Composition operators in generalized holomorphic Hölder spaces

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1. Modulus of continuity and Zygmund type conditions

Definition 1.1. *A function $\omega : [0,2] \to \mathbb{R}_+$ is called modulus of continuity if*

(1) ω *is cont. in a neighborhood of the origin and* $\omega(0) = 0$,
(2) ω *is almost increasing and bounded on* $[0,2]$,
(3) $\frac{\omega(h)}{h}$ *is almost decreasing on* $[0,2]$.

Definition 1.2. *Let $\omega : [0,2] \to \mathbb{R}_+$ be a bounded measurable function such that $\omega(0) = 0$. The function ω is called either Dini or b_1 - weight if there exists $C > 0$ such that*

$$\int_0^t \frac{\omega(s)}{s} ds \leqslant C\omega(t), \quad 0 < t \leqslant 2, \text{ (Dini condition)}, \tag{1.1}$$

$$\int_t^2 \frac{\omega(s)}{s^2} ds \leqslant C\frac{\omega(t)}{t}, \quad 0 < t \leqslant 2, \ (b_1- \text{ condition}). \tag{1.2}$$

respectively, where C does not depend on t.

As typical example of modulus of continuity ω, which satisfies the conditions (1.1) and (1.2), one can take $\omega(h) = h^\lambda \ln^{-\beta} \frac{e}{h}$, where $\lambda \in (0,1), \beta \in \mathbb{R}$ while the function $\omega(t) = \ln^{-\beta} \frac{e}{t}$, $\beta > 1$ satisfies the b_1-weight condition, but does not satisfy the Dini condition.

Remark 1.1. *Observe that for modulus of continuity the Dini condition guarantees that $\omega(t)/t \in L^1([0,2])$ and that $\int_0^t \frac{\omega(s)}{s} ds \approx \omega(t)$, $0 < t \leqslant 2$. Similarly if ω is a modulus of continuity then ω satisfies the b_1 condition iff $t\int_t^2 \frac{\omega(s)}{s} ds \approx \omega(t)$ for $0 < t \leqslant 1$.*

2. Auxiliary weights (modulus of continuity)

Definition 2.1. *For a function ω such that $\omega(t)/t \in L^1([0,2])$ we define*

$$W(t) := W_\omega(t) = \int_0^t \frac{\omega(s)}{s} ds, \quad t \in [0,2]. \tag{2.1}$$

Proposition 2.1. *Let ω be a modulus of continuity such that $\omega(t)/t \in L^1([0,2])$. Then W_ω is a modulus of continuity such that*

$$\omega(t) \leqslant C W_\omega(t), \quad 0 < t \leqslant 2, \tag{2.2}$$

where $C > 0$ is a constant.

Moreover $\omega \approx W_\omega$ iff ω is a Dini weight.

Definition 2.2. *Let $\omega : [0,2] \to \mathbb{R}_+$ be a bounded measurable function. Define*

$$U(t) := U_\omega(t) = t \int_{\min\{t,1\}}^2 \frac{\omega(s)}{s^2} ds, \quad t \in (0,2]. \tag{2.3}$$

Proposition 2.2. *Let ω be a modulus of continuity. Then U_ω is a modulus of continuity such that*

$$\omega(t) \leqslant C U_\omega(t), \quad 0 < t \leq 2, \tag{2.4}$$

where $C > 0$ is a constant.

Moreover $\omega \approx U_\omega$ iff ω is a b_1 weight.

3. Hölder type spaces of holomorphic functions

Let $\omega : [0,2] \to \mathbb{R}_+$ be a modulus of continuity. By $L^\omega(\mathbb{D})$ we denote the space of measurable functions defined in $\mathbb{D}$ such that

$$|f(z) - f(w)| \le C\omega(|z-w|), \quad z, w \in \mathbb{D}. \tag{3.1}$$

The semi-norm and norm of $f \in L^\omega(\mathbb{D})$ are given by

$$\|f\|_{\#,L^\omega(\mathbb{D})} = \sup_{z,w\in\mathbb{D}} \frac{|f(z)-f(w)|}{\omega(|z-w|)}, \ \|f\|_{L^\omega(\mathbb{D})} = \|f\|_{\#,L^\omega(\mathbb{D})} + \|f\|_{L^\infty(\mathbb{D})}.$$

The generalized Hölder type space of holomorphic functions in the unit disc with prescribed modulus of continuity:

$$A^\omega(\mathbb{D}) = L^\omega(\mathbb{D}) \cap H(\mathbb{D}).$$

We use the notation $\|f\|_{\#,A^\omega(\mathbb{D})} = \|f\|_{\#,L^\omega(\mathbb{D})}$.

By $B^\omega(\mathbb{D})$ denote the space of functions holomorphic in $\mathbb{D}$ such that

$$|f'(z)| \le C\frac{\omega(1-|z|)}{1-|z|}, \quad z \in \mathbb{D},$$

where C does not depend on z. The semi-norm and norm of a function $f \in B^\omega(\mathbb{D})$ are given by

$$\|f\|_{\#,B^\omega(\mathbb{D})} = \sup_{z\in\mathbb{D}} |f'(z)|\frac{1-|z|}{\omega(1-|z|)}, \ \|f\|_{B^\omega(\mathbb{D})} = \|f\|_{\#,B^\omega(\mathbb{D})} + \|f\|_{L^\infty(\mathbb{D})}.$$

Let us write $A^\omega(\mathbb{T})$ for the space of functions continuous in $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$ such that

$$|f(\xi) - f(\eta)| \leqslant C\omega(|\xi - \eta|), \ \xi, \eta \in \mathbb{T},$$

with the seminorm and norm given by

$$\|f\|_{\#,A^\omega(\mathbb{T})} = \sup_{\xi,\eta\in\mathbb{T}} \frac{|f(\xi)-f(\eta)|}{\omega(|\xi-\eta|)}, \quad \|f\|_{A^\omega(\mathbb{T})} = \|f\|_{\#,A^\omega(\mathbb{T})} + \|f\|_{L^\infty(\mathbb{T})}.$$

Of course, we have

$$A^\omega(\mathbb{D}) \subseteq A^\omega(\mathbb{T}). \tag{3.2}$$

4. Relation between $A^\omega(\mathbb{D})$, $B^\omega(\mathbb{D})$ and $A^\omega(\mathbb{T})$.

Theorem 4.1. *Let ω be a modulus of continuity. Then*

$$A^\omega(\mathbb{D}) \subseteq A^\omega(\mathbb{T}) \subseteq A^U(\mathbb{D}),$$

where $U = U_\omega$ is given by (2.3). In particular if ω is a b_1 weight then $A^\omega(\mathbb{D}) = A^\omega(\mathbb{T}) = A^U(\mathbb{D})$.

Theorem 4.2. *Let ω be a modulus of continuity. Then*

$$A^\omega(\mathbb{T}) \subseteq B^U(\mathbb{D}).$$

If in addition $\omega(t)/t \in L^1([0,2])$, then

$$B^\omega(\mathbb{D}) \subseteq A^W(\mathbb{T}).$$

In particular, if ω satisfies Dini condition then $B^\omega(\mathbb{D}) \subseteq A^\omega(\mathbb{T})$ and if ω satisfies b_1 condition then $A^\omega(\mathbb{T}) \subseteq B^\omega(\mathbb{D})$.

More generally:

Proposition 4.1. *Let ω_1 and ω_2 be modulus of continuity. The following statements hold.*

(1) *If there exists $C > 0$ such that*

$$\int_0^t \frac{\omega_1(s)}{s} ds \leqslant C\omega_2(t), \quad 0 < t < 2,$$

then $B^{\omega_1}(\mathbb{D}) \subseteq A^{\omega_2}(\mathbb{D})$.

(2) *If there exists $C > 0$ such that*

$$\int_t^2 \frac{\omega_1(s)}{s^2} ds \leqslant C\frac{\omega_2(t)}{t}, \quad 0 < t < 2,$$

then $A^{\omega_1}(\mathbb{D}) \subseteq B^{\omega_2}(\mathbb{D})$.

5. Constructions of functions in $A^\omega(\mathbb{D})$ and $B^\omega(\mathbb{D})$

Here we provide some characterizations of Dini and b_1 weights ω when they are also assumed to be modulus of continuity.

Definition 5.1. *Let ω be a modulus of continuity and $a \in \mathbb{C}$ with $|a| = 1$. Let us define*

$$\widetilde{\omega}(j) = \int_0^1 \omega(1-t)t^j dt, \quad j \in \mathbb{N} \cup \{0\}. \tag{5.1}$$

and

$$H_a^\omega(z) = z\int_0^1 \frac{\omega(1-t)}{1-taz}dt = \sum_{j=0}^{\infty} \widetilde{\omega}(j)a^j z^{j+1}, \quad z \in \mathbb{D}. \tag{5.2}$$

Proposition 5.1. *Let $|a| = 1$ and ω be a modulus of continuity such that $\omega(t)/t \in L^1([0,2])$.*
Then

$$H_a^\omega \in B^{U_\omega}(\mathbb{D}).$$

Moreover $\|H_a^\omega\|_{B^{U_\omega}(\mathbb{D})} = \|H_1^\omega\|_{B^{U_\omega}(\mathbb{D})}$ for any $|a| = 1$.

Proposition 5.2. *Let ω be a modulus of continuity with $\omega(t)/t \in L^1([0,2])$. The following statements are equivalent.*
(i) ω is a b_1-weight.
(ii) $H_a^\omega \in B^\omega(\mathbb{D})$.

6. Composition operators: boundedness

Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. We denote by C_ϕ the linear composition operator

$$C_\phi f(z) := f \circ \phi(z) = f(\phi(z)), \quad z \in \mathbb{D}.$$

Let us denote:

$$\kappa_{\omega_1,\omega_2,\phi} \equiv \sup_{z\in\mathbb{D}} \left\{ |\phi'(z)| \frac{\omega_1(1-|\phi(z)|)}{1-|\phi(z)|} \frac{1-|z|}{\omega_2(1-|z|)} \right\}. \tag{6.1}$$

In the particular case that $\|\phi\|_{L^\infty(\mathbb{D})} = \alpha < 1$ then

$$C^{-1}\omega_1(1-\alpha) \le \frac{\omega_1(1-|\phi(z)|)}{1-|\phi(z)|} \le C \frac{\omega_1(1-\alpha)}{1-\alpha}$$

and the quantity $\kappa_{\omega_1,\omega_2,\phi} \approx \kappa_{\omega_2,\phi}$ where

$$\kappa_{\omega_2,\phi} \equiv \sup_{z\in\mathbb{D}} \left\{ |\phi'(z)| \frac{1-|z|}{\omega_2(1-|z|)} \right\}. \tag{6.2}$$

Note that for a holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ we have

$$\kappa_{\omega_2,\phi} < \infty \Leftrightarrow \phi \in B^{\omega_2}(\mathbb{D}).$$

Depending on a context, in some places below instead of writing $\kappa_{\omega_2,\phi} < \infty$ we will use $\phi \in B^{\omega_2}(\mathbb{D})$.

Proposition 6.1. *Let ω_1, and ω_2 be modulus of continuity and let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} < 1$. Then $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ (respectively $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$) is bounded iff $\phi \in B^{\omega_2}(\mathbb{D})$ (respectively $\phi \in A^{\omega_2}(\mathbb{D})$.)*

7. Composition operators: boundedness

Theorem 7.1. *Let ω_1, and ω_2 be modulus of continuity and let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic function. The following statements hold.*

1. *If $\kappa_{\omega_1,\omega_2,\phi} < \infty$ then $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is bounded.*
2. *Assume that ω_1 is a b_1-weight. If the operator $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is bounded then $\kappa_{\omega_1,\omega_2,\phi} < \infty$.*

Here, as we defined before,

$$\kappa_{\omega_1,\omega_2,\phi} \equiv \sup_{z\in\mathbb{D}} \left\{ |\phi'(z)| \frac{\omega_1(1-|\phi(z)|)}{1-|\phi(z)|} \frac{1-|z|}{\omega_2(1-|z|)} \right\}. \tag{7.1}$$

Theorem 7.2. *Let ω_1 and ω_2 be modulus of continuity and $\phi : \mathbb{D} \to \mathbb{D}$ holomorphic. Assume that ω_2 is a Dini weight. If*

$$\sup_{z\in\mathbb{D}} \left\{ |\phi'(z)| \int_{1-|\phi(z)|}^{2} \frac{\omega_1(t)}{t^2} dt \, \frac{1-|z|}{\omega_2(1-|z|)} \right\} < \infty, \tag{7.2}$$

then the operator $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is bounded.

Theorem 7.3. *Let ω_1 and ω_2 be modulus of continuity and $\phi : \mathbb{D} \to \mathbb{D}$ holomorphic. Assume that ω_1 satisfies Dini and b_1-weight conditions. If $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is bounded, then*

$$\sup_{z\in\mathbb{D}} \left\{ |\phi'(z)| \frac{\omega_1(1-|\phi(z)|)}{1-|\phi(z)|} \frac{1}{\int_{1-|z|}^{2} \frac{\omega_2(t)}{t^2} dt} \right\} < \infty. \tag{7.3}$$

8. Composition operators: boundedness

In view of Schwarz-Pick Lemma we outline two corollaries.

Corollary 8.1. *Let ω_1 and ω_2 be modulus of continuity and $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic. If*

$$\sup_{z\in\mathbb{D}} \left\{ \frac{\omega_1(1-|\phi(z)|)}{\omega_2(1-|z|)} \right\} < \infty, \tag{8.1}$$

then the operator $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is bounded.

Corollary 8.2. *Let ω_1 and ω_2 be modulus of continuity and $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Assume that ω_2 is a Dini weight. If*

$$\sup_{z\in\mathbb{D}} \left\{ \frac{1-|\phi(z)|}{\omega_2(1-|z|)} \int_{1-|\phi(z)|}^{2} \frac{\omega_1(t)}{t^2} dt \right\} < \infty \tag{8.2}$$

then $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is bounded.

We conclude this slide with one related example.

$$\omega(h) = \omega_{\lambda,\beta}(h) = h^\lambda \ln^\beta \frac{e}{h}, \quad \text{where } \lambda \in (0,1) \text{ and } \beta \in \mathbb{R}.$$

Theorem 8.1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$ such that $\sup_{z\in\mathbb{D}} |\phi(z)| = 1$. Let $\omega_{\lambda,\beta}(h) = h^\lambda \ln^\beta \frac{e}{h}$, where $\lambda \in (0,1)$ and $\beta \in \mathbb{R}$. If C_ϕ is bounded from $B^{\omega_{\lambda,\beta}}(\mathbb{D})$ to $B^{\omega_{\lambda,\beta}}(\mathbb{D})$ (equivalently from $A^{\omega_{\lambda,\beta}}(\mathbb{D})$ to $A^{\omega_{\lambda,\beta}}(\mathbb{D})$) then*

$$\kappa_{\omega_{\lambda,\beta},\phi} \equiv \sup_{z\in\mathbb{D}} \left\{ |\phi'(z)| \frac{1-|z|}{\omega_{\lambda,\beta}(1-|z|)} \frac{\omega_{\lambda,\beta}(1-|\phi(z)|)}{1-|\phi(z)|} \right\} < \infty. \tag{8.3}$$

9. Composition operators: compactness

We study compactness of C_ϕ as operator acting from $A^{\omega_1}(\mathbb{D})$ to $A^{\omega_2}(\mathbb{D})$, and from $B^{\omega_1}(\mathbb{D})$ to $B^{\omega_2}(\mathbb{D})$, as well.

Definition 9.1. *We say that a sequence* $\{f_n\}$ *of elements* f_n *in a Banach space* X *converges to* f *weakly in* X *if* $\lim_{n\to\infty} Lf_n = Lf$ *for every linear functional* L *on* X.

Definition 9.2. *We say that a bounded linear operator* $T : X \to Y$ *is w-compact if* $\|Tf_n\|_Y \to 0$ *as* $n \to \infty$ *whenever* $\{f_n\}$ *converges to* 0 *weakly in* X.

Theorem 9.1. *Let* ω_1 *and* ω_2 *be modulus of continuity. Let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function in* $\mathbb{D}$ *and* $\sup_{z\in\mathbb{D}} |\phi(z)| < 1$.
The following are equivalent:
(i) $\phi \in B^{\omega_2}(\mathbb{D})$.
(ii) $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ *is bounded.*
(iii) $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ *is w-compact.*

Theorem 9.2. *Let* ω_1 *and* ω_2 *be modulus of continuity. Let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function in* $\mathbb{D}$ *and* $\sup_{z\in\mathbb{D}} |\phi(z)| < 1$.
The following are equivalent:
(i) $\phi \in A^{\omega_2}(\mathbb{D})$.
(ii) $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ *is bounded.*
(iii) $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ *is w-compact.*

10. Composition operators: compactness

Let us now analyze the w-compactness for operators C_ϕ with $\|\phi\|_{L^\infty(\mathbb{D})} = 1$. Denote

$$k^0_{\omega_1,\omega_2,\phi} \equiv \limsup_{|z|\to 1^-} |\phi'(z)| \frac{\omega_1(1-|\phi(z)|)}{1-|\phi(z)|} \frac{1-|z|}{\omega_2(1-|z|)}. \tag{10.1}$$

Theorem 10.1. *Let ω_1 and ω_2 be modulus of continuity. Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} = 1$. The following statements hold:*

1. *Assume that $\omega_2(t)/t \in L^1([0,2])$. If $k^0_{\omega_1,\omega_2,\phi} = 0$ then the operator $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is w-compact.*
2. *Assume that ω_1 satisfies b_1 condition and ω_2 satisfies the Dini condition. If $k^0_{\omega_1,\omega_2,\phi} = 0$ then the operator $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is w-compact.*

Remark 1. *There are compact operators for which*

$$k^0_{\omega_1,\omega_2,\phi} \neq 0.$$

We prove this for the case of the spaces $B^\omega(\mathbb{D})$ leaving similar result for $A^\omega(\mathbb{D})$ to an interested reader. It suffices to take a function ϕ with $\|\phi\|_{L^\infty(\mathbb{D})} < 1$, $\phi \in B^{\omega_2}(\mathbb{D})$ and $k^0_{\omega_1,\omega_2,\phi} \neq 0$.

Theorem 10.2. *Let ω_1 and ω_2 be modulus of continuity with ω_2 satisfying b_1- condition. There exist a holomorphic function $\phi = \phi_{\omega_2} : \mathbb{D} \to \mathbb{D}$ such that $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is w-compact and $k^0_{\omega_1,\omega_2,\phi} \neq 0$.*

We will use the function constructed in Proposition 5.1. Denote

$$\phi_{\omega_2}(z) = \delta_0 H_1^{\omega_2}(z) = \delta_0 z \int_0^1 \frac{\omega_2(1-t)}{1-tz} dt, \quad z \in \mathbb{D},$$

where δ_0 is such that $\sup_{z\in\mathbb{D}} |\phi_{\omega_2}(z)| < 1$.

11. Boundedness: further characterizations

Definition 11.1. *Let ω be a modulus of continuity and let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$. We define*

$$f_{\phi,\omega}(z) = |\phi(z)| \int_0^1 \frac{\omega(1-t)dt}{1-t|\phi(z)|}, \quad z \in \mathbb{D}, \tag{11.1}$$

Straightforward calculus shows that

$$|\nabla f_{\phi,\omega}(z)| = |\phi'(z)| \int_0^1 \frac{\omega(1-t)}{(1-t|\phi(z)|)^2} dt, \quad z \in \mathbb{D}. \tag{11.2}$$

Definition 11.2. *Let ω be a modulus of continuity. We write $DL^{\omega}(\mathbb{D})$ for the space of functions $f : \mathbb{D} \to \mathbb{C}$ which are differentiable in $\mathbb{D}$ and there exists a constant $C > 0$ such that*

$$|\nabla f(z)| \le C \frac{\omega(1-|z|)}{1-|z|}, \quad z \in \mathbb{D}. \tag{11.3}$$

Theorem 11.1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$. Let ω_1 and ω_2 be modulus of continuity. Then the following statements hold:*

1. *If $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$ then the operator $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is bounded.*
2. *Assume that ω_1 satisfies (b1) and $\omega_1(t)/t \in L^1(0,2)$. If the operator $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is bounded then $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$.*

Theorem 11.2. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$. Let ω_1 and ω_2 be modulus of continuity. Then the following statements hold:*

1. *Assume ω_2 satisfies the condition (Dini). If $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$ then the operator $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is bounded.*
2. *Assume that ω_1 satisfies (Dini) and (b1). If $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is bounded then $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$.*

12. Boundedness: derivative-free characterization

Theorem 12.1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$. Let ω_1 and ω_2 be modulus of continuity. Then the following assertions are true:*

(1) *If $f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{D})$ then $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$.*
(2) *Let ω_2 satisfies the condition (Dini). If $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$ then $f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{D})$.*

Theorem 12.2. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$, ω_1 be a modulus of continuity satisfying (b1) and $\omega_1(t)/t \in L^1(0, 2)$, and ω_2 be modulus of continuity satisfying (Dini). Then the following statements are equivalent:*

(1) *$f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$.*
(2) *$C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ is bounded.*
(3) *$f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{D})$.*

Theorem 12.3. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$, ω_1 be a modulus of continuity satisfying (Dini) and (b1) and ω_2 be modulus of continuity satisfying (Dini). Then the following statements are equivalent:*

(1) *$f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$.*
(2) *$C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ is bounded.*
(3) *$f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{D})$.*

13. Boundedness: further characterizations

Recall that

$$f_{\phi,\omega_1}(z) = |\phi(z)| \int_0^1 \frac{\omega(1-t)}{1-t|\phi(z)|}dt.$$

Collecting all the above results we can conclude the following list of characterizations.

Theorem 13.1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function in $\mathbb{D}$. Let ω_1 and ω_2 satisfy the conditions (Dini) and (b1). Then the following statements are equivalent:*

(1) $C_\phi : A^{\omega_1}(\mathbb{D}) \to A^{\omega_2}(\mathbb{D})$ *is bounded.*
(2) $C_\phi : B^{\omega_1}(\mathbb{D}) \to B^{\omega_2}(\mathbb{D})$ *is bounded.*
(3) $f_{\phi,\omega_1} \in DL^{\omega_2}(\mathbb{D})$.
(4) $f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{D})$.
(5) $f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{T})$ *and there exists* $C > 0$ *such that*

$$|f_{\phi,\omega_1}(\xi) - f_{\phi,\omega_1}(r\xi)| \le C\omega_2(1-r), \quad \xi \in \mathbb{T}, \quad 0 < r < 1.$$

(6) $f_{\phi,\omega_1} \in L^{\omega_2}(\mathbb{T})$ *and there exists* $C > 0$ *such that*

$$\left|f_{\phi,\omega_1}(z) - Pf_{\phi,\omega_1}(z)\right| \le C\omega_2(1-|z|), \quad z \in \mathbb{D}.$$

Here the Poisson integral of a function $f \in C(\mathbb{T})$ is defined by:

$$Pf(z) = \int_{\mathbb{T}} f(\tau)\frac{1-|z|^2}{|\tau - z|^2}d\tau, z \in \mathbb{D}, \text{ and } Pf(\tau) = f(\tau), \tau \in \mathbb{T}.$$

14. Composition operators on holomorphic variable exponent spaces

Let $\lambda : \mathbb{D} \to [0,1]$ be a continuous function. We say that λ satisfies the log-condition (log-Hölder condition) on $\mathbb{D}$ if

$$(14.1)\qquad |\lambda(z)-\lambda(w)| \le \frac{C}{\ln\frac{1}{|z-w|}}, \quad z,w\in\mathbb{D},\ |z-w|<\frac{1}{2},$$

where C is independent of z,w.

Let $\lambda \in \Lambda(\mathbb{D})$. By $L^{\lambda(\cdot)}(\mathbb{D})$, we denote the space of functions f measurable in $\mathbb{D}$ such that

$$|f(z)-f(w)| \le C|z-w|^{\lambda(z)}, \quad \text{for all} \quad z,w\in\mathbb{D},$$

or what is equivalent

$$|f(z)-f(w)| \le C|z-w|^{\lambda(w)}, \quad \text{for all} \quad z,w\in\mathbb{D},$$

where C is independent of z,w. The semi-norm and norm of a function $f\in L^{\lambda(\cdot)}(\mathbb{D})$ are given respectively by

$$\|f\|_{\#,L^{\lambda(\cdot)}(\mathbb{D})} = \sup_{z,w\in\mathbb{D}} \frac{|f(z)-f(w)|}{|z-w|^{\lambda(z)}},\ \|f\|_{L^{\lambda(\cdot)}(\mathbb{D})} = \|f\|_{\#,L^{\lambda(\cdot)}(\mathbb{D})}+\|f\|_{L^{\infty}(\mathbb{D}}$$

The variable exponent generalized Hölder spaces of holomorphic functions in $\mathbb{D}$, denoted as $A^{\lambda(\cdot)}(\mathbb{D})$, is the space of functions f from $L^{\lambda(\cdot)}(\mathbb{D})$, which are holomorphic in $\mathbb{D}$, with the notation $\|f\|_{\#,A^{\lambda(\cdot)}(\mathbb{D})} = \|f\|_{\#,L^{\lambda(\cdot)}(\mathbb{D})}$.

By $B^{\lambda(\cdot)}(\mathbb{D})$, we denote the space of functions f holomorphic in $\mathbb{D}$ such that

$$|f'(z)| \le C(1-|z|)^{\lambda(z)-1}, \quad z\in\mathbb{D},$$

where C is independent of z. The semi-norm and norm of a function $f\in B^{\lambda(\cdot)}(\mathbb{D})$ are given respectively by

$$\|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{D})} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^{1-\lambda(z)},\ \|f\|_{B^{\lambda(\cdot)}(\mathbb{D})} = \|f\|_{\#,B^{\lambda(\cdot)}(\mathbb{D})}+\|f\|$$

15. Composition operators on holomorphic variable exponent spaces

We also set

$$\kappa_{\lambda_1,\lambda_2,\phi} \equiv \sup_{z\in\mathbb{D}} \left\{ |\phi'(z)|(1-|\phi(z)|)^{\lambda_1(z)-1}(1-|z|)^{1-\lambda_2(z)} \right\}.$$

Notice that if $\|\phi\|_{L^\infty(\mathbb{D})} = \beta < 1$ then

$$\frac{1}{2} \leq (1-|\phi(z)|)^{\lambda_1(z)-1} \leq \frac{1}{1-\beta}, \quad z \in \mathbb{D},$$

and hence $\kappa_{\lambda_1,\lambda_2,\phi} \approx \kappa_{\lambda_2,\phi}$ where

$$\kappa_{\lambda,\phi} \equiv \sup_{z\in\mathbb{D}} \left\{ |\phi'(z)|(1-|z|)^{1-\lambda(z)} \right\}. \tag{15.1}$$

We now outline the following important fact.

Remark 15.1. *Let $\lambda \in \Lambda(\mathbb{D})$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} < 1$. Then*

$$\kappa_{\lambda,\phi} < +\infty \quad \textit{if and only if} \quad \phi \in B^{\lambda(\cdot)}(\mathbb{D}).$$

Theorem 15.1. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} < 1$. Then $C_\phi : B^{\lambda_1(\cdot)}(\mathbb{D}) \to B^{\lambda_2(\cdot)}(\mathbb{D})$ (respectively $C_\phi : A^{\lambda_1(\cdot)}(\mathbb{D}) \to A^{\lambda_2(\cdot)}(\mathbb{D})$) is bounded if and only if $\phi \in B^{\lambda_2(\cdot)}(\mathbb{D})$ (respectively $\phi \in A^{\lambda_2(\cdot)}(\mathbb{D})$).*

16. Composition operators on holomorphic variable exponent spaces

We note that the above result can be clearly refined in particular cases when either $\sup_{z\in\mathbb{D}}\lambda(z)<1$, or $0<\inf_{z\in\mathbb{D}}\lambda(z)$, or both these conditions are satisfied.

Recall that

$$\kappa_{\lambda_1,\lambda_2,\phi}\equiv\sup_{z\in\mathbb{D}}\left\{|\phi'(z)|(1-|\phi(z)|)^{\lambda_1(z)-1}(1-|z|)^{1-\lambda_2(z)}\right\}.$$

Theorem 16.1. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi:\mathbb{D}\to\mathbb{D}$ be holomorphic function. The following statements hold.*

(1) *If $\kappa_{\lambda_1,\lambda_2,\phi}<+\infty$ then $C_\phi:B^{\lambda_1(\cdot)}(\mathbb{D})\to B^{\lambda_2(\cdot)}(\mathbb{D})$ is bounded.*
(2) *If $C_\phi:B^{\lambda_1(\cdot)}(\mathbb{D})\to B^{\lambda_2(\cdot)}(\mathbb{D})$ is bounded then*

$$\sup_{z\in\mathbb{D}}\left\{|\phi'(z)|\frac{(1-|z|)^{1-\lambda_2(z)}}{(1-|\phi(z)|)^{1-\lambda_1^{',+}}}\right\}<+\infty, \tag{16.1}$$

where $\lambda_1^{',+}:=\sup_{\sigma\in\mathbb{T}}\lambda_1(\sigma)$.

Theorem 16.2. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi:\mathbb{D}\to\mathbb{D}$ be a holomorphic function. Assume that $\sup_{z\in\mathbb{D}}\lambda_1(z)<1$ and $\inf_{z\in\mathbb{D}}\lambda_2(z)>0$. If $\kappa_{\lambda_1,\lambda_2,\phi}<+\infty$ then $C_\phi:A^{\lambda_1(\cdot)}(\mathbb{D})\to A^{\lambda_2(\cdot)}(\mathbb{D})$ is bounded.*

Theorem 16.3. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi:\mathbb{D}\to\mathbb{D}$ be a holomorphic function. Assume that $\sup_{z\in\mathbb{D}}\lambda_2(z)<1$ and $\inf_{z\in\mathbb{D}}\lambda_1(z)>0$. If $C_\phi:A^{\lambda_1(\cdot)}(\mathbb{D})\to A^{\lambda_2(\cdot)}(\mathbb{D})$ is bounded, then the condition (16.1) holds.*

17. Composition operators on holomorphic variable exponent spaces

The next two corollaries follow by the Schwarz-Pick Lemma.

Corollary 17.1. *Let* λ_1 *and* λ_2 *belong to* $\Lambda(\mathbb{D})$, *and let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function. If*

$$\sup_{z\in\mathbb{D}} \left\{ \frac{(1-|\phi(z)|)^{\lambda_1(z)}}{(1-|z|)^{\lambda_2(z)}} \right\} < +\infty, \tag{17.1}$$

then $C_\phi : B^{\lambda_1(\cdot)}(\mathbb{D}) \to B^{\lambda_2(\cdot)}(\mathbb{D})$ *is bounded.*

Corollary 17.2. *Let* λ_1 *and* λ_2 *belong to* $\Lambda(\mathbb{D})$, *and let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function. Assume that* $\sup_{z\in\mathbb{D}} \lambda_1(z) < 1$ *and* $\inf_{z\in\mathbb{D}} \lambda_2(z) > 0$. *If the condition* (17.1) *holds then* $C_\phi : A^{\lambda_1(\cdot)}(\mathbb{D}) \to A^{\lambda_2(\cdot)}(\mathbb{D})$ *is bounded.*

18. The w-compactness of C_ϕ under the condition $\|\phi\|_{L^\infty(\mathbb{D})} < 1$.

Theorem 18.1. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} < 1$. Then the following statements are equivalent:*

(1) $\phi \in B^{\lambda_2(\cdot)}(\mathbb{D})$.
(2) $C_\phi : B^{\lambda_1(\cdot)}(\mathbb{D}) \to B^{\lambda_2(\cdot)}(\mathbb{D})$ *is bounded.*
(3) $C_\phi : B^{\lambda_1(\cdot)}(\mathbb{D}) \to B^{\lambda_2(\cdot)}(\mathbb{D})$ *is w-compact.*

Theorem 18.2. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} < 1$. The following statements are equivalent:*

(1) $\phi \in A^{\lambda_2(\cdot)}(\mathbb{D})$.
(2) $C_\phi : A^{\lambda_1(\cdot)}(\mathbb{D}) \to A^{\lambda_2(\cdot)}(\mathbb{D})$ *is bounded.*
(3) $C_\phi : A^{\lambda_1(\cdot)}(\mathbb{D}) \to A^{\lambda_2(\cdot)}(\mathbb{D})$ *is w-compact.*

19. The w-compactness of C_ϕ under the condition $\|\phi\|_{L^\infty(\mathbb{D})} = 1$.

Let

$$(19.1)\quad \gamma_{\lambda_1,\lambda_2,\phi} \equiv \limsup_{|z|\to 1^-} |\phi'(z)|(1-|\phi(z)|)^{\lambda_1(z)-1}(1-|z|)^{1-\lambda_2(z)}.$$

Theorem 19.1. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$, and let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic function with $\|\phi\|_{L^\infty(\mathbb{D})} = 1$. The following statements are true.*

1. *Suppose that $\inf_{z\in\mathbb{D}} \lambda_2(z) > 0$. If $\gamma_{\lambda_1,\lambda_2,\phi} = 0$ then the operator $C_\phi : B^{\lambda_1(\cdot)}(\mathbb{D}) \to B^{\lambda_2(\cdot)}(\mathbb{D})$ is w-compact.*
2. *Suppose that $\inf_{z\in\mathbb{D}} \lambda_2(z) > 0$ and $\sup_{z\in\mathbb{D}} \lambda_1(z) < 1$. If $\gamma_{\lambda_1,\lambda_2,\phi} = 0$ then the operator $C_\phi : A^{\lambda_1(\cdot)}(\mathbb{D}) \to A^{\lambda_2(\cdot)}(\mathbb{D})$ is w-compact.*

We conclude the presentation showing that there exists compact operators for which $\gamma_{\lambda_1,\lambda_2,\phi} \neq 0$. To this end we construct the function ϕ such that $\|\phi\|_{L^\infty(\mathbb{D})} < 1$, $\phi \in B^{\lambda_2(\cdot)}(\mathbb{D})$ and $\gamma_{\lambda_1,\lambda_2,\phi} \neq 0$.

Theorem 19.2. *Let λ_1 and λ_2 belong to $\Lambda(\mathbb{D})$ and $\lambda_2^{',+} = \sup_{\sigma\in\mathbb{T}} \lambda_2(\sigma) > 0$. There exits a holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ which satisfies $\sup_{z\in\mathbb{D}} |\phi(z)| < 1$, C_ϕ is w-compact from $B^{\lambda_1(\cdot)}(\mathbb{D})$ to $B^{\lambda_2(\cdot)}(\mathbb{D})$ and $\gamma_{\lambda_1,\lambda_2,\phi} \neq 0$.*

Proof. Consider the holomorphic function $\phi(z) = \frac{1}{3}(1-z)^{\lambda_2^{',+}}$ defined for $z \in \mathbb{D}$. It is clear that $\phi \in B^{\lambda_2(\cdot)}(\mathbb{D})$ and $\sup_{z\in\mathbb{D}} |\phi(z)| < 1$. By Theorem 18.1, C_ϕ is w-compact from $B^{\lambda_1(\cdot)}(\mathbb{D})$ to $B^{\lambda_2(\cdot)}(\mathbb{D})$. Without loss of generality we can assume that $\lambda_2(1) = \lambda_2^{',+}$. $\square$

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Thank you.