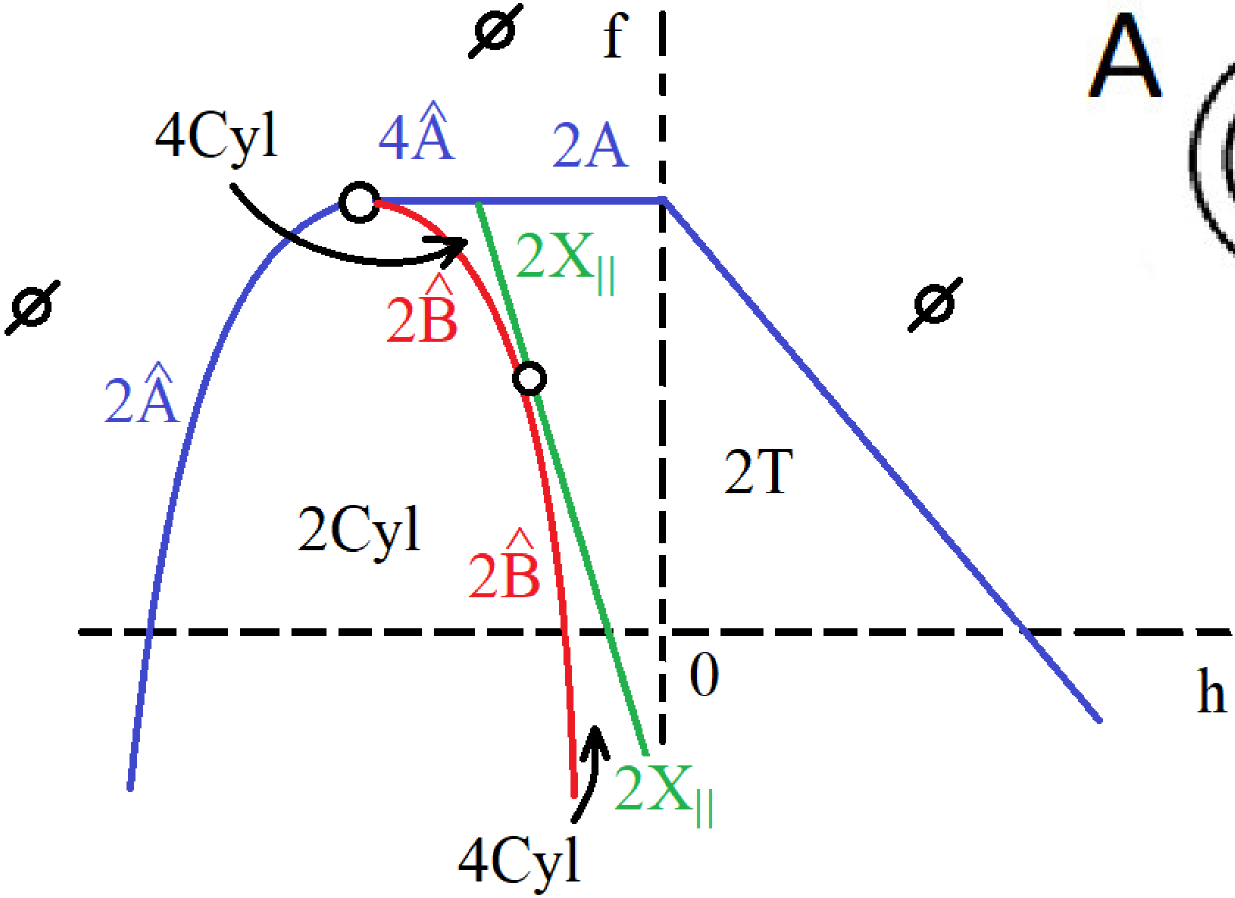


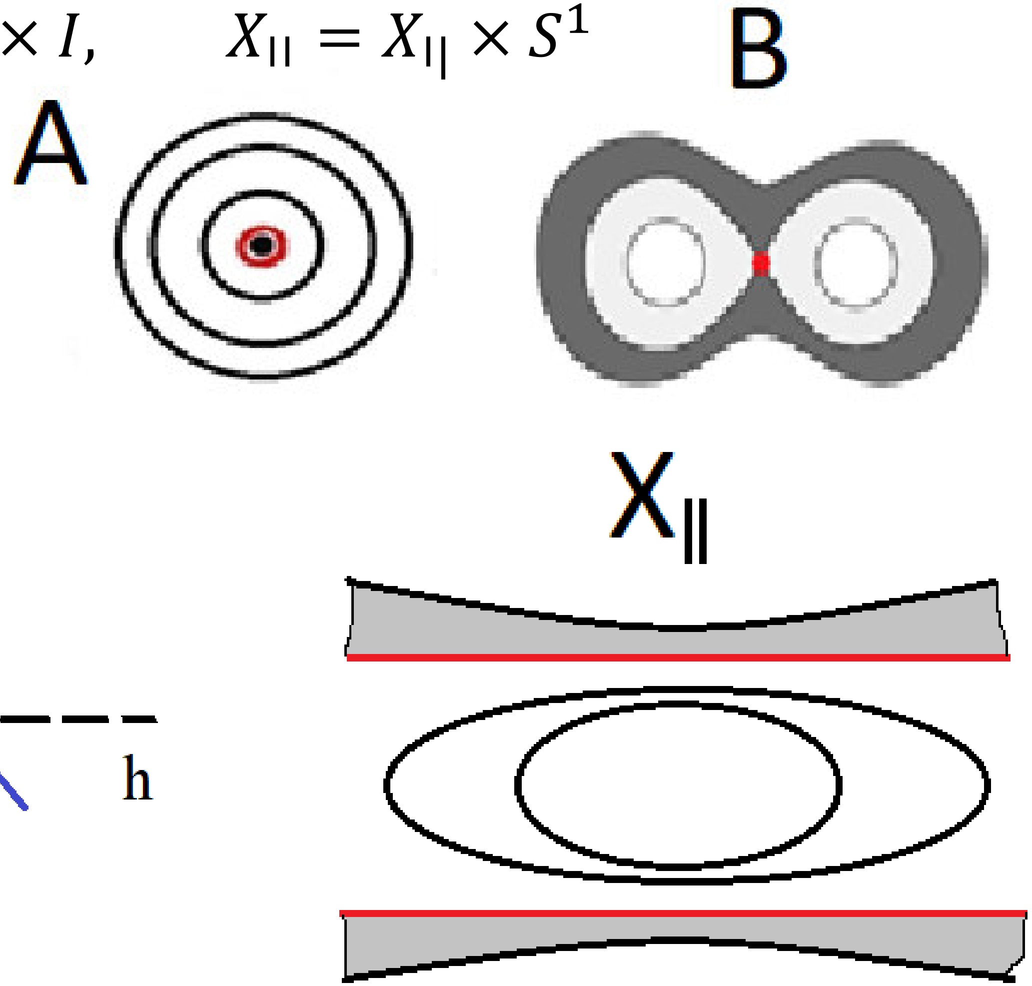
Sokolov $e(3)$ system

- $H = -\frac{\kappa}{\alpha} J_1^2 + \alpha J_2^2 + J_1 x_2 - J_2 x_1, \qquad Q = J \times x$
- $K = Q_3 (\kappa J^2 - x^2) - \alpha Q_1^2 + \frac{\kappa}{\alpha} Q_2^2 + \left(\frac{\kappa}{\alpha} - \alpha\right) Q_3^2$

D.V. Novikov, PhD Thesis, 2013:

$\hat{A} = A_2 \times I, \qquad \hat{B} = B_2 \times I, \qquad X_{||} = X_{||} \times S^1$





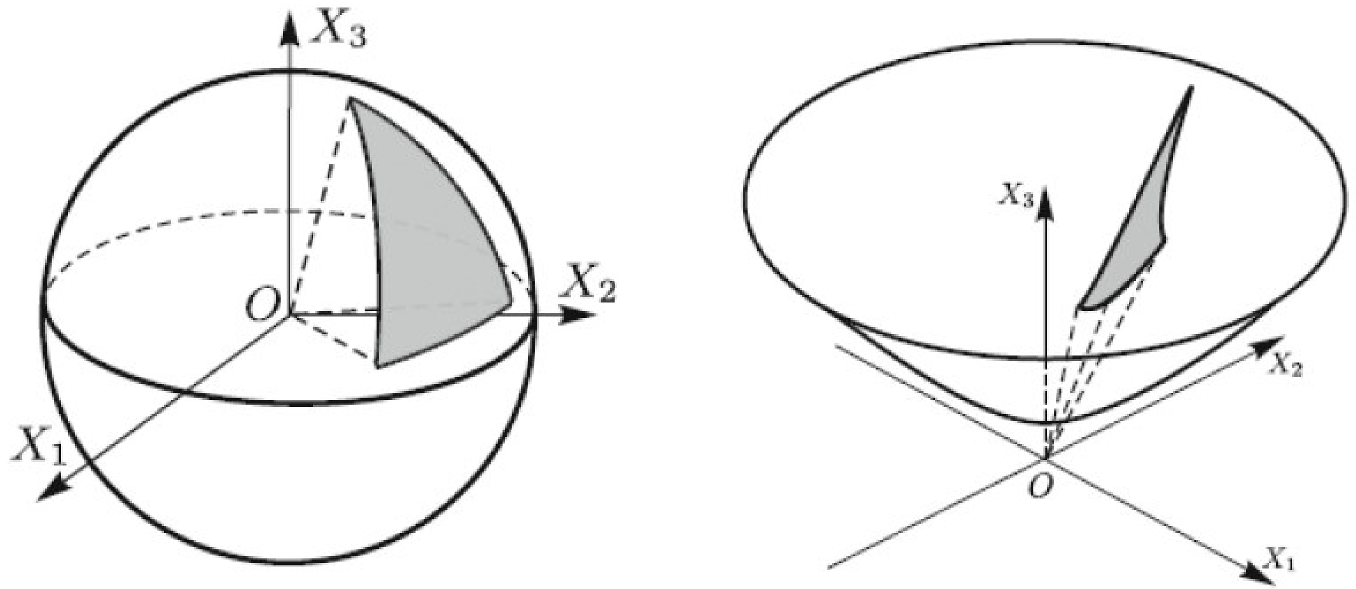
Non-compact foliations (1)

- Review: *A.Fomenko, D.Fedoseev, 16*: Review with a lot of examples of such bifurcations in concrete systems
- *S.Nikolaenko:* - noncompact Goryachev case (2017)
  - noncomplete flows on M^2 (including multi-saddle case), analog of f-graph for such "2-atoms" (20)
  - "3-atoms" w. complete flows & noncompact foliations, (2021)
- $D.Novikov$: Sokolov case for Lie algebras $e(3)$, $so(3,1)$ – 2011, 2014
- *K.Alseshkin*: analog of Liouville theorem for non-complete flows, e.g. if only one of n flows is not complete (2014).
- *A.Borisov, I.Mamaev: analogs of rigid body dynamics systems in pseudo-Euclidean spaces (2016). S.V. Sokolov: separation of variables for pseudo-Euclidean Kovalevskaya system.*
- *E.Kudryavtseva, D.Fedoseev, O.Zagryadsky:* Bertrand problem and its generalizations *(12, 13, 15, 16,18).*

Two analogs of rigid body dynamics

- Motion of a rigid “plate” on S^2 – rotation of a body in R^3 around a fixed point O
- Transformation in C^6: $R^6 \to \tilde{R}^6$

$$x_1 \to ix_1, \quad x_2 \to ix_2, \quad x_3 \to x_3, \quad J_1 \to iJ_1, \quad J_2 \to iJ_2, \quad J_3 \to J_3.$$



A.V.Borisov, I.S.Mamaev, “Rigid Body dynamics in non-Euclidean spaces” // Rus. J. of Math. Phys., **23**:4 (2016), 431-454

Transformations of R^6

- $R^6(J_1, J_2, J_3, x_1, x_2, x_3) \subset C^6$ and its mapping preserving J_3, x_3 :

$$J_1 \to i\, J_1, \qquad J_2 \to i\, J_2, \qquad x_1 \to i\, x_1, \qquad x_2 \to i\, x_2$$

Poisson bracket for so(3,1)-e(3)-so(4) remains real-valued.

- For a wide class real-analytic IHS, this map produces another real-analytic system: Euler, Lagrange, Kovalevskaya, Goryachev, Hess (BM16)
- Several other integrable systems remain real-valued: Zhukovsky, Steklov, Klebsch
- Another construction: Poisson morphism between e(3) and so(3,1) written by Komarov-Sokolov-Tsiganov

  Under this map, Kovalevskaya so(3,1) → Sokolov e(3):

$$\begin{pmatrix} 0 & -J_3 & -J_2 & 0 & -x_3 & -x_2 \\ +J_3 & 0 & J_1 & +x_3 & 0 & x_1 \\ J_2 & -J_1 & 0 & x_2 & -x_1 & 0 \\ 0 & -x_3 & -x_2 & 0 & -\kappa J_3 & -\kappa J_2 \\ +x_3 & 0 & x_1 & +\kappa J_3 & 0 & J_1 \\ x_2 & -x_1 & 0 & J_2 & -J_1 & 0 \end{pmatrix}$$

- $f_1 = x_1^2 + x_2^2 - x_3^2 + \kappa(J_1^2 + J_2^2 - J_3^2) = a, \quad f_2 = J_1 x_1 + J_2 x_2 - J_3 x_3 = b$

Euler $\quad H = \frac{J_1^2}{A_1} + \frac{J_2^2}{A_2} - \frac{J_3^2}{A_3} = h, \qquad F = J_1^2 + J_2^2 - J_3^2 = f,$

Kovalevskaya $\quad H = J_1^2 + J_2^2 - 2 J_3^2 + 2x_1 = h,$

$$F = (J_1^2 - J_2^2 - 2x_1 + \kappa)^2 + (2 J_1 J_2 - 2x_2)^2 = f.$$

- Zhukovsky: $H = \frac{(J_1+\lambda_1)^2}{2A_1} + \frac{(J_2+\lambda_2)^2}{2A_2} - \frac{(J_3+\lambda_3)^2}{2A_3}, \quad \mathrm{K} = J_1^2 + J_2^2 - J_3^2$
- Klebsch $\quad H = \frac{J_1^2}{2A_1} + \frac{J_2^2}{2A_2} - \frac{J_3^2}{2A_3} + \frac{\varepsilon}{2}(A_1 x_1^2 + A_2 x_2^2 - A_3 x_3^2)$

$$\mathrm{K} = \frac{\varepsilon}{2}(J_1^2 + J_2^2 - J_3^2) - \frac{\varepsilon}{2}(A_2 A_3 x_1^2 + A_3 A_1 x_2^2 - A_1 A_2 x_3^2).$$

- Steklov : $\quad H = a_1 J_1^2 + a_2 J_2^2 - a_3 J_3^2 + 2(a_1^2 x_1 J_1 + a_2^2 x_2 J_2 - a_3^2 x_3 J_3) +$
$+ \frac{\varepsilon}{2}(a_1^3 x_1^2 + a_2^3 x_2^2 - a_3^3 x_3^2)$

$\mathrm{K} = J_1^2 + J_2^2 - J_3^2 - 2(a_1 x_1 J_1 + a_2 x_2 J_2 - a_3 x_3 J_3) - 3(a_1^2 x_1^2 + a_2^2 x_2^2 - a_3^2 x_3^2)$

Analog of the Euler top

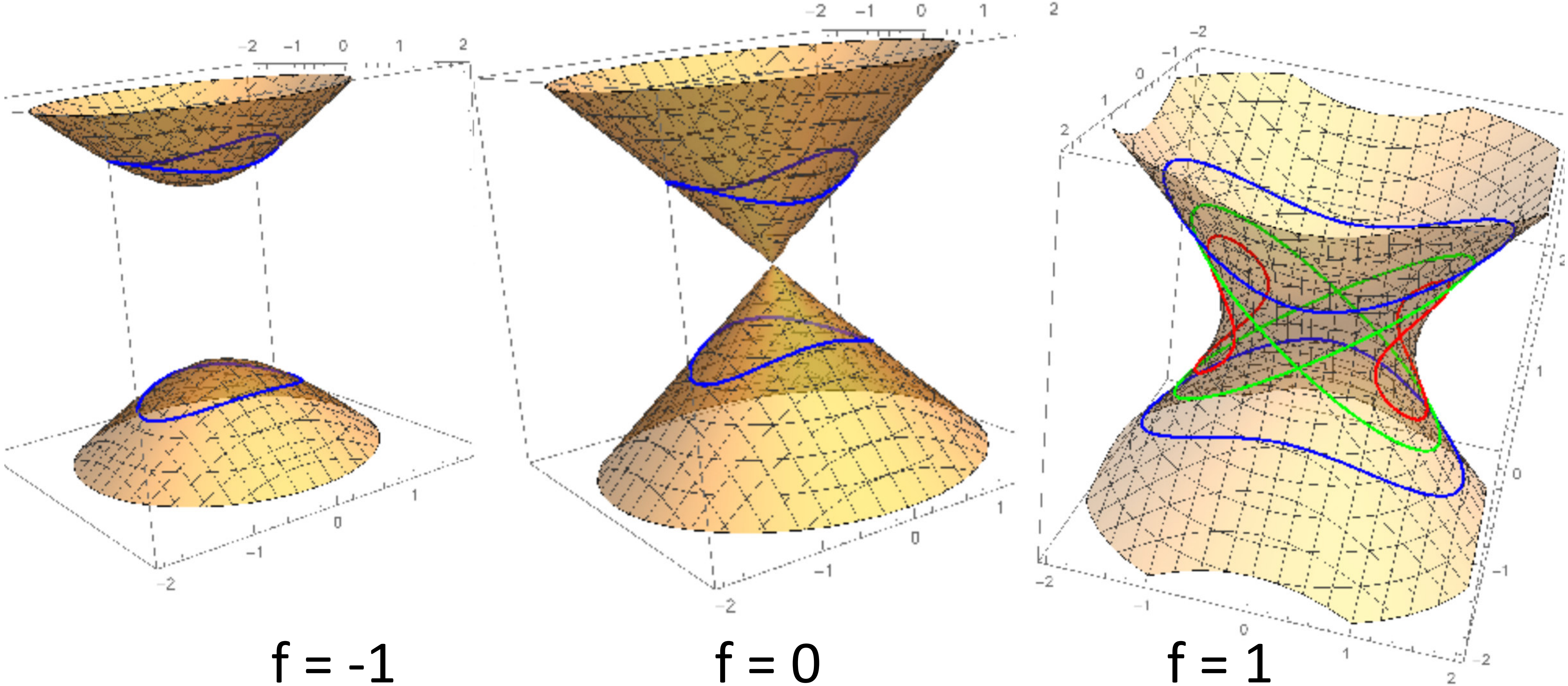
- H, F depends on J_1, J_2, J_3 and are quadratic polynomials

- Liouville fiber $T_{a,b,h,f}$: $f_1 = a, f_2 = b,\ H = h, F = f$ as a bundle:
  - $H = h, F = f$ in $R^3(\vec{\boldsymbol{J}})$ as the base
  - $f_1 = a, f_2 = b$ in $R^3(\vec{\boldsymbol{x}})$ as the fiber upon a point $\vec{\boldsymbol{J}}$

- First integral F and Casimir function f_1 have same form:

$$F = J_1^2 + J_2^2 - J_3^2, \qquad K = x_1^2 + x_2^2 - x_3^2$$

Space $R^3(\vec{J})$ of velocities in the case: $A_1 > A_2 > A_3$

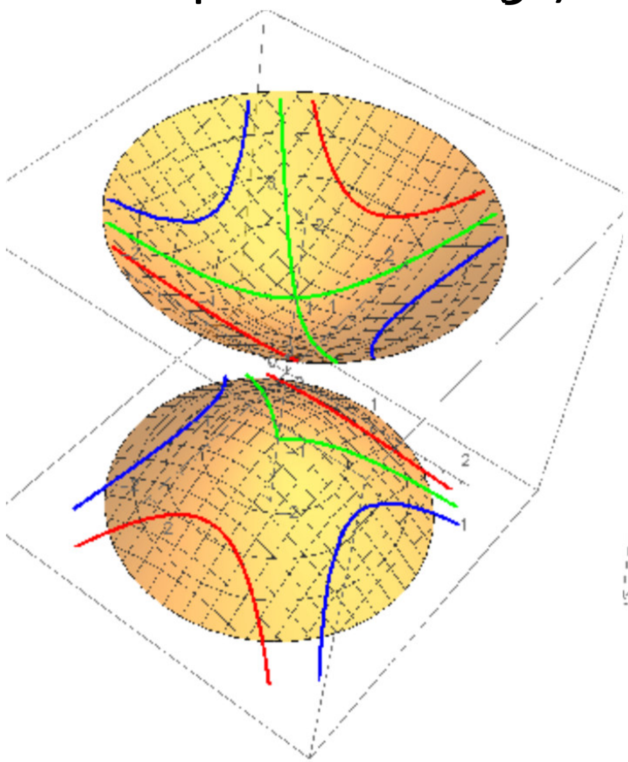


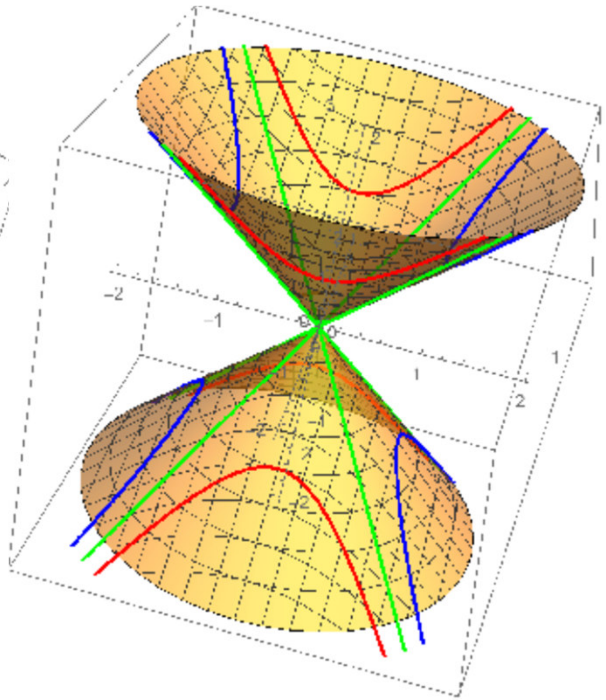
f = -1 f = 0 f = 1

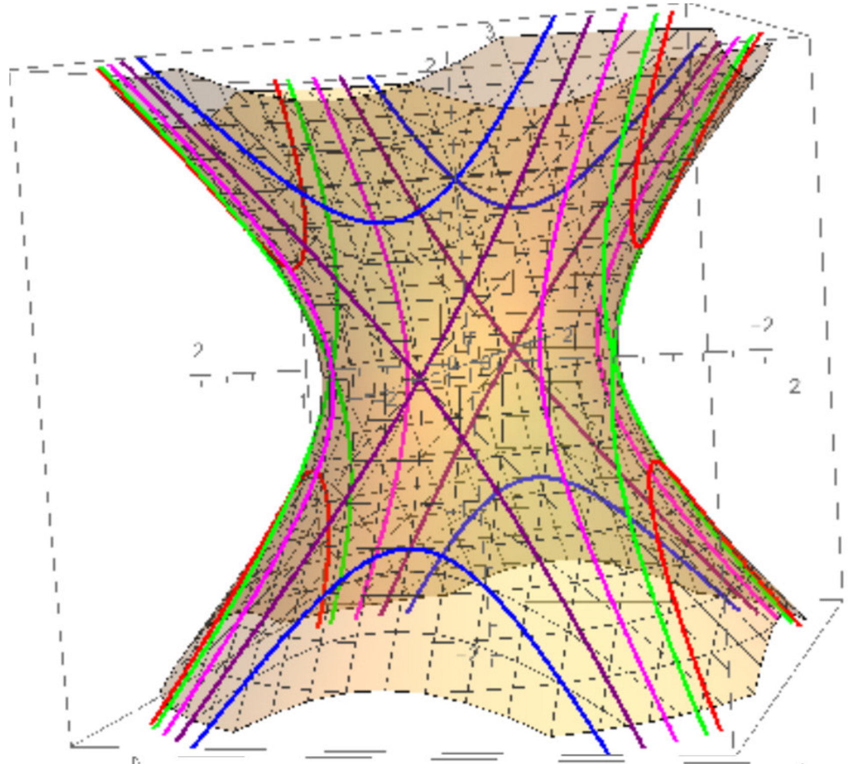
$$K = J_1^2 + J_2^2 - J_3^2 = k, \qquad H = \frac{J_1^2}{A_1} + \frac{J_2^2}{A_2} - \frac{J_3^2}{A_3}$$

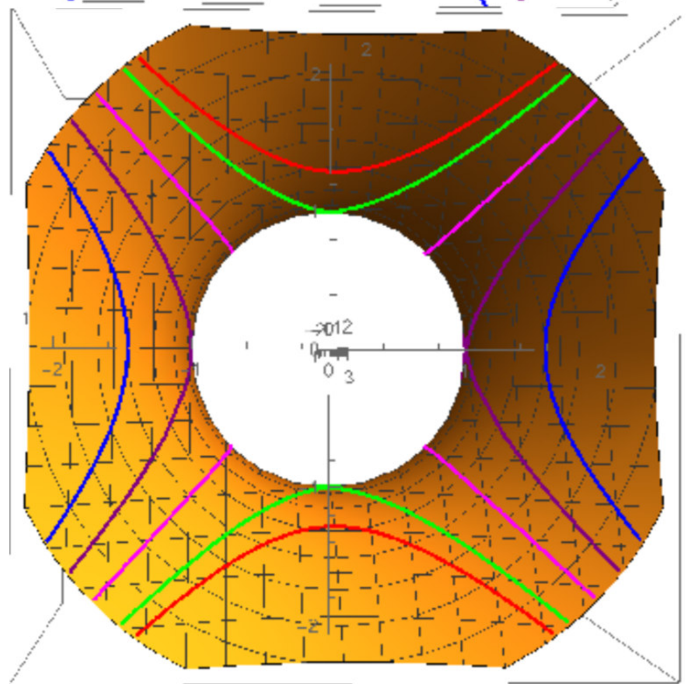
$$h_- < h_{crit} < h_+$$

Space $R^3(\vec{J})$ of velocities in the case: $A_1 > A_3 > A_1$









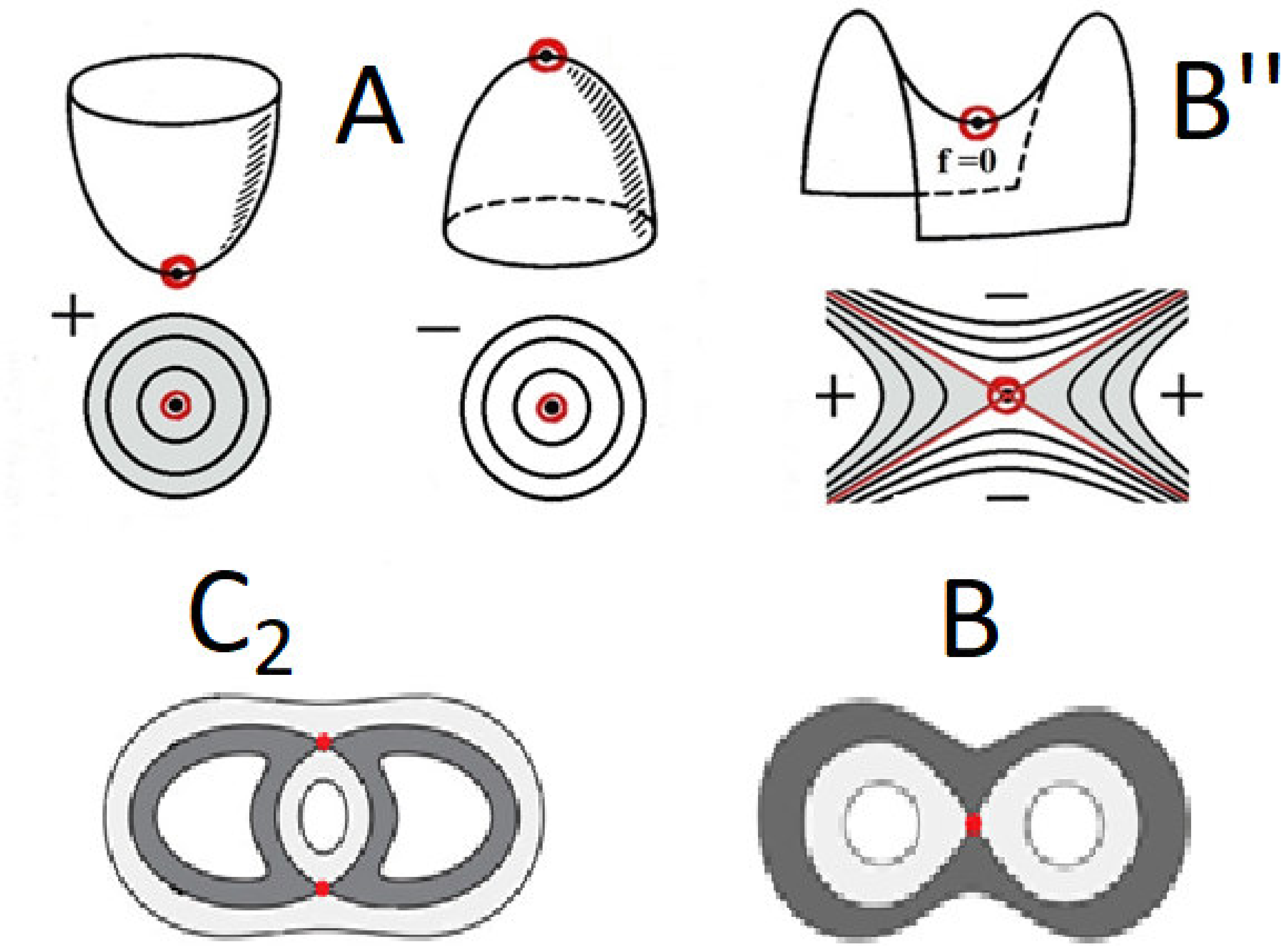
- f = -1 f = 0 f = 1

$$F = J_1^2 + J_2^2 - J_3^2 = f,$$

$$H = \frac{J_1^2}{A_1} + \frac{J_2^2}{A_2} - \frac{J_3^2}{A_3}$$

$$h_- < h_{crit\,1} < h_{mid} < h_{crit\,2} < h_+$$

Examples of 2-atoms



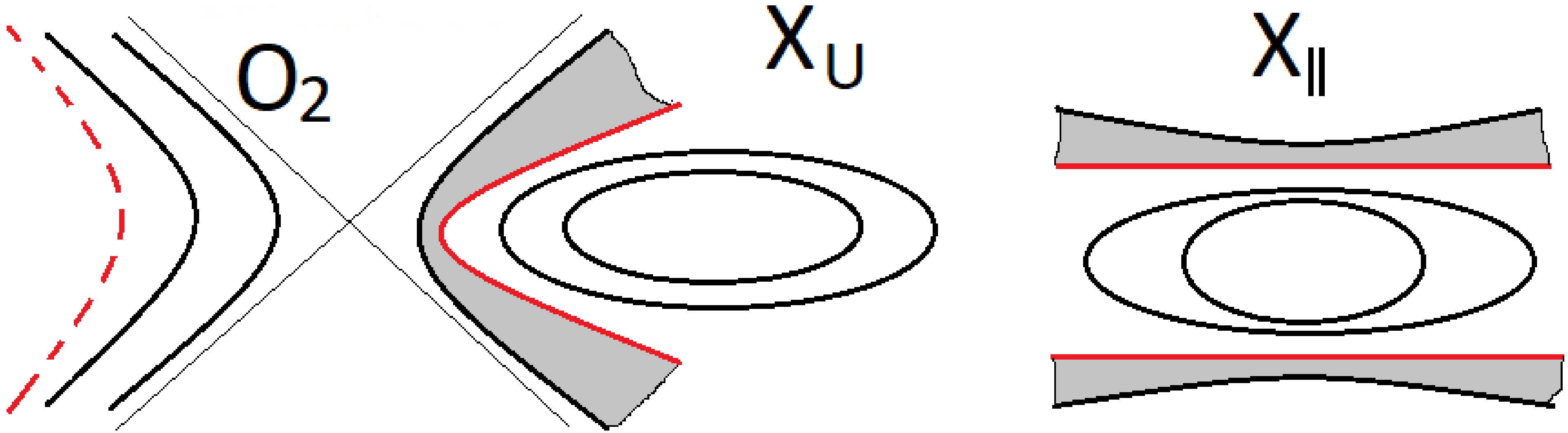
Singularities of pseudo-Euclidean Euler case.

Theorem (Kibkalo, 2021). Singularities of pseudo-Euclidean Euler case have following types. Here ^ means a product of 2-atom and I=R, B'' is a Morse saddle, C_2 is a 2-atom like in the classical Euler case.

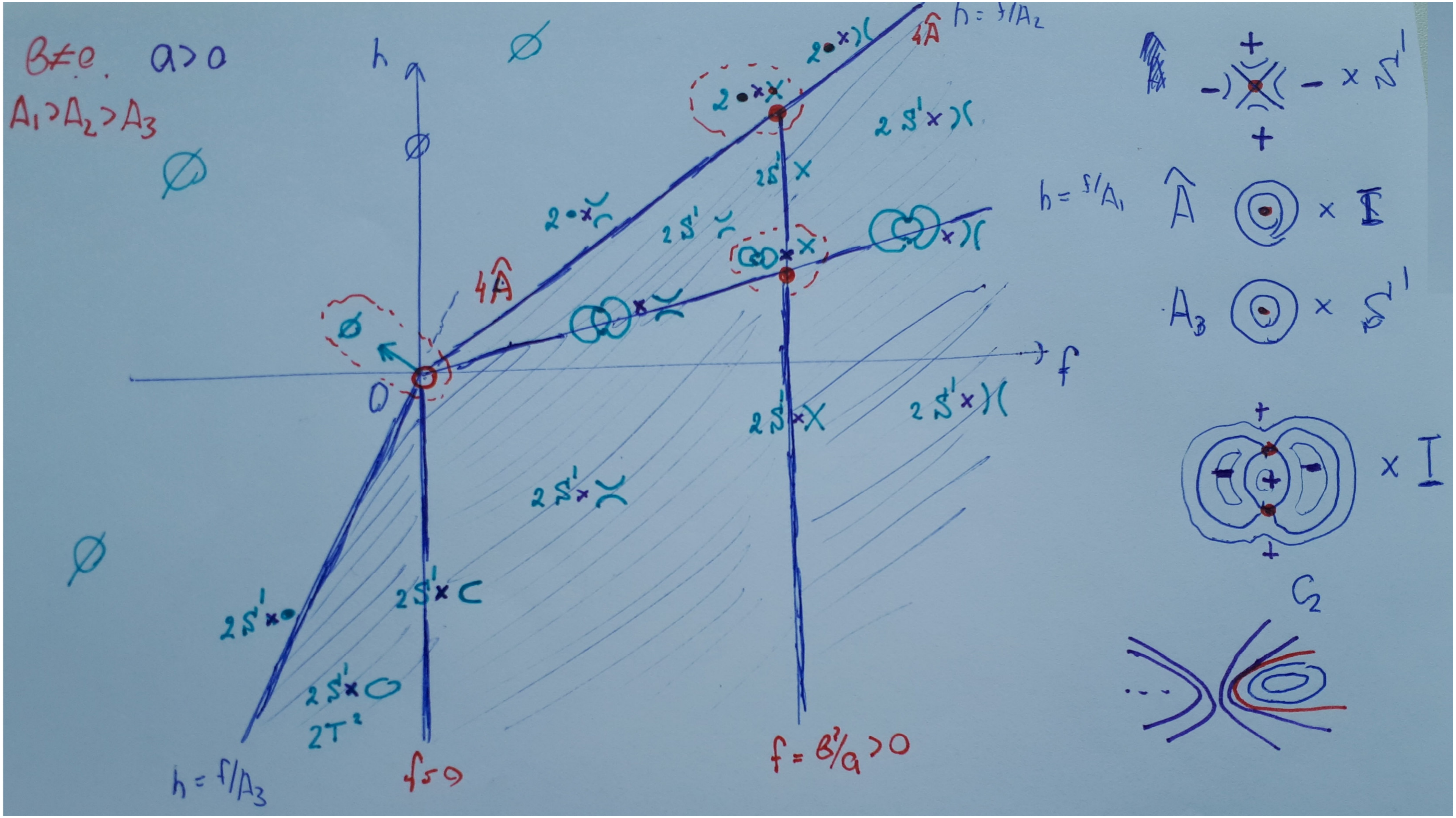
Regular fiber is homeomorphic to a torus T^2, cylinder Cyl or R^2

$A = A_2 \times S^1$, $\widehat{A} = A_2 \times I$, $\widehat{C_2} = C_2 \times I$, $B'' = B'' \times S^1$ $\widehat{B''} = B'' \times I$

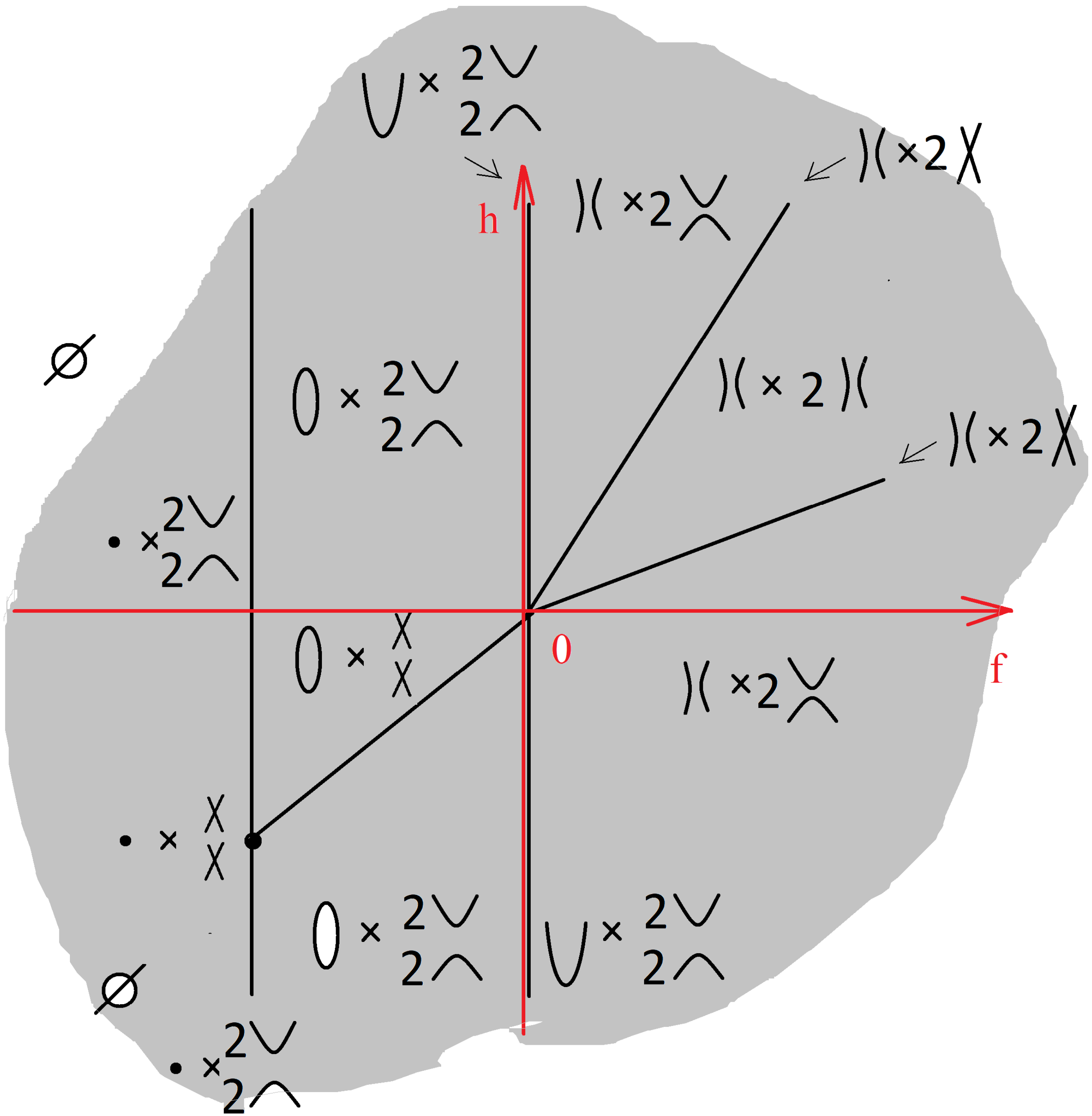
$X_U = X_U \times S^1$ $\widehat{X_U} = X_U \times I$, $X_{||} = X_{||} \times$ S^1



Euler case for $A_1 > A_2 > A_3,\ b \neq\ 0,\ a\ > 0$

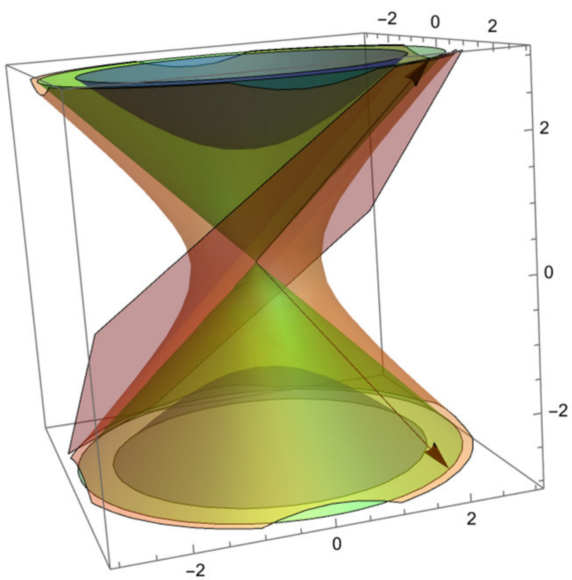


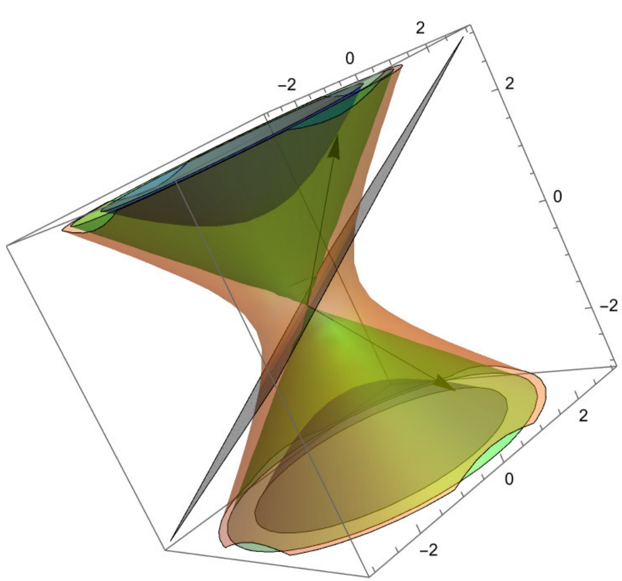
Euler case for $A_1 > A_3 > A_1,\ b \neq 0,\ a < 0$

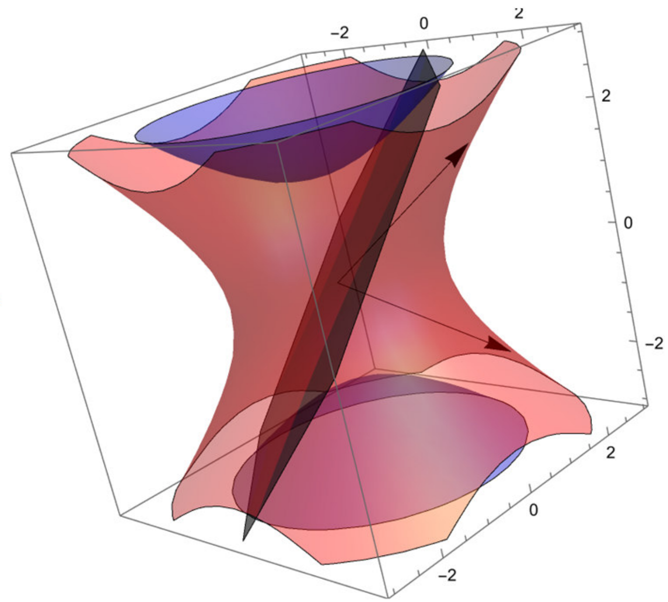


Case of $a \cdot b = 0$

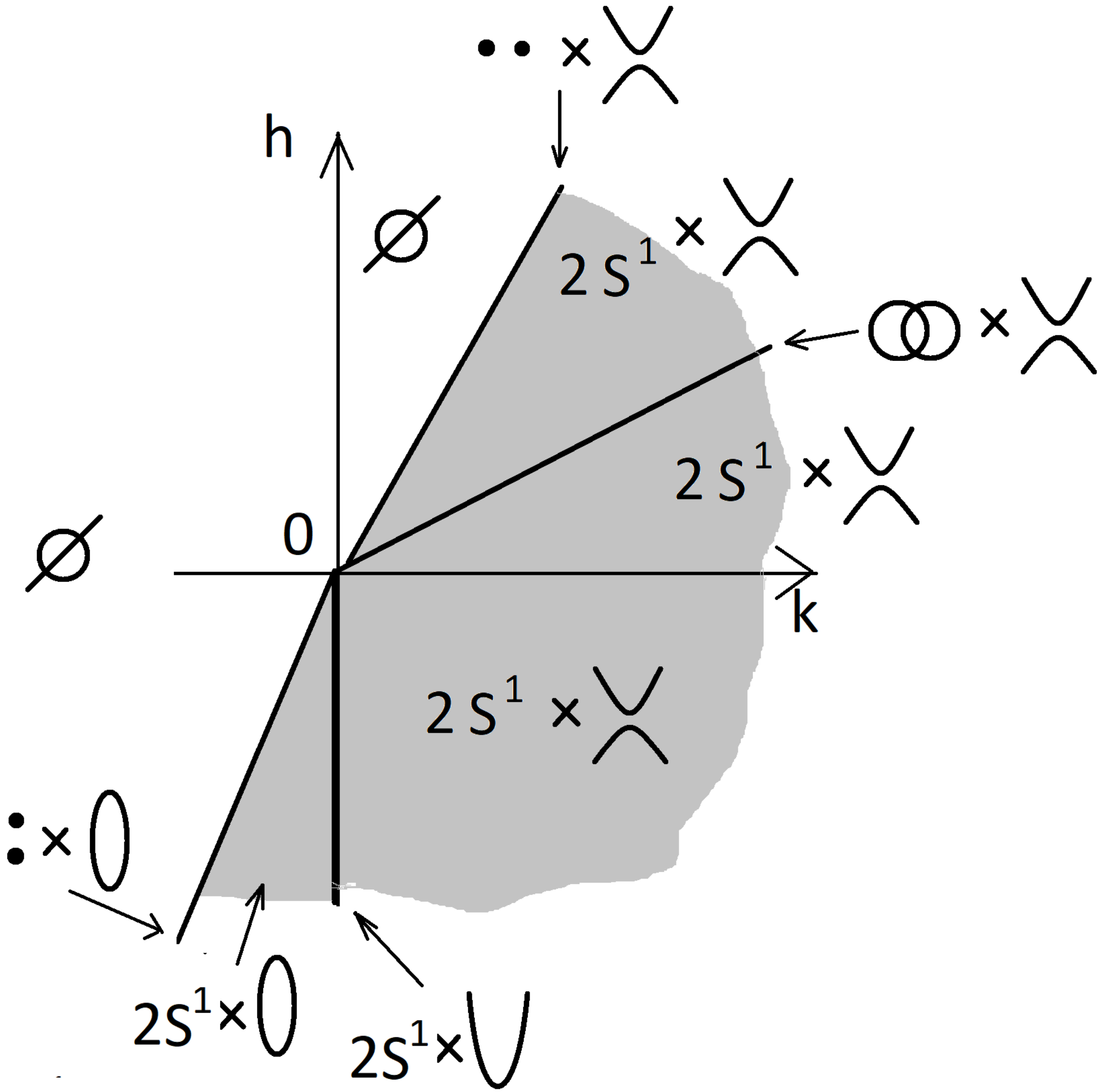
- Kibkalo, Altyev, 2021:
- Bifurcation diagrams, analogs of 3-atoms, rough molecules (bases of Liouville foliation).
- Difference:
  - the plane $f_2 = \langle \overrightarrow{J}, \vec{x} \rangle_g = b$ passes through $\vec{x} = 0$
  - or the quadric $f_1 = a$ is a cone, not a hyperboloid



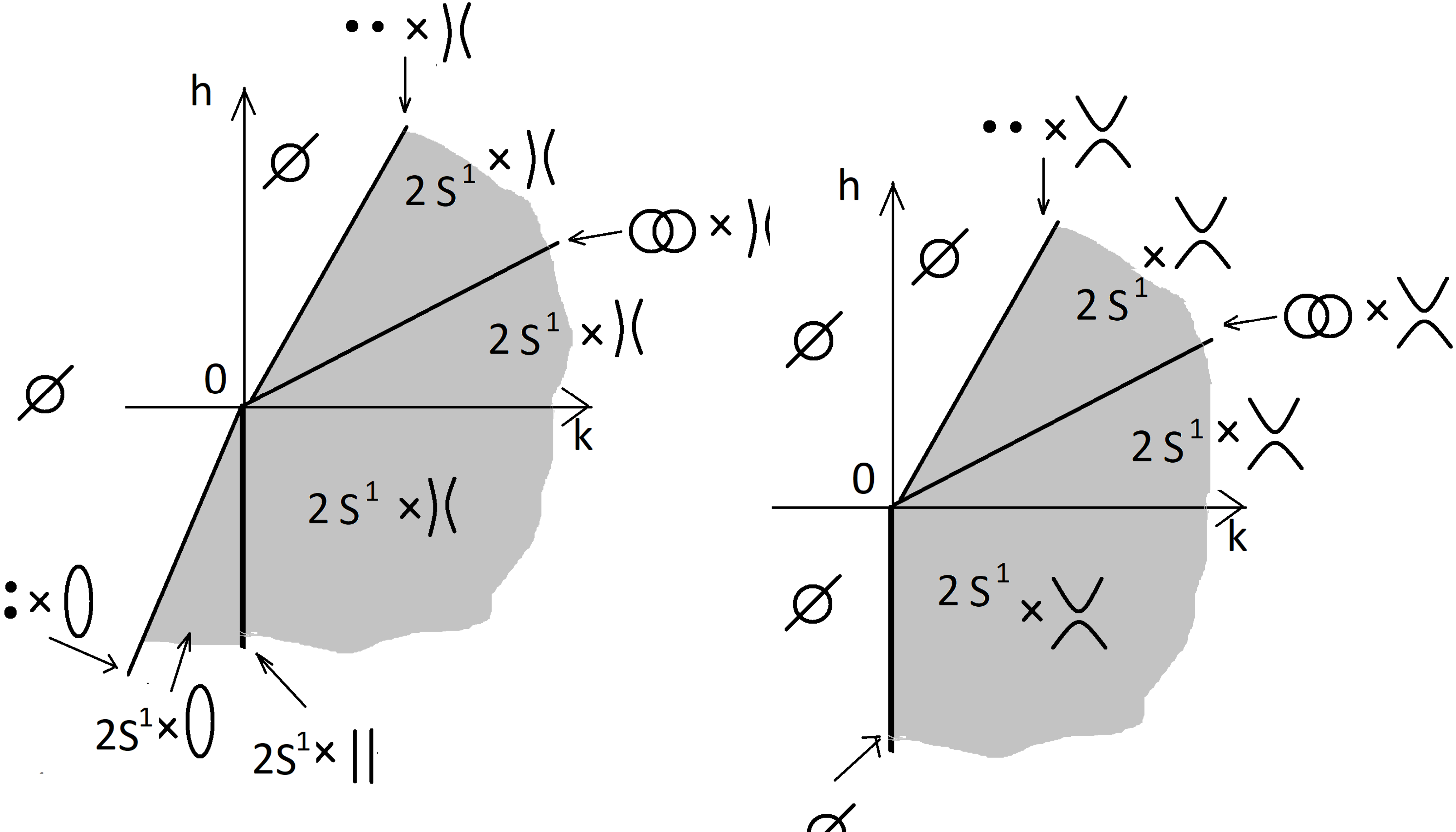




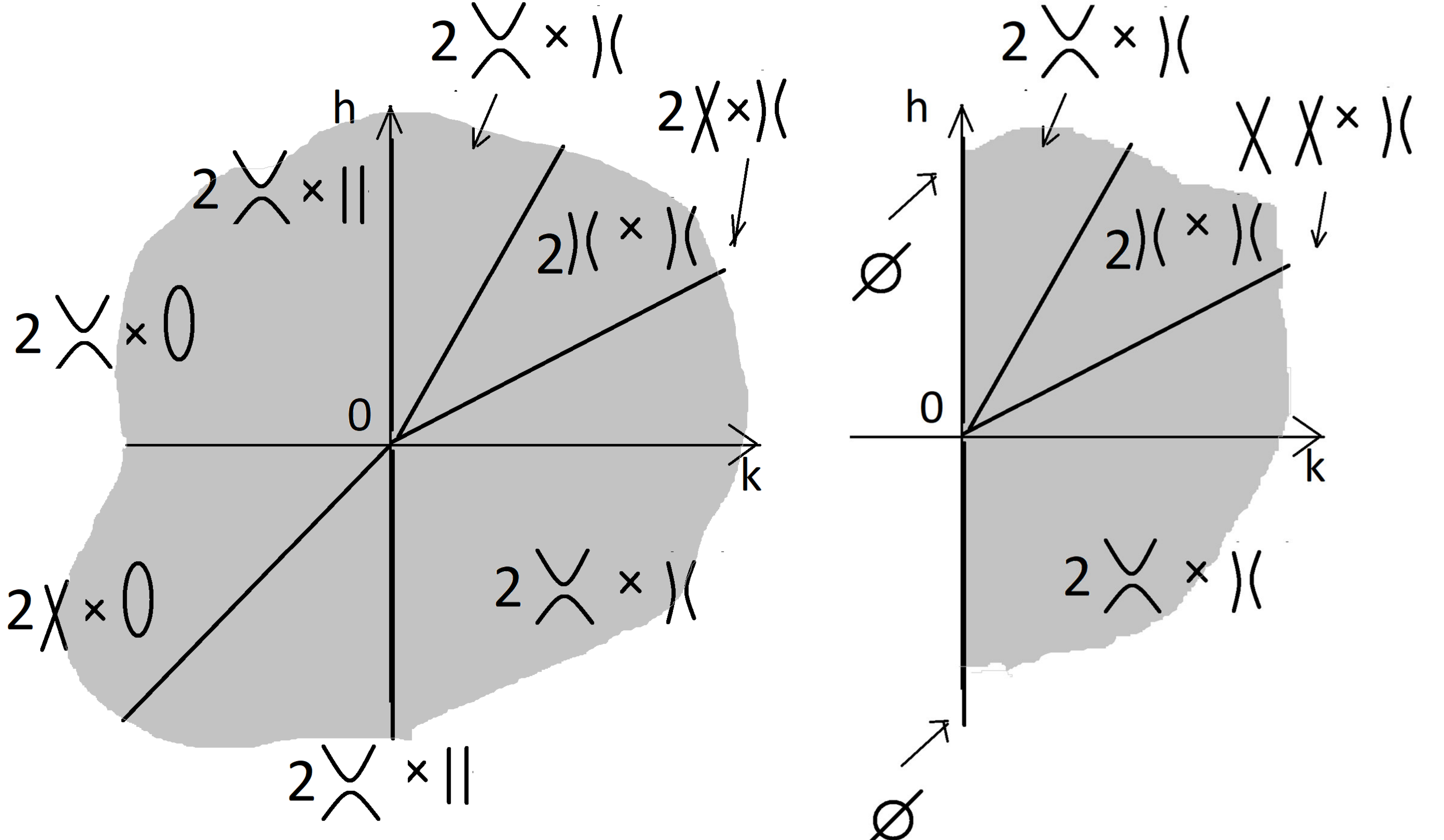
Bif. diagr. $b \neq 0,\ a = 0,\ A_1 > A_2 > A_3$



Bif. diagr. $b = 0, A_1 > A_2 > A_3, a > 0$ *or* $a < 0$



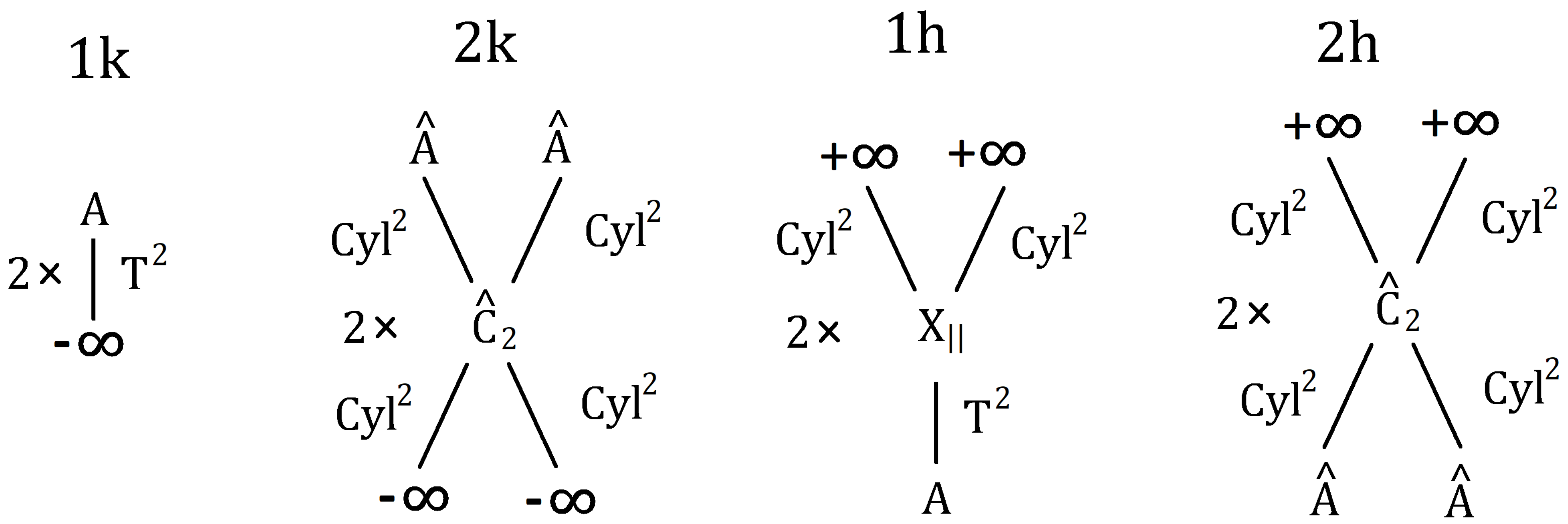
Bif. diagr: Euler for $A_1 > A_3 > A_2$, $b = 0$, $a > 0$ (left) and $a < 0$ (right)



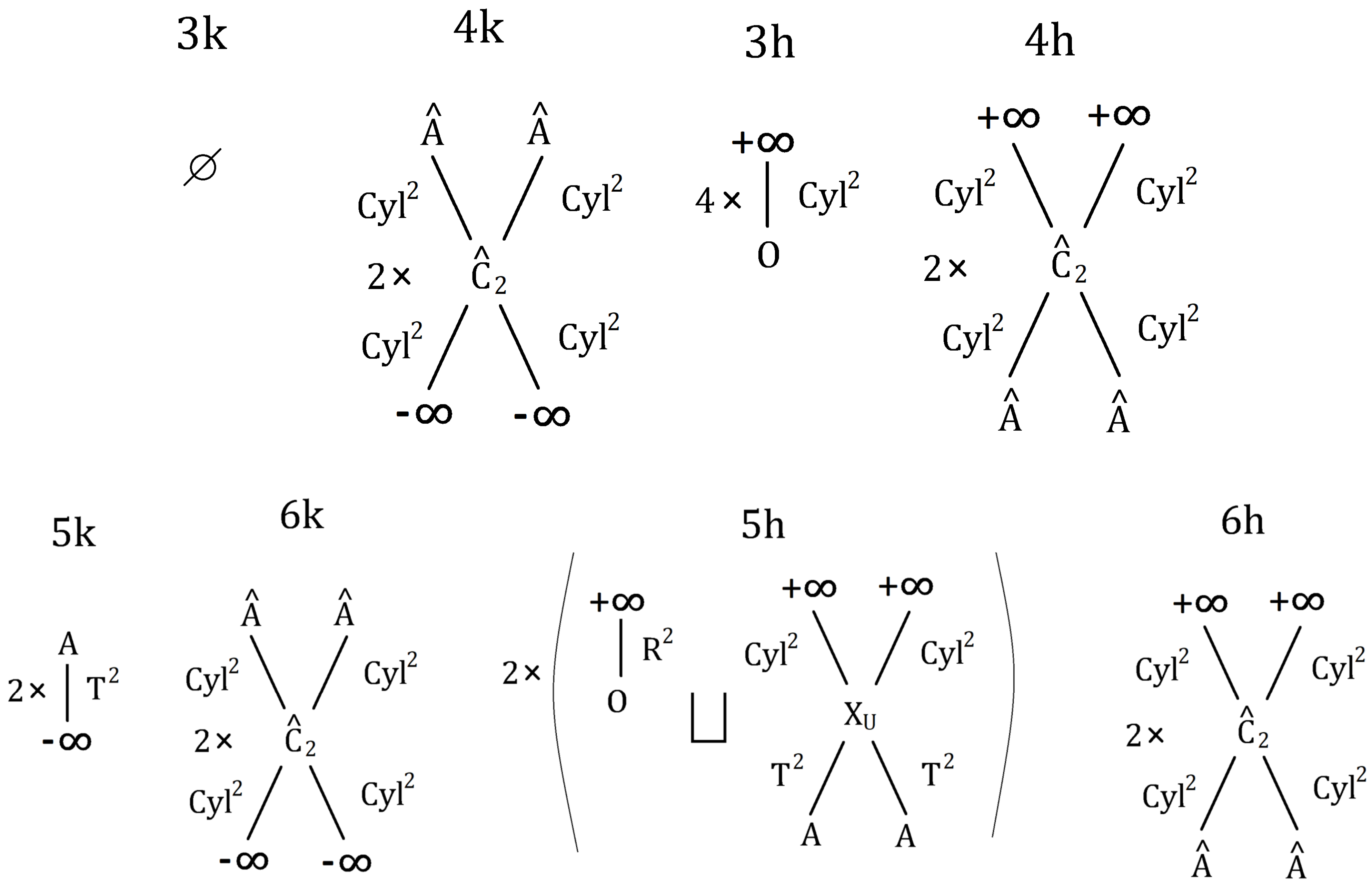
Rough molecules for ps.-Eucl. Euler

Theorem (Kibkalo, Altyev, 2021).

Bases of Liouville foliations on non-singular $Q^3_{a,b,h}$ or $Q^3_{a,b,k}$ of pseudo-Euclidean Euler top with $\boldsymbol{A_1 > A_2 > A_3}$ and **a**$\cdot$ $\boldsymbol{b = 0}$ are the following.



Rough molecules for ps.-Eucl. Euler (2)



Kovalevskaya top analog

Kovalevskaya top: rigid body with a fixed point

Princ. moments of inertia $2:2:1$, energy $H = J_1^2 + J_2^2 + 2J_3^2 + 2c_1x_1$

Problem formulation

- $T_{a,b,h,f}$ is a commone level surface of f_1, f_2, H, F. Is it a compact set?

$$T_{a,b,h,f} = \{y \in \mathbb{R}^6 \mid f_1 = a, f_2 = b, H = h, F = f\}$$

$$f_1 = x_1^2 + x_2^2 - x_3^2 = a,$$

$$f_2 = x_1 J_1 + x_2 J_2 - x_3 J_3 = b,$$

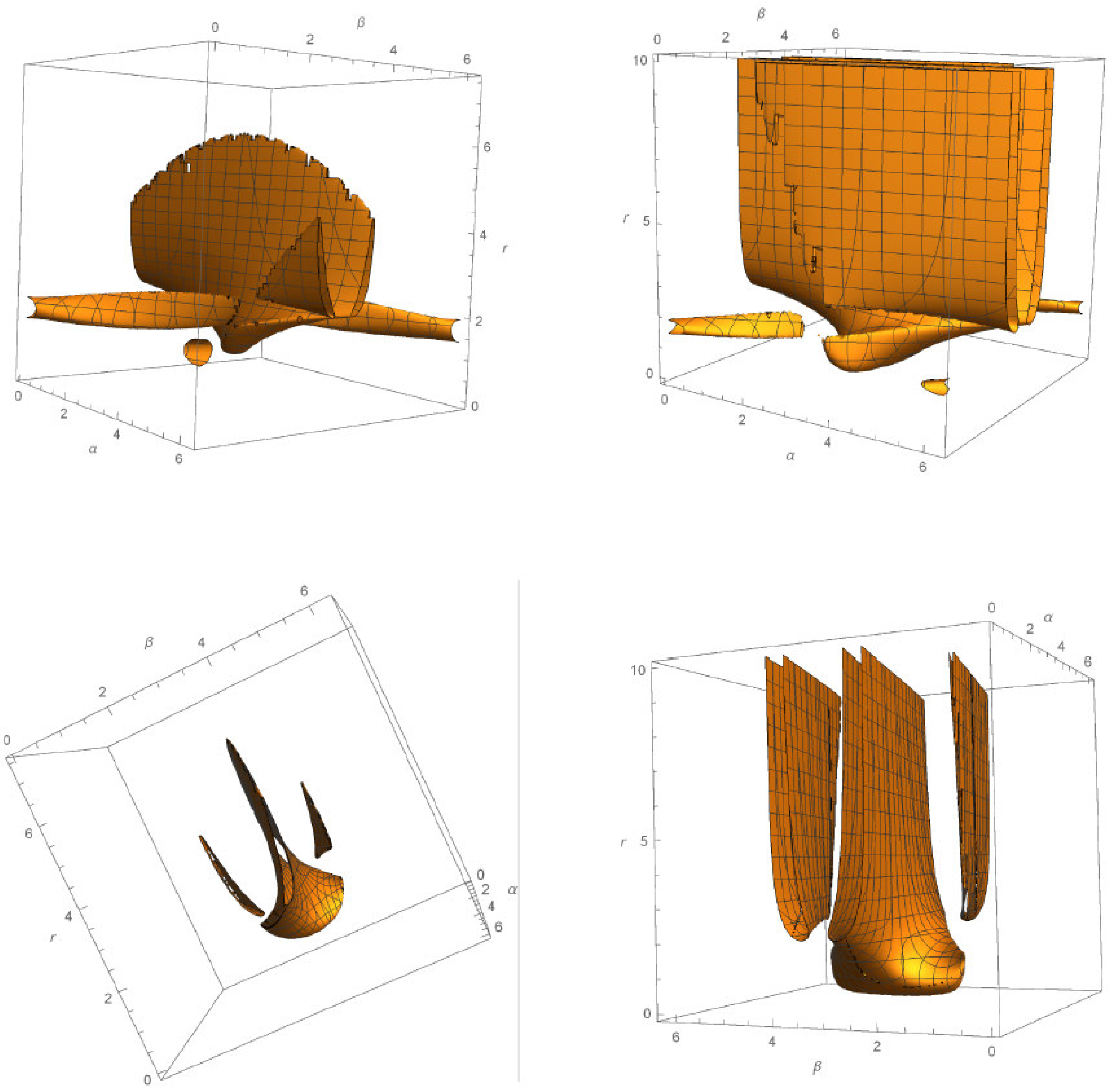
$$H = \frac{1}{2}(J_1^2 + J_2^2 - 2J_3^2) - x_1 = h,$$

$$F = \frac{1}{4}\left(J_1^2 - J_2^2 + 2x_1\right)^2 + \frac{1}{4}(2J_1 J_2 + 2x_2)^2 = f.$$

- Problem is a nontrivial one: i.e. at the case $\varkappa = 0$ we do not have

$$x_1^2 + x_2^2 + x_3^2 < |a|, \qquad J_1^2 + J_2^2 + 2J_3^2 < |h| + g(a).$$

Examples: S and $T_{1,1,h,f}$ for h=4, F=1.8, 2.2



Main ideas

- A closed subset of R^6 is noncompact $\leftrightarrow$ it is unbounded (e.g. its projection on some coordinate)
- Viète theorem: let $a_n \neq 0$. Then roots $x_i(a_0, \ldots\ a_{n-1})$ of a polynomial $P\ = \sum_{i=0}^{n} a_i x^i$ are continuous on other coefficients.
- If all coefficients depend continuously on a point of a compact (closed) surface y $\in M$, then $P(x, y) = 0$ is unbounded iff $a_n(y) = 0$ for some $y \in M$.

Transformations of coordinates

- $$H = \frac{1}{2}(J_1^2 + J_2^2) - J_3^2 - x_1 \; = \; h,$$
$$F = \tfrac{1}{4}\left(J_1^2 - J_2^2 + 2x_1 + \varkappa b_1^2\right)^2 + \tfrac{1}{4}\left(2J_1J_2 + 2x_2\right)^2 \; = \; f\,.$$

- $4F = (J_1^2 - J_2^2)^2 + 4J_1^2J_2^2 + \ldots = (J_1^2 + J_2^2)^2 + \ldots, \qquad 4F = \xi_1^2 + \xi_2^2.$

- Linear in ξ_1, ξ_2 and x_1, x_2 change of coordinates

$$\xi_1 = J_1^2 - J_2^2 + 2x_1, \quad \xi_2 = 2J_1J_2 + 2x_2 \qquad \text{here } \xi_1^2 + \xi_2^2 = 4f$$

- Two polar coordinate transformations (degenerate at $r = 0$ and $\sqrt{f} = 0$)

$$\xi_1 = 2\sqrt{f}\,\cos\alpha, \quad \xi_2 = 2\sqrt{f}\,\sin\alpha, \qquad J_1 = r\,\cos\beta, \quad J_2 = r\,\sin\beta.$$

At $f \neq 0$: a product of torus $T^2(\alpha, \beta)$ at semi-axis of $r > 0$.

Level surface in new coordinates

- Перепишем f_1, f_2, H в новых координатах.
  - From H: $J_1^2 - J_3^2 = h + \xi_1/2$
  - Thus from f_1

$$J_3^2 = -h - \sqrt{f}\ \cos\alpha + r^2(\cos\beta)^2$$

$$4x_3^2 = r^4 - 4\sqrt{f}\cos(\alpha - 2\beta)r^2 + 4(f - a).$$

- Squaring the equation $-b + x_1 J_1 + x_2 J_2 = x_3 J_3$

- obtain $P(r) = 0$: a polynomial of degree 4 in r

$$8P(r) = g_4(\alpha,\beta)r^4 + g_3(\alpha,\beta)r^3 + g_2(\alpha,\beta)r^2 + g_1(\alpha,\beta)r + g_0(\alpha,\beta) = 0.$$

с переменными от α, β, $h, f, b, a, \varkappa$ коэффициентами.

Coeff. $g_j(\alpha, \beta)$ of $P(r)$ are limited at A, i.e. at torus T^2

$$A = [0, 2\pi] \times [0, 2\pi]$$

They depend continuously on angle coordinates α, β and integral values a, b, h, f:

$$g_4 = 2h + 2\sqrt{f}\ \cos(\alpha - 4\beta),$$

$$g_3 = 8b\ \cos\beta,$$

$$g_2 = 4a - 8\sqrt{f}h\cos(\alpha - 2\beta) + 2(2a - 4f)\cos 2\beta,$$

$$g_1 = -16b \cdot \sqrt{f}\ \cos(\alpha - \beta),$$

$$g_0 = 32b^2 + 2(h + \sqrt{f}\ \cos\alpha)(f - a).$$

Compactness of a commo level surface

- Set $V = A \times \mathbb{R}^+ / \sim$ (torus if $r > 0$, circle if $r = 0$):

Denote $S \subset V$ a surface of roots of $P(r) = 0$

- For $f \neq 0$: have a projection $\pi : T_{a,b,h,f} \longrightarrow S$.

Lemma 1. Preimage $\pi^{-1}(x)$ of $x \in S$ in $T^2_{a,b,h,\tilde{f}}$

- is empty $\Leftrightarrow x_3^2(x) < 0$ or $J_3^2(x) < 0$
- is a one point $\Leftrightarrow x_3(x) = J_3(x) = 0.$
- is a pair of points if $x_3(x) \cdot J_3(x) \neq 0$

$$(|x_3(x)|, |J_3(x)|),\ (-|x_3(x)|, -|J_3(x)|),$$

or

$$(|x_3(x)|, -|J_3(x)|),\ (-|x_3(x)|, |J_3(x)|).$$

- **Theorem** (K., 20). Let $h^2 > f$. Then common level surface $T_{a,b,h,f}$ of pseudo-Eucl. Koval. System is compact (for every a, b, $\kappa \in R$).

Compactness criterion and noncritical bifurcations

Theorem (Kibkalo, 2020)

Let the value of angular momentum (area integral) $b \neq 0$.

- Then a common level surface $T_{a,b,h,f}$ is noncompact iff $-\sqrt{f} \leq h \leq \sqrt{f}$.
- An invariant neighbourhood of the fiber $h = \pm\sqrt{f}$ in $Q^3_{a,b,h}$ contains a bifurcation of both compact (or empty) and non-compact fibers.

Bifurcation curves of Kovalevskaya case

Three bifurcation curves of Kovalevskaya pseduo-Euclidean system have same formulas as bifurcation curves of the classiacal Kovalevskaya case

1. Line $k = 0;$
2. Parametric curve

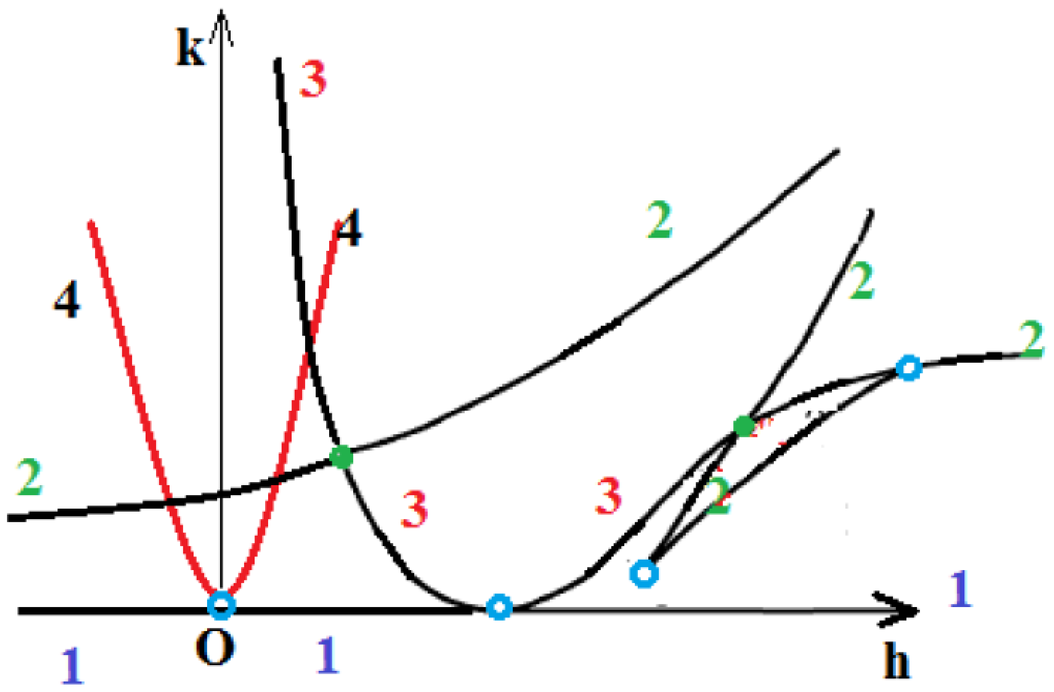
$$k(z) = \left(4a - \frac{4b^2}{z} + \frac{b^4}{z^4}\right), \quad h(z) = \frac{b^2}{z^2} + 2z, \quad \text{где } z \in \mathbb{R} - \{0\}.$$

3. Parabola $k = \left(h - \frac{2b^2}{a}\right)^2.$
4. Parabola $k = h^2$ — has non-critical bifurcations in it's pre image

Such analog of Kovalevskaya system has non-critical bifurcations.

Their detection can not be done by analyzing of $rk\ dF$.

Bifurcation curves on $R^2(h, f)$



1. blue 1: line Oh
2. green 2: parametric curve $(h(z), k(z))$
3. red 3: "critical" parabola
4. black 4: parabola with noncompact noncritical bifurcations

Zhukovsky case: Euler + gyrostat

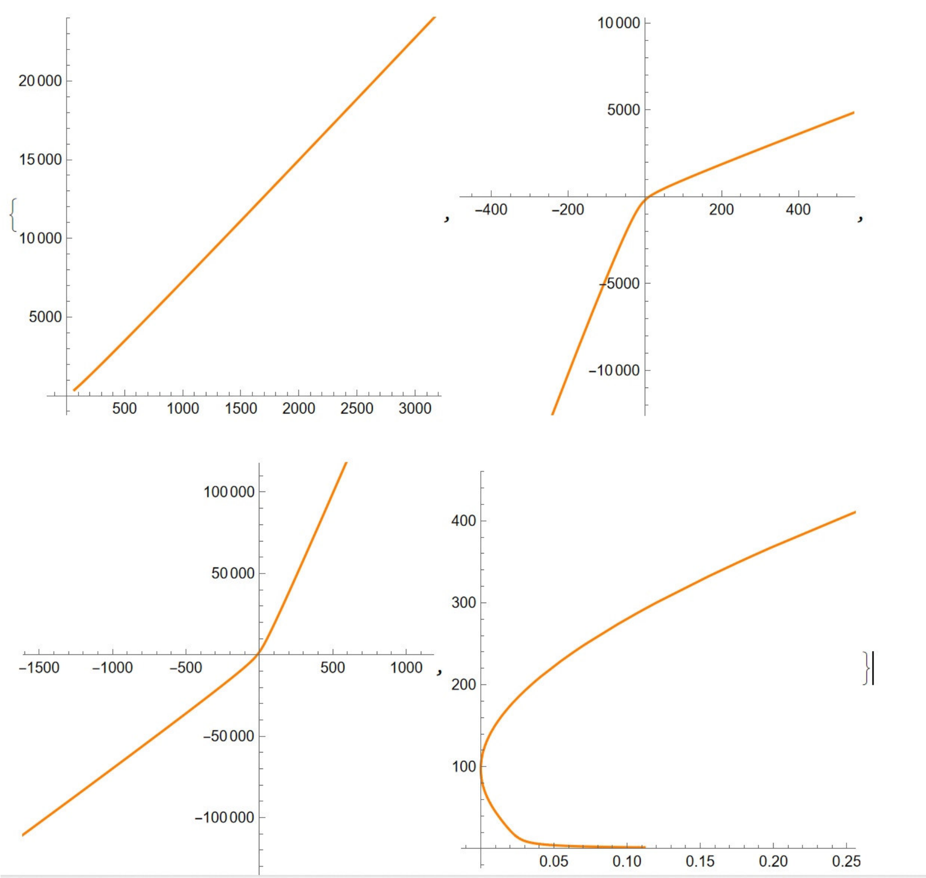
$$f_1 = x_1^2 + x_2^2 - x_3^2 = a\,, \qquad f_2 = J_1x_1 + J_2x_2 - J_3x_3 = b$$

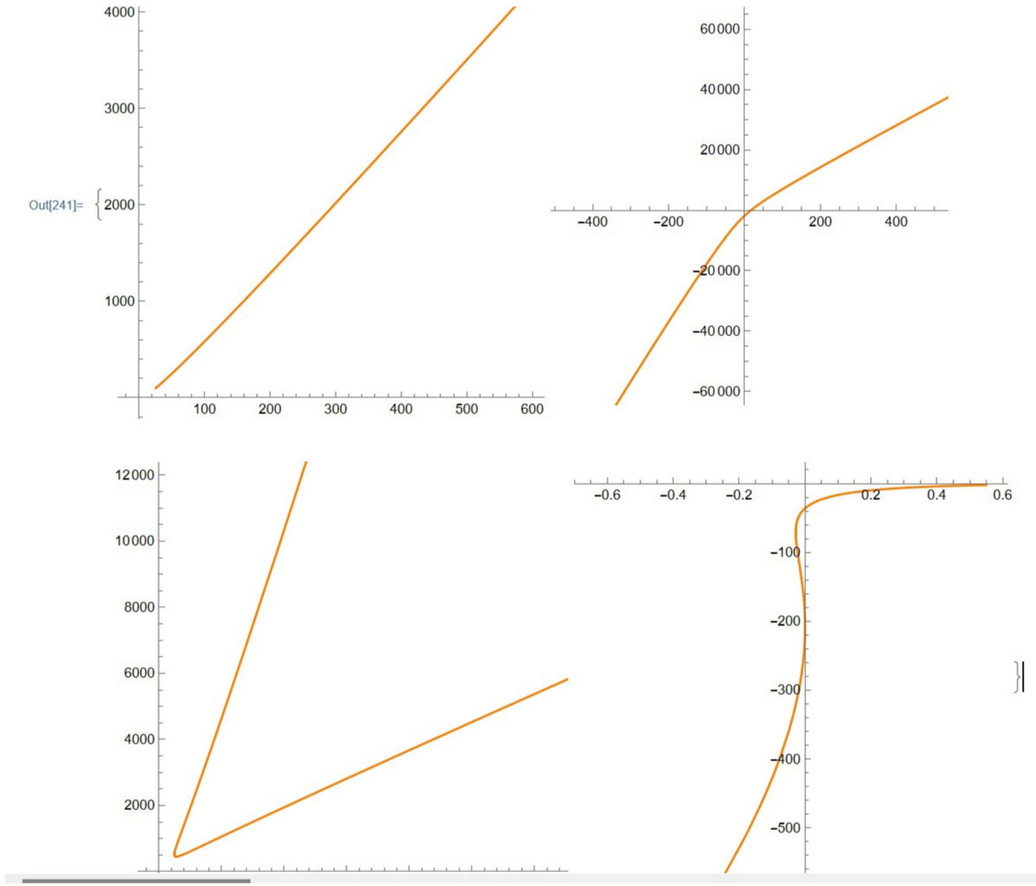
$$H = \frac{(J_1+\lambda_1)^2}{2A_1} + \frac{(J_2+\lambda_2)^2}{2A_2} - \frac{(J_3+\lambda_3)^2}{2A_3} = h, \qquad K = J_1^2 + J_1^2 - J_3^2 = k$$

- **Theorem** (E.Agureeva, 22) For orbit $M^4_{a,b}$ s.th. $a \cdot b \neq 0$, union of three following curves contain the bifurcation diagram of the pseudo-Euclodean Zhukovsly system:
  - k = 0 (for k<0 each fibers are non-compact)
  - $k = \frac{b^2}{a}$
  - $h(t) = \frac{t^2}{2} \cdot \left(\frac{A_1\lambda_1^2}{(1+2A_1t)^2} + \frac{A_2\lambda_2^2}{(1+2A_2t)^2} - \frac{A_3\lambda_3^2}{(1+2A_3t)^2} \right)$

$$k(t) = \frac{A_1^2\lambda_1^2}{(1+2A_1t)^2} + \frac{A_2^2\lambda_2^2}{(1+2A_2t)^2} - \frac{A_3^2\lambda_3^2}{(1+2A_3t)^2}$$

Ps.-Eucl. Zhukovsky case: param. bifurcation curve





A_1, A_2, A_3=(1,2,3)

$t \in [-0,6;\ \ 0,6]$

$(\lambda_1, \lambda_2, \lambda_3) = (4, 5, 6)$

A_1, A_2, A_3=(1,3,2)

$t \in [-0,6;\ \ 0,6]$

$(\lambda_1, \lambda_2, \lambda_3) = (4, 5, 6)$

Open problems:

- Pseudo-Euclidean analogs of
  - Zhukovsky system (Euler top with a gyrostat)
  - Euler systems on so(4) and so(3,1) (when $\kappa \neq 0$)
  - Lagrange top w. a smooth potential $U(x_3)$ and gyrostat
  - Kovalevskaya tops for all $\kappa \in R$

- Modeling of transitions between tori and spheres with handles and punctures using pseudo-integrable billiards (having angles of $3\pi/2$).

- Topology of Liouville foliations of billiard systems inside quadrics in R^3 with the Minkowski metrics.

Thank you
for your attention!

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