HOPF-TYPE THEOREMS FOR f-NEIGHBORS

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November, 11, 2021

Introduction

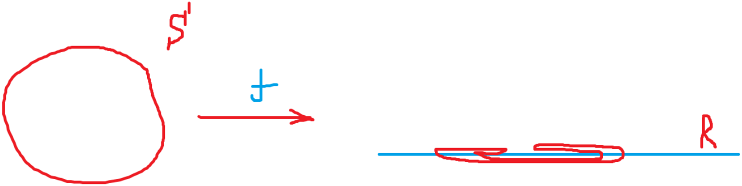


Figure Schematic continuous map $f \colon \mathbb{S}^1 \to \mathbb{R}^1$

Introduction

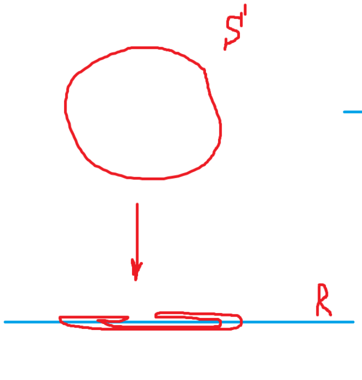
Theorem (K. Borsuk, S. Ulam, 1933)

Under any continuous map f of a standard Euclidean sphere $\mathbb{S}^n$ to $\mathbb{R}^n$ some two opposite points are mapped to a single point.

Theorem (H. Hopf, 1944)

Let n be a positive integer, let M be a compact Riemannian manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a continuous map. Then for any prescribed $\delta > 0$, there exists a pair $\{x,y\} \in M \times M$ such that $f(x) = f(y)$ and x and y are joined by a geodesic of length δ.

Get rid of restriction on the codomain dimension?



From 'usual' f-neighbors to 'visual' f-neighbors

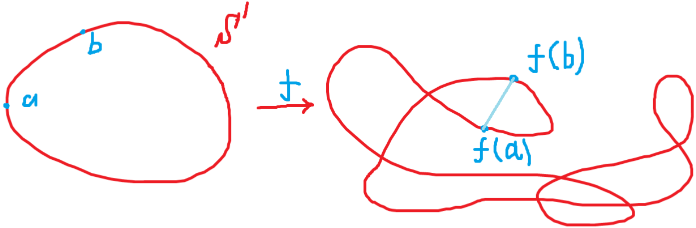


Figure a and b are visual f-neighbors under continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$

... to 'spherical' f-neighbors

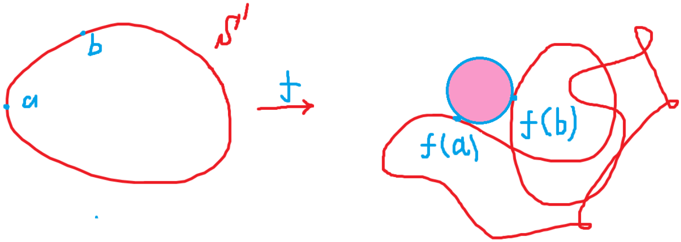


Figure a and b are spherical f-neighbors under continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$

... to 'topological' f-neighbors

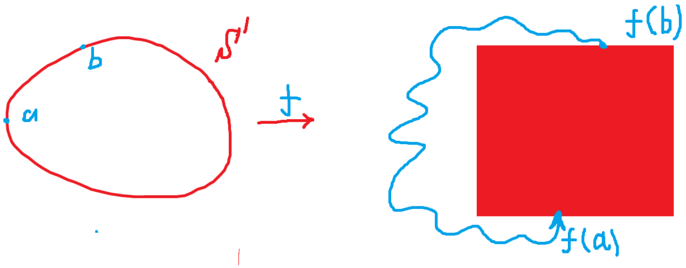


Figure a and b are topological f-neighbors under continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$

Formal definitions of f-neighbors of different types

Definition

Let $(\mathcal{M}, d)$ be a metric space, and let $f\colon \mathcal{M} \to \mathbb{R}^m$ be a continuous map. Distinct points a and b in $\mathcal{M}$ are called

1. *f-neighbors* if $f(a) = f(b)$;
2. *spherical f-neighbors* if either $f(a) = f(b)$ or there exists a Euclidean ball $B^m \subset \mathbb{R}^m$ such that $\{f(a), f(b)\} \subset \partial B^m$ and $f(\mathcal{M}) \cap B^m = \emptyset$;
3. *visual f-neighbors* if either $f(a) = f(b)$ or the segment with endpoints $f(a)$ and $f(b)$ intersects $f(\mathcal{M})$ only at these two points;
4. *topological f-neighbors* if either $f(a) = f(b)$ or some path in $\mathbb{R}^m$ with endpoints at $f(a)$ and $f(b)$ intersects $f(\mathcal{M})$ only at these two points.

How to 'measure' collections of f-neighbors of various types?

Remark

The Borsuk-Ulam theorem in its usual setting gives f-neighbors a and b in $\mathbb{S}^n$ with $d(a,b)=\pi$, while the Hopf theorem states that there are f-neighbors a and b with any $d(a,b)\in(0,\pi]$ (for the case of continuous maps of the Euclidean sphere).

To 'measure' collections of f-neighbors of various types we will use the sets of distances that are realized as distances between f-neighbors.

How to 'measure' collections of f-neighbors of various types?

Definition

We introduce the following notation:

1. $\Omega_f = \{d(a,b) \in \mathbb{R} \mid a \text{ and } b \text{ in } \mathcal{M} \text{ are } f\text{-neighbors}\}$;
2. $\Omega_f^{sph} = \{d(a,b) \in \mathbb{R} \mid a \text{ and } b \text{ in } \mathcal{M} \text{ are spherical } f\text{-neighbors}\}$;
3. $\Omega_f^{vis} = \{d(a,b) \in \mathbb{R} \mid a \text{ and } b \text{ in } \mathcal{M} \text{ are visual } f\text{-neighbors}\}$;
4. $\Omega_f^{top} = \{d(a,b) \in \mathbb{R} \mid a \text{ and } b \text{ in } \mathcal{M} \text{ are topological } f\text{-neighbors}\}$.

Then $\Omega_f \subseteq \Omega_f^{sph} \subseteq \Omega_f^{vis} \subseteq \Omega_f^{top}$. The Hopf theorem implies that, in its settings each of the sets Ω_f, Ω_f^{sph}, Ω_f^{vis}, and Ω_f^{top} is continual and contains an interval adjacent to 0.

Problem formulation

Theorem (A. V. Malyutin, O. R. Musin, 2021)

Let m and n be positive integers such that $n < m$, let $\mathbb{S}^n$ be a Euclidean unit n-sphere in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$, and let $\mathbb{S}^n \to \mathbb{R}^m$ be a continuous map. Then there are spherical f-neighbors x and y in $\mathbb{S}^n$ such that the Euclidean distance $|x-y|$ is at least $\sqrt{(n+2)/n}$.

Problem

Investigate non-trivial properties of the sets $\Omega_f^{sph} \subseteq \Omega_f^{vis} \subseteq \Omega_f^{top}$ for the case of continuous maps $f : \mathbb{M}^n \to \mathbb{R}^m$ with $m > n$.

Negative results

Proposition (Example 1)

Let $\mathbb{S}^1$ be a Euclidean circle in $\mathbb{R}^2$, and let d be the angular distance on $\mathbb{S}^1$. Then for any prescribed $\varepsilon \in (0, \frac{\pi}{3})$, there exists a continuous map $f_\varepsilon \colon \mathbb{S}^1 \to \mathbb{R}^2$ such that $\Omega^{vis}_{f_\varepsilon} = (0, \varepsilon] \cup [\frac{2\pi}{3} - \varepsilon, \pi]$.

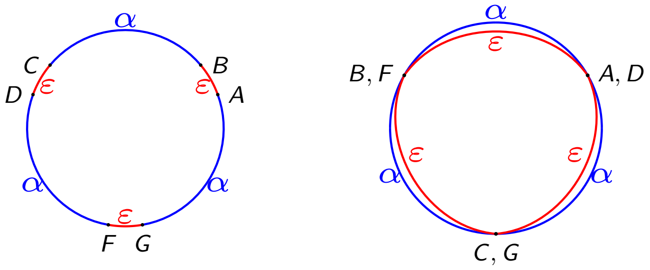
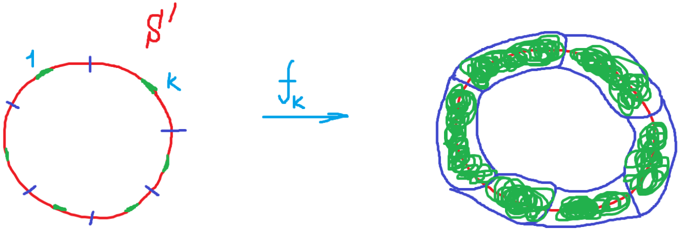


Figure $\mathbb{S}^1$ left and $f_\varepsilon(\mathbb{S}^1)$ right

Negative results

Proposition (Example 2)

Let n be a positive integer, let $\mathbb{S}^n$ be a Euclidean n-sphere in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$, and let d be the angular distance on $\mathbb{S}^n$. Then there exists a sequence of continuous maps $(f_k\colon \mathbb{S}^n \to \mathbb{R}^{n+1})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \mu(\Omega_{f_k}^{top}) = 0$, where μ is the Lebesgue measure on $\mathbb{R}$.



Negative results

Proposition (Example 3. Torus knot diagrams)

There exists a sequence of continuous maps $f_n : \mathbb{S}^1 \to \mathbb{R}^2$ *such that* $\lim_{n\to\infty} \mu(\Omega^{top}_{f_n}) = 0$.

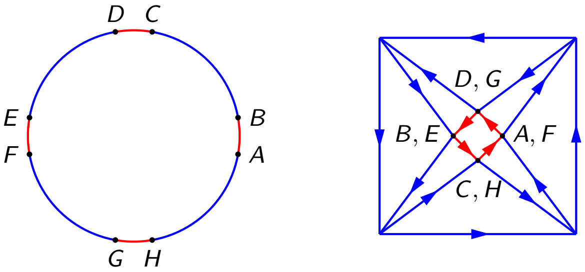


Figure 3. Example of f_4: $\mathbb{S}^1$ left and $f_\alpha(\mathbb{S}^1)$ right, which is a (3,4)-torus knot diagram.

Example 3. Proof by „coloring“

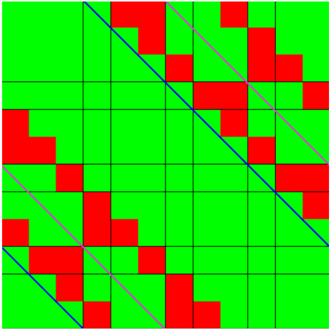


Figure 4. The set of pairs of points $S^1 \times S^1 = T^2$. Color a pair (a,b) red if a and b are f-neighbors of corresponding type and green otherwise.

Positive results

Theorem

Let M be a compact Riemannian manifold, let $d\colon M \times M \to [0,\infty)$ be any intrinsic metric on M compatible with its topology, and let $f\colon M \to \mathbb{R}^m$ be a continuous map. Then Ω_f^{vis} for (M,d) is infinite.

Theorem

Let S^1 be a topological circle, let d be an intrinsic metric on S^1 compatible with its topology, and let $f\colon S^1 \to \mathbb{R}^2$ be a continuous map. Then Ω_f^{sph} for (S^1,d) is infinite.

Positive results: quantitative generalization of the Hopf theorem

Definition

Let n be a positive integer, and let δ be a positive real number. Let M be a compact Riemannian manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a continuous map. We denote by $\mathcal{F}(\delta)$ the subset of $M \times M$ such that $\{a,b\} \in \mathcal{F}(\delta)$ if and only if $f(a) = f(b)$ and the points a and b are joined by a geodesic of length δ.

Definition

We call points $a, b \in M$ *δ-conjugate* if they are joined by an infinite number of geodesics of length δ. We denote the set of such points by $\mathcal{C}(\delta)$.

Positive results: quantitative generalization of the Hopf theorem

Theorem

Let n be a positive integer such that $n > 1$, and let δ be a positive real number. Let M be a compact Riemannian manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a continuous map. If $\mathcal{C}(\delta)$ is empty, then $\mathcal{F}(\delta)$ is uncountable.

Open problems

Problem

Let $f : \mathbb{S}^1 \to I^2$ be a continuous surjective map on the unit square $I^2 \subset \mathbb{R}^2$. Does Ω_f is uncountable?

Problem

Let $\mathbb{S}^2$ be a Euclidean sphere and let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be a continuous map. Does $\mathcal{F}(\delta)$ contains a loop for every $\delta \in (0, \pi)$?