Every suspension and every homology sphere are $2H$-spaces

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All spaces are path-connected Hausdorff and with a base point. All maps and homotopies are pointed.

Def. (H-space, in a broad sense) A pair (X, μ), where $\mu \colon X \times X \to X$ is a multiplication, is called an H-space if it satisfies
the Unit Axiom: $\mu(e, x) = \mu(x, e) = x$ for all $x \in X$.
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Simple Facts. (1) $\mathrm{Sym}^n \mathbb{C} \cong \mathbb{C}^n$ (roots of a polynomial of degree n)
(2) $\mathrm{Sym}^2(S^1) =$ Mobius strip
(3) $\mathrm{Sym}^n(\mathbb{R}^m)$ is NOT a TOP manifold (with or without boundary) (it is only a polyhedron), for $n \geq 2$ and $m \geq 3$

Base point $[e, e, \ldots, e] \in \mathrm{Sym}^n X$, and n-valued multiplication $\mu\colon X \times X \to \mathrm{Sym}^n X$

Def. (nH-space) A pair (X, μ) is called an nH-space, if it satisfies the n-valued Unit Axiom: $\mu(e, x) = \mu(x, e) = [x, x, \ldots, x]$ for all $x \in X$.
(So, H-space is just an $1H$-space)

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we get $\lambda\colon X \times X \to \mathrm{Sym}^{n+m} X$,
$\lambda(x,y) := [z_1, \ldots, z_n, u_1, \ldots, u_m] \in \mathrm{Sym}^{n+m} X$, where
$\mu(x,y) = [z_1, \ldots, z_n]$ and $\nu(x,y) = [u_1, \ldots, u_m]$

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Thm(Buchstaber, 1990) *The sphere S^2 is a $2H$-space. Moreover, it is a 2-valued commutative topological group.*

Idea of the proof. Take T^2 and an involution $\tau\colon T^2 \to T^2, \tau(\varphi, \psi) = (-\varphi, -\psi)$.

We use the additive notation $a \in T^2, \tau(a) = -a$.
Then the quotient space $T^2/\tau \cong S^2$. Denote by $\pi\colon T^2 \to S^2$ the canonical projection.
Set for $x, y \in S^2, x = \pi(a), y = \pi(b)$ the 2-valued commutative multiplication $\mu\colon S^2 \times S^2 \to \mathrm{Sym}^2 S^2$ by

$$\mu(x, y) = [\pi(a + b), \pi(a - b)]$$

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Some negative results

Thm(G., 2012) *For any integer* $m \geq 2$ *the space* $\mathbb{C}P^m$ *is not a* $2H$*-space.*

Thm(G., 2011) *Any finite connected CW complex* X *with the fundamental group* $\pi_1(X) = \pi_1(M_g^2)$ *is not a* $2H$*-space, where* M_g^2 *is a compact Riemann surface of genus* $g \geq 2$*.*

Positive results

Thm1(G., 2022) *For any connected finite or countable polyhedron* Y *its reduced suspension* $X = \Sigma Y$ *is a strictly commutative* *nH-space* *for all* $n \geq 2$.

Thm2(G., 2022) *Any smooth homology sphere* $X^m, m \geq 3,$ *is a strictly commutative* *nH-space* *for all* $n \geq 2$.

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Cor. *For any f.p. superperfect group* G

$$H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$$

there exist closed smooth manifolds X *such that* $\pi_1(X) = G$ *and* X *is a* *2H-space*.

Fact. *Fundamental group of an H-space is always abelian. Moreover, for any finitely generated abelian group G there exists a compact connected Lie group X^m with $\pi_1(X^m) = G$.*

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Open question.
Classify f.p. groups G that are fundamental groups of $2H$-spaces (finite CW complexes or smooth manifolds).

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Def. ($n\Delta$-space) X is called an $n\Delta$-space for some $n \geq 2$, if there exists a map $f_n\colon X \to X$ s.t. the diagonal

$\Delta_n\colon X \to \mathrm{Sym}^n X, \Delta_n(x) = [x, x, \dots, x]$, is homotopic to the map

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Simple fact. *The wedge sum (finite or countable) of $n\Delta$-spaces is again an $n\Delta$-space.*

Lemma 1. *($n\Delta$-spaces) $\Rightarrow$ (strictly commutative nH-spaces)*

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Set $\tilde{\mu} \colon X \times X \to \mathrm{Sym}^2 X, \tilde{\mu}(x, y) = [f(x), f(y)]$.

It is homotopic to the required $\mu \colon X \times X \to \mathrm{Sym}^2 X$ with unit axiom.

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Lemma 2. ($n\Delta$-spaces) $\Rightarrow$ (cup product in $\tilde{H}^*(X; \mathbb{Q})$ is zero)

Cor. *The torus T^m of dimension $m \geq 2$ is NOT an $n\Delta$-space for any $n \geq 2$.*

Theorem [Morton, 1967]. *Consider* $\mathrm{Sym}^n S^1$ *with the map*

$$s_n\colon \mathrm{Sym}^n S^1 \to S^1,\ \ s_n[t_1, t_2, \ldots, t_n] = t_1 + t_2 + \ldots + t_n.$$

Then for all $n \geq 2$ *the map* s_n *is a fibre bundle with the fiber* D^{n-1}*, trivial for odd* n *and nonoriented for even* n*. In particular, the map* s_n *is a homotopy equivalence for all* $n \geq 2$*.*

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Lemma 3. *The circle* S^1 *is an* $n\Delta$*-space for all* $n \geq 2$*.*

Proof. Set $f(t) = nt$. Then $t \mapsto [nt, 0, 0, \ldots, 0] \sim t \mapsto [t, t, \ldots, t]$

Lemma 4. *For any connected finite or countable polyhedron* Y *its reduced suspension* $X = \Sigma Y$ *is an* $n\Delta$*-space for all* $n \geq 2$*.*

Thm1(G., 2022) *For any connected finite or countable polyhedron* Y *its reduced suspension* $X = \Sigma Y$ *is a strictly commutative* *nH-space* *for all* $n \geq 2$.

Cor. *Spheres* $S^m, m \geq 1$, *are strictly commutative* *nH-spaces* *for all* $n \geq 2$.

Thm1(G., 2022) *For any connected finite or countable polyhedron Y its reduced suspension $X = \Sigma Y$ is a strictly commutative nH-space for all $n \geq 2$.*

Cor. *Spheres $S^m, m \geq 1$, are strictly commutative nH-spaces for all $n \geq 2$.*

Lemma 5. Fix some $n \geq 2$. Suppose X and Y are connected finite CW complexes, and $f\colon X \to Y$ is a map s.t.

(1) $\mathrm{Sym}^n f\colon \mathrm{Sym}^n X \to \mathrm{Sym}^n Y$ is a homotopy equivalence;
(2) Y is a strictly commutative nH-space.

Then X is also a strictly commutative nH-space.

Lemma 6. *Suppose* X *and* Y *are connected finite CW complexes, and* $f\colon X \to Y$ *is a map s.t.* $f_*\colon H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ *is an isomorphism. Then for all* $n \geq 2$ *the corresponding maps*

$$\mathrm{Sym}^n f\colon \mathrm{Sym}^n X \to \mathrm{Sym}^n Y$$

also induce an isomorphism of integral homology.

Thm2(G., 2022) *Any smooth homology sphere* $\Sigma^m, m \geq 3$, *is a strictly commutative nH-space for all* $n \geq 2$.

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Idea of the proof.

Set $f\colon \Sigma^m \to S^m$ of degree 1. f itself is not a homotopy equivalence. But, due to Lemma 6 $\mathrm{Sym}^n f\colon \mathrm{Sym}^n\Sigma^m \to \mathrm{Sym}^n S^m$ is a homotopy equivalence for all $n \geq 2$. Lemma 5 concludes the proof.

Thm3(G., 2022) *Suppose* X *is a connected finite CW complex of dimension* $\dim X = d \geq 2$ *s.t.* $\pi_1(X)$ *is a perfect group (i.e.* $\pi_1(X)^{ab} = H_1(X;\mathbb{Z}) = 0$*). Then* X *is an strictly commutative nH-space for all* $n \geq d$*.*

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Cor. *Any f.p. perfect group* G *may be realized as a fundamental group of 2-dimensional connected finite polyhedron* X *which is a* $2H$*-space.*

Open Problems

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1. Find all closed unorientable surfaces $X = M^2$ (with the exception $X = \mathbb{R}P^2$) which are $2H$-spaces.
2. Construct some simply-connected closed smooth manifold $X = M^m, m \geq 4$, which is a $2H$-space and is not a sphere and a product of a sphere and a simply connected H-space.
3. Is the product $S^5 \times S^5$ a $2H$-space or not?

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Thank you for your attention!