Connected components in the Prym eigenform loci in genus 5

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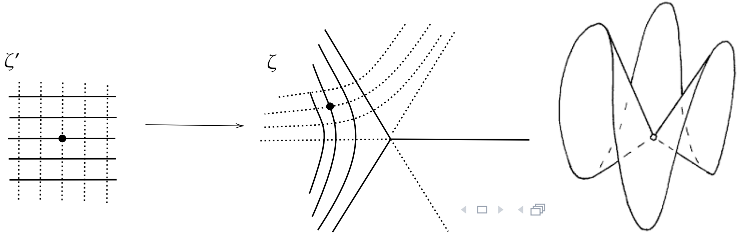
Flat surfaces

- An abelian differential ω — a holomorphic 1-form on a compact Riemann surface X.
- $\omega \not\equiv 0$, in some local coordinate $\omega_z = z^k dz$
- For Σ— the set of zeroes of ω, $X - \Sigma$ admits an atlas of charts to $\mathbb{C}$ whose transition maps are translations.
- $X - \Sigma$ admits a structure of a flat manifold, since translations preserve the standard flat (Euclidean) metric on $\mathbb{C}$.
- the flat metric near $p \in \Sigma$ is the pull back of the flat metric on $\mathbb{C}$ under the map $z \to z^k + 1$



Flat surfaces

- H_g — the space of all flat surfaces of genus g; $H(\kappa)$ — strata by degrees of zeroes, $\kappa \vdash 2g-2$
- A **saddle connection** is a geodesic for the flat metric(straight line) joining two singularities
- maximal collection of saddle connections triangulates $(X, \omega) \Rightarrow$ points in $H(\kappa_1, \ldots, \kappa_k)$ correspond to polygons in the complex plane:
  - even number of sides, split in parallel pairs, which are identified
  - total angles after identification $2\pi(\kappa_i + 1)$

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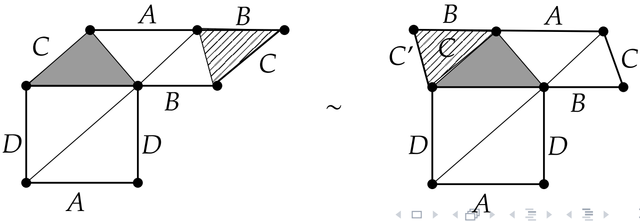
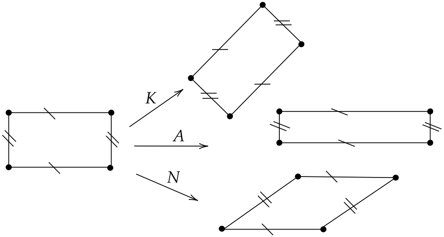
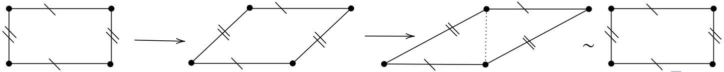


$SL_2(\mathbb{R})$ action up to cut&paste equivalence

Iwasawa decomposition: $\forall M \in SL_2\mathbb{R}\ \exists K, A, N \in SL_2\mathbb{R}$ s.t. $M = KAN$: K=$\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$, A=$\left[\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix}\right]$, N=$\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$. The $SL_2(\mathbb{R})$ acts on polygons on the complex plane, while identifications and the number of vertices is preserved:



Cut and paste equivalence:



Affine invariant submanifolds

Theorem(Eskin-Mirzakhani-Mohammadi)

Any closed $SL_2(\mathbb{R})$-invariant set is a finite union of affine-invariant submanifolds. Affine-invariant submanifolds are $SL_2(\mathbb{R})$-invariant.

- M – an open connected manifold, $f\colon M \to H(\kappa)$ – a proper immersion
- An **affine invariant submanifold** is the image $f(M)(\kappa)$ s.t. $\forall p \in M$ $\exists U(p)$ with $f(U)$ determined by real linear equations in period coordinates and constant term 0.
- Affine invariant submanifolds have dimension at least 2.

Prym variety

- X — a closed Riemann surface of genus g
- $\tau : X \to X$ — an involution of X, $\tau^2 = \mathrm{id}$
- $\Omega(X)$ is the space of holomorphic 1-forms on X
- $\Omega(X,\tau)^- = ker(\tau + \mathrm{id}) \subset \Omega(X)$
- $H_1^-(X,\mathbb{Z})$ is the anti-invariant homology of X with respect to τ
- **Prym variety** of (X,τ) is defined as:

$$Prym(X,\tau) = \Omega(X,\tau)^{-*}/H_1^-(X,\mathbb{Z})$$

Prym eigenforms

- **Quadratic order**: $O_D \simeq \mathbb{Z}[x]/(x^2+bx+c)$, the discriminant of the order is defined by $D = b^2 - 4c$
- **Real multiplication** by O_D if:
  - $\exists i : O_D \to \mathrm{End}(Prym(X,\tau))$ injective
  - $i(O_D) \subset \mathrm{End}(Prym)$ proper (if $f \in \mathrm{End}(A)$ and $\exists n \in \mathbb{N}^*$ s.t. $nf \in i(O_D) \to f \in i(O_D)$))
  - $i(O_D) \subset End(Prym)$ self-adjoint
- **Prym eigenform**: (X,ω) which admits an involution $\tau : X \to X$ s.t.:
  - $Prym(X,\tau)$ admits a real multiplication by some O_D
  - $\omega \in \Omega(X,\tau)^-$ is an eigen-vector of O_D

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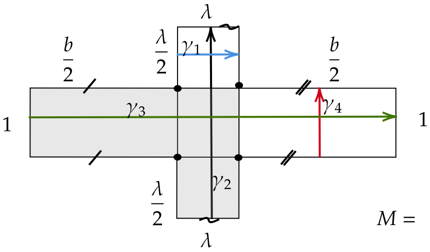
Prym eigenform loci

- $i(O_D)$ is generated by 1 element, say M
- For (X, ω, τ) choose $\gamma_1, \dots, \gamma_g \in H_1(X, \mathbb{Z})$, s.t.$\{\gamma_i, M^*\gamma_i\}$ is a basis of $H_1(X, \mathbb{Z})$, then $\int_{M^*\gamma_i} \omega = \sqrt{D} \int_{\gamma_i} \omega$, $\int_{\tau_*\gamma_i} \omega + \int_{\gamma_i} \omega = 0$
- The problem is to describe flat surfaces, which admit τ as above and there exists and integer 4×4 matrix (M), s.t.:
  - M is self-adjoint w.r.t to the intersection form on $\{\gamma\}$.
  - $M(\int_{\gamma_i} \omega) = \sqrt{D}(\int_{\gamma_i} \omega)$



$$\omega(\gamma_1) = \lambda$$
$$\omega(\gamma_2) = i(\lambda + 1)$$
$$\omega(\gamma_3) = b$$
$$\omega(\gamma_4) = i$$

$$M = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{array} \begin{array}{c} \begin{array}{cccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Prym eigenform loci

- ΩE_D — the locus of Prym eigenforms (X, ω) in $H(\kappa)$ for fixed $D \equiv 0$ or $1 \mod 4$.

Theorem (McMullen)

The locus $\Omega_D E \subset H_g$ is a closed $SL_2(\mathbb{R})$-invariant submanifold

- How many connected components are there in

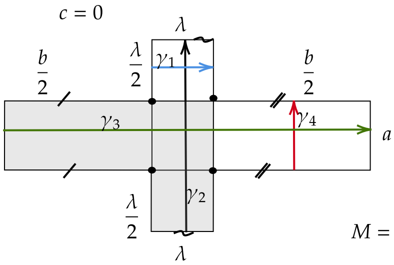
$$\Omega E_D(\kappa) = \Omega_D E \cap H(\kappa)?$$

Genus 2 with 1 zero: $H(2)$

Proposition(McMullen)

Surfaces in $\Omega E_D(2)$ are characterised by tuples (a, b, c, e), s.t. $gcd(a, b, c, e) = 1,\ a, b, c > 0,\ c + e < b, 0 \leq a < gcd(b, c),\ D = e^2 + 4bc.$

Set $\lambda = (e + \sqrt{e^2 + 4b}/2)$:



$$\omega(\gamma_1) = \lambda$$
$$\omega(\gamma_2) = i(\lambda + a) + c$$
$$\omega(\gamma_3) = b$$
$$\omega(\gamma_4) = ia + c$$

$$M = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{array} \begin{array}{c} \begin{array}{cccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

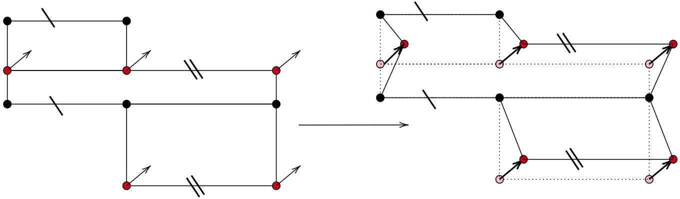
Connected components of $\Omega E_D(\kappa)$

- Prym eigenform loci only exist up to genus 5
- the results for smaller genera the works of C.Mcmullen, E.Lanneau, D.Nguyen and our result refers to $\Omega E_D(4,4)$:

Strata	g	# connected components
$\Omega E_D(1,1)$	2	1, nonempty
$\Omega E_D(2)$	2	1 or 2 , $D \neq 4$
$\Omega E_D(2,2)^{odd}$	3	1 or 2, $D \equiv 0, 1, 4 \bmod 8 (*)$
$\Omega E_D(1,1,2)$	3	
$\Omega E_D(4)$	3	1, $D = 8, 12$, for $D > 17$ $(*)$
$\Omega E_D(6)$	4	1, $D \neq 4, 9$
$\boldsymbol{\Omega E_D(4,4)}$	**5**	**1, nonempty**

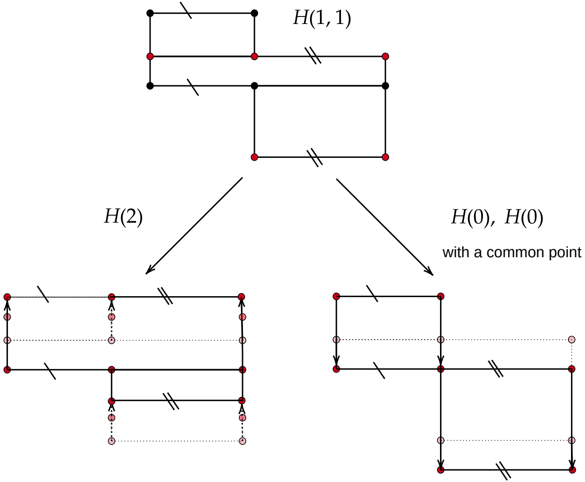
Isoperiodic deformations

- Saddle connections and their unions, that connect only 1 zero compose a lattice of **absolute periods** for each zero
- Saddle connections joining distinct zeroes- **absolute periods**
- Shifting only one absolute period lattice by a small vector v, adding v to corresponding relative periods - **isoperiodic deformation**



Collapsing singularitites

- the result may be of lower genus with several points identified



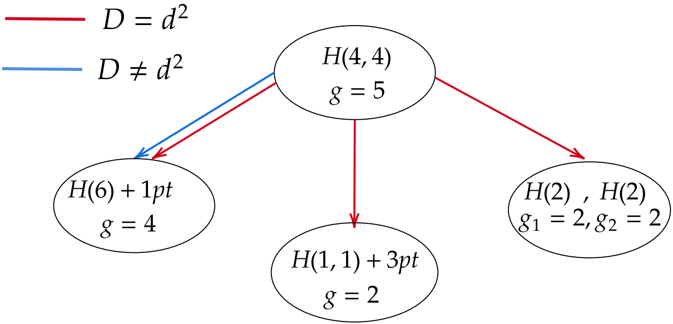
Prym eigenforms in genus 5

- How many connected components are there in $\Omega E_D(4,4)$?
- The locus $\Omega E_D(8)$ is empty for any D
- Collapsing a saddle connection we necessarily obtain a surface of smaller genus

Theorem

$\Omega E_D(4,4)$ is connected for every $D \equiv 0, 1 \mod 4$, $D \geq 4$

Results of horizontal collapsing



Strategy of the proof

- The three cases produce three families of genus 5 surfaces
- Surfaces within a family are connected by isoperiodic moves
- Surfaces from different families are connected by isoperiodic moves of a different kind
- In surfaces for which collapsing the relative periods results in a genus 4 we show that the $SL_2(\mathbb{R})$ in $\Omega E_D(6)$ action lifts to $\Omega E_D(4,4)$. Since the loci is connected the result forllows for $D \neq 6, 9$
- For $D = 6, 9$ we apply similar logic,in relation to $\Omega E_D(1,1)$, which are connected for $D \geq 4$, hence the result follows.