Method for constructing solutions to integrable PΔEs

Sotiris Konstantinou-Rizos

Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia

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Outline

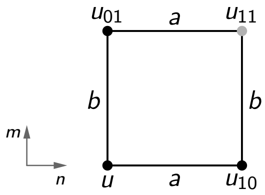
- Nonlinear partial difference equations (PΔEs).
  - Equations on quad-graphs;
  - Integrability (Lax pairs, $3D$ consistency).
- A new discrete Darboux-Lax scheme for solving integrable nonlinear PΔEs (with X. Fisenko and P. Xenitidis, 2022, Chaos, Solitons and Fractals **158** (2022)).
- Conclusions.

Nonlinear partial difference equations: Equations on quad-graphs.

- A quad-graph equation is an equation of the form

$$Q(u, u_{10}, u_{01}, u_{11}; a, b) = 0,$$

where Q is affine multi-linear.



- The discrete potential KdV equation

$$(u - u_{11})(u_{10} - u_{01}) = a - b.$$

- Let $u = u(n, m)$. By u_{ij} we denote $u_{ij} = u(n + i, m + j)$, $i, j \in \mathbb{Z}$. That is, $u_{00} \equiv u$, $u_{10} = u(n + 1, m)$, $u_{01} = u(n, m + 1)$ and $u_{11} = u(n + 1, m + 1)$ etc.

Lax representation

- Integrability means that there exists a pair of matrices $L = L(u, u_{10}; a, \lambda)$ and $M = M(u, u_{01}; b, \lambda)$, such that equation

$$Q(u, u_{10}, u_{01}, u_{11}; a, b) = 0$$

can equivalently be written as

$$L(u_{01}, u_{11}; a, \lambda)M(u, u_{01}; b, \lambda) = M(u_{10}, u_{11}; b, \lambda)L(u, u_{10}; a, \lambda).$$

In many cases, $L = M$.

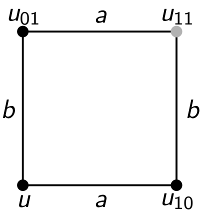
- For example, the dpKdV equation has Lax representation with $L = M$ and L is given by

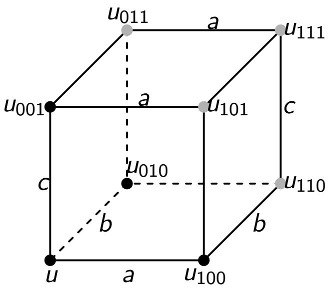
$$L(u, u_{10}, a) = \begin{pmatrix} u & \lambda - a - uu_{10} \\ 1 & -u_{10} \end{pmatrix}.$$

3D-consistency

We rewrite $Q(u, u_{10}, u_{01}, u_{11}; a, b) = 0$ on the bottom, the front and the left side of the cube

$$Q(u, u_{100}, u_{010}, u_{110}; a, b) = 0 \implies u_{110} = \dots$$
$$Q(u, u_{100}, u_{001}, u_{101}; a, c) = 0 \implies u_{101} = \dots$$
$$Q(u, u_{001}, u_{010}, u_{011}; c, b) = 0 \implies u_{011} = \dots$$





Discrete Potential KdV (dpKdV)

$$(u-u_{11})(u_{10}-u_{01})+b-a=0. \tag{1}$$

Writing (1) on bottom, front and left face of the cube:

$$(u-u_{110})(u_{100}-u_{010})=a-b, \tag{2a}$$
$$(u-u_{101})(u_{100}-u_{001})=a-c, \tag{2b}$$
$$(u-u_{011})(u_{001}-u_{010})=c-b, \tag{2c}$$

we can solve for u_{110}, u_{101} and u_{011}, namely

$$u_{110}=u+\frac{a-b}{u_{010}-u_{100}}, \tag{3a}$$
$$u_{101}=u+\frac{a-c}{u_{001}-u_{100}}, \tag{3b}$$
$$u_{011}=u+\frac{b-c}{u_{001}-u_{010}}, \tag{3c}$$

respectively.

Now, if we shift (3a) in the k-direction, and then substitute u_{101} and u_{011} by (3b), (3c), we deduce

$$u_{111} = -\frac{(a-b)u_{100}u_{010} + (b-c)u_{010}u_{001} + (c-a)u_{100}u_{001}}{(a-b)u_{001} + (b-c)u_{100} + (c-a)u_{010}}. \tag{4}$$

The dpKdV equation is 3D-consistent.

There are some straightforward consequences of the $3D$ consistency property.

- $3D$ consistency implies Lax representation.
- A Bäcklund transformation can be derived.

Set $v = u_{001}$.
Equations (2b) and (2c) can be written as

$$(u - v_{10})(u_{10} - v) = a - c,$$
$$(u - v_{01})(v - u_{01}) = c - b.$$

It can be easily shown that if u is a solution of the dpKdV equation, so does v.

The dpKdV equation

$$(u - u_{11})(u_{10} - u_{01}) = a - b$$

admits a trivial solution

$$u(n, m) = an + bm + const.$$

Using the Bäcklund transformation, a nontrivial solution can be constructed, namely

$$v(n, m) = na + mb + c\frac{1 + \rho_{n,m}}{1 - \rho_{n,m}}, \quad \rho_{n,m} = \gamma \left(\frac{a + c}{a - c}\right)^n \left(\frac{b + c}{b - c}\right)^m.$$

A new discrete Darboux-Lax scheme for solving integrable nonlinear PΔEs.

Consider an integrable quad-graph system,

$$\mathbf{Q}(\mathbf{f}_{00}, \mathbf{f}_{10}, \mathbf{f}_{01}, \mathbf{f}_{11}; a, b) = 0$$

namely, it is equivalent to

$$\mathrm{L}(\mathbf{f}_{01}, \mathbf{f}_{11}; a, \lambda)\mathrm{M}(\mathbf{f}_{00}, \mathbf{f}_{01}; b, \lambda) = \mathrm{M}(\mathbf{f}_{10}, \mathbf{f}_{11}; b, \lambda)\mathrm{L}(\mathbf{f}_{00}, \mathbf{f}_{10}; a, \lambda),$$

for some matrices L and M.

Definition

Darboux transformation is a gauge-like, spectral parameter-dependent transformation that leaves matrices L and M covariant, i.e.

$$\mathrm{L}(\mathbf{f}, \mathbf{f}_{10}; a, \lambda) \longmapsto \mathrm{B}_{10}\mathrm{L}(\mathbf{f}, \mathbf{f}_{10}; a, \lambda)\mathrm{B}^{-1} = \tilde{\mathrm{L}} := \mathrm{L}(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_{10}; a, \lambda);$$

$$\mathrm{M}(\mathbf{f}, \mathbf{f}_{01}; b, \lambda) \longmapsto \mathrm{B}_{01}\mathrm{M}(\mathbf{f}, \mathbf{f}_{01}; b, \lambda)\mathrm{B}^{-1} = \tilde{\mathrm{M}} := \mathrm{M}(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_{01}; \mathrm{b}, \lambda).$$

- We start by assuming an initial form for matrix B.

$$\mathrm{B} = \lambda \mathrm{B}^{(1)} + \mathrm{B}^{(0)}.$$

- We determine the elements of $\mathrm{B}^{(1)}$ and $\mathrm{B}^{(0)}$ by

$$\mathrm{L}(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_{10}; \lambda)\mathrm{B} = \mathrm{B}_{10}\mathrm{L}(\mathbf{f}, \mathbf{f}_{10}; \lambda), \quad \mathrm{M}(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_{01}; \lambda)\mathrm{B} = \mathrm{B}_{01}\mathrm{M}(\mathbf{f}, \mathbf{f}_{01}; \lambda).$$

- The derived Darboux matrix depends on $\mathbf{f}$, $\tilde{\mathbf{f}}$, λ, ε, $g(n,m)$:

$$\mathrm{B} = \lambda \mathrm{B}^{(1)}(\mathbf{f}_{00}, \tilde{\mathbf{f}}_{00}, g; \varepsilon) + \mathrm{B}^{(0)}(\mathbf{f}_{00}, \tilde{\mathbf{f}}_{00}, g; \varepsilon).$$

- In the construction of the Darboux matrix there are some algebraic relations, as well as some difference equations for its elements:

$$\mathcal{B}^{(n)}(\mathbf{f}_{00}, \mathbf{f}_{10}, \tilde{\mathbf{f}}_{00}, \tilde{\mathbf{f}}_{10}, g; \varepsilon) = 0, \qquad \mathcal{B}^{(m)}(\mathbf{f}_{00}, \mathbf{f}_{01}, \tilde{\mathbf{f}}_{00}, \tilde{\mathbf{f}}_{01}, g; \varepsilon) = 0,$$

  which constitute the n- and the m-parts of an auto-Bäcklund transformation. We denote this transformation by $\mathcal{B}(\mathbf{f}, \tilde{\mathbf{f}}, g; \varepsilon) = 0$.

- The Bianchi commuting diagram, aka superposition principle:

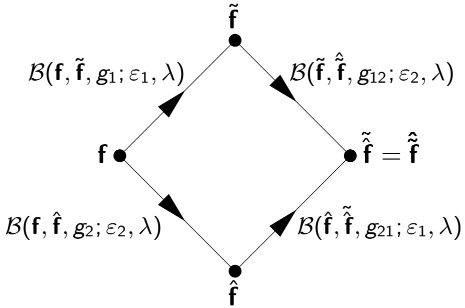


Figure: Bianchi commuting diagram.

- Demanding $\hat{\tilde{\mathbf{f}}} = \tilde{\hat{\mathbf{f}}}$ we construct algebraically a new solution.

$$\mathrm{B}(\tilde{\mathbf{f}}, \hat{\tilde{\mathbf{f}}}, g_{12}; \varepsilon_2, \lambda)\mathrm{B}(\mathbf{f}, \tilde{\mathbf{f}}, g_1; \varepsilon_1, \lambda) = \mathrm{B}(\hat{\mathbf{f}}, \tilde{\hat{\mathbf{f}}}, g_{21}; \varepsilon_1, \lambda)\mathrm{B}(\mathbf{f}, \hat{\mathbf{f}}, g_2; \varepsilon_2, \lambda).$$

An example: The Adler–Yamilov system.

- Consider the Adler–Yamilov system which is a discretisation of the NLS equation (SKR, A.V. Mikhailov, P. Xenitidis: J. Phys. A **56**):

$$p_{01} = p_{10} - \frac{a-b}{1+p_{00}q_{11}} p_{00}, \qquad q_{01} = q_{10} + \frac{a-b}{1+p_{00}q_{11}} q_{11}.$$

- This system has the Lax pair

$$\mathrm{L}(\mathrm{p}, \mathrm{q}_{10}; \mathrm{a}, \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a + pq_{10} & p \\ q_{10} & 1 \end{pmatrix};$$

$$\mathrm{L}(\mathrm{p}, \mathrm{q}_{01}; \mathrm{b}, \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b + pq_{01} & p \\ q_{01} & 1 \end{pmatrix}.$$

- We seek matrix B such that

$$\mathrm{LB} = \mathrm{B}_{10}\tilde{\mathrm{L}}, \qquad \mathrm{MB} = \mathrm{B}_{01}\tilde{\mathrm{M}}.$$

- The simplest initial assumption for the form of B is linear in λ, thus we assume that B has the form

$$\mathrm{B} = \lambda \mathrm{B}^1 + \mathrm{B}^0.$$

Theorem

The following matrix

$$\mathrm{B}(q_{00}, \tilde{p}_{00}; \varepsilon) = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\tilde{p}_{00} \\ -q_{00} & \varepsilon + \tilde{p}_{00} q_{00} \end{pmatrix},$$

is the Darboux matrix for the Adler–Yamillov system if and only if its entries satisfy the system of difference equations (Bäcklund transform)

$$\tilde{p}_{10} = p_{00} + \frac{\alpha - \varepsilon}{1 + \tilde{p}_{00} q_{10}} \tilde{p}_{00}, \quad \tilde{q}_{10} = q_{00} - \frac{\alpha - \varepsilon}{1 + \tilde{p}_{00} q_{10}} q_{10}, \tag{5a}$$

$$\tilde{p}_{01} = p_{00} + \frac{\beta - \varepsilon}{1 + \tilde{p}_{00} q_{01}} \tilde{p}_{00}, \quad \tilde{q}_{01} = q_{00} - \frac{\beta - \varepsilon}{1 + \tilde{p}_{00} q_{01}} q_{01}. \tag{5b}$$

The superposition principle can be written as:

$$\hat{\tilde{p}}_{00} = -\frac{\tilde{p}_{00} - \hat{p}_{00}}{\varepsilon_1 - \varepsilon_2 + (\tilde{p}_{00} - \hat{p}_{00}) q_{00}}, \quad \tilde{q}_{00} - \hat{q}_{00} = \left(\varepsilon_1 - \varepsilon_2 + (\tilde{p}_{00} - \hat{p}_{00}) q_{00}\right) q_{00}. \tag{6}$$

How to construct a new solution using the Bäcklund transform.

- We start with the simple solution $p_{00} = 0, \quad q_{00} = \alpha^{-n}\beta^{-m}$ to the Adler–Yamilov system.
- We substitute this solution to the Bäcklund transformation and obtain:

$$\tilde{p}_{10} = \frac{\alpha - \varepsilon}{1 + \tilde{p}_{00}\alpha^{-n-1}\beta^{-m}}\tilde{p}_{00}, \quad \tilde{p}_{01} = \frac{\beta - \varepsilon}{1 + \tilde{p}_{00}\alpha^{-n}\beta^{-m-1}}\tilde{p}_{00}$$

- We can linearise these Ricatti equations by setting $\tilde{p}_{00} = \alpha^n\beta^m/g_{00}$,

$$(\alpha - \varepsilon)g_{10} = \alpha g_{00} + 1, \quad (\beta - \varepsilon)g_{01} = \beta g_{00} + 1.$$

- The general solution of this linear system is

$$g_{00} = \frac{\alpha^n}{(\alpha - \varepsilon)^n}\frac{\beta^m}{(\beta - \varepsilon)^m}c - \frac{1}{\varepsilon}, \tag{7}$$

- Thus,

$$\tilde{p}_{00} = \frac{\varepsilon\,\alpha^n\beta^m(\alpha-\varepsilon)^n(\beta-\varepsilon)^m}{c\,\varepsilon\,\alpha^n\beta^m-(\alpha-\varepsilon)^n(\beta-\varepsilon)^m}, \quad \tilde{q}_{00} = \frac{c\,\varepsilon^2}{c\,\varepsilon\,\alpha^n\beta^m-(\alpha-\varepsilon)^n(\beta-\varepsilon)^m}.$$

- The product $p_{00}q_{00}$ represents a soliton solution.

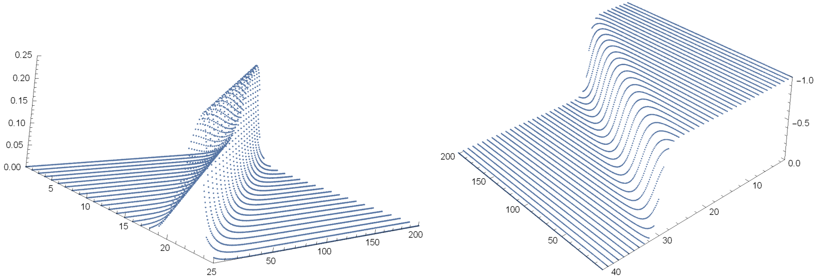


Figure: The one soliton solution of the Adler–Yamilov system and the potential $1/g_{00}$. In both cases $\alpha = 8$, $\beta = 4$, $\varepsilon = 1$ and $c = -2$.

We can algebraically construct a new solution using the Bianchi diagram.

$$\hat{\hat{p}}_{00} = \frac{\varepsilon_1\varepsilon_2\delta_0\left(c_1\delta_2 - c_2\delta_1\right) + (\varepsilon_1 - \varepsilon_2)\delta_1\delta_2}{c_1c_2\varepsilon_1\varepsilon_2(\varepsilon_1 - \varepsilon_2)\delta_0 - c_1\varepsilon_1^2\delta_2 + c_2\varepsilon_2^2\delta_1}, \quad \hat{\hat{q}}_{00} = \frac{c_1c_2\varepsilon_1^2\varepsilon_2^2(\varepsilon_1 - \varepsilon_2)}{c_1c_2\varepsilon_1\varepsilon_2(\varepsilon_1 - \varepsilon_2)\delta_0 - c_1\varepsilon_1^2\delta_2 + c_2\varepsilon_2^2\delta_1}.$$

where $\delta_0 := \alpha^n\beta^m, \quad \delta_1 := (\alpha - \varepsilon_1)^n(\beta - \varepsilon_1)^m, \quad \delta_2 := (\alpha - \varepsilon_2)^n(\beta - \varepsilon_2)^m.$

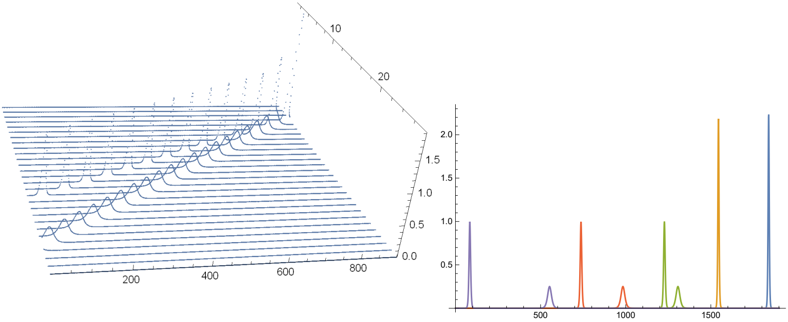


Figure: The two-soliton solution of the Adler–Yamilov system with $\alpha = 8$, $\beta = 4$, $\varepsilon_1 = 1$, $c_1 = -2$, $\varepsilon_2 = 3$ and $c_2 = 8$.

Conclusions.

- We presented a systematic method for solving equations on quad-graphs.
- From the solutions of the quad-graph equations one can derive solutions to nonlinear integrable PDEs via continuum limits.
- These results can be generalised for noncommutative discrete integrable systems (with P. Xenitidis).
- One can construct the corresponding noncommutative Yang–Baxter maps (with A. Nikitina).

Thank you!