Симплектические частично-гиперболические автоморфизмы на T^6: динамика и классификация

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Содержание I

Our focus here is the topological dynamics of automorphisms of a 6-dimensional torus generated by an integer symplectic transformation of $\mathbb{R}^6$ for the case of partial hyperbolicity. Partially diffeomorphisms on any smooth manifold M were introduced and studied first by Brin and Pesin. The hyperbolic case is now rather well understood (Anosov, Franks, Manning, others).
There are many results about partially hyperbolic diffeomorphisms (Birns-Wilkinson, Hammerlindl, Hasselblatt-Pesin, Hertz). Concerning partially hyperbolic automorphisms on a torus $\mathbb{T}^n$, the most detailed study was done by Hertz, who solved partially the question of their stable ergodicity posed by Hirsch-Pugh-Shub. Recall that stable ergodicity of a C^r-smooth diffeomorphism f of a manifold M, being ergodic with respect to a smooth Lebesgue measure on M, means the existence of a neighborhood U of f in the space of C^r-smooth diffeomorphisms such that each $g \in U$ is ergodic. Our goal here is to classify, with respect to topological conjugacy, possible types of orbit behavior of symplectic automorphisms of $\mathbb{T}^6$. We trust this will be useful as a source of interesting examples.

Let L be a linear transformation between two normed linear spaces of finite dimension. The norm, respectively conorm, of L are defined as

$$||L|| := \sup ||Lv||, ||v|| = 1, \; m(L) := \inf ||Lv||, ||v|| = 1.$$

Definition

A diffeomorphism $f : M \to M$ is partially hyperbolic, if there is a continuous Df-invariant splitting $TM = E^u \oplus E^c \oplus E^s$, in which both E^u and E^s are nontrivial sub-bundles, and for some Riemannian metrics on M the following estimates hold

$$m(D^u f) > 1 > ||D^s f||,$$
$$m(D^u f) > ||D^c f|| \geq m(D^c f) > ||D^s f||,$$

where $D^\sigma f$ is the restriction of Df to E^σ for $\sigma = s, c$ or u.

In the case of a n-dimensional torus, a partially hyperbolic automorphism is defined by an integer uni-modular matrix having eigenvalues with real parts both greater than one and lesser than one and also on the unit circle of the complex plane. The corresponding eigen-spaces W^s, W^u and central subspace W^c, when projecting on a torus and shifting them at any its point, give the required decomposition of the definition.

On an even-dimensional torus, $n = 2m$, $\mathbb{T}^{2m}$, one can introduce the standard symplectic structure using coordinates $(x_1, \ldots, x_{2m})$ in $\mathbb{R}^{2m}$: $\Omega = dx_1 \wedge dx_{m+1} + \cdots + dx_m \wedge dx_{2m}$, and consider symplectic automorphisms of the torus which preserve this symplectic structure. A symplectic automorphism f_A is defined by a symplectic matrix A with integer entries. Such matrices satisfy the identity $A^\top I A = I$ with a skew-symmetric matrix I of the form

$$I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$

This identity implies the symplectic matrices to form the Lie group $\mathrm{Sp}(2n, \mathbb{R})$ w.r.t. the operation of matrix multiplication.

In our earlier paper, we presented a classification of automorphisms f_A of the four-dimensional torus $\mathbb{T}^4$ generated by symplectic integer matrices $A \in Sp(4, \mathbb{Z})$. These automorphisms can possess either a transitive unstable one-dimensional foliation or they are decomposable. In the first case two such automorphisms are (topologically) conjugate, if their matrices are integrally similar (conjugate in $\mathrm{M}_n(\mathbb{Z})$). In the second case, a decomposable f_A is conjugated to the direct product of two 2-dimensional automorphisms acting on $\mathbb{T}^2 \times \mathbb{T}^2$, one of which is an Anosov automorphism of the 2-torus and another one is periodic on a 2-torus.
It is natural to extend these results to the higher-dimensional case. This is more involved problem because several different possibilities can occur: the dimensions of a center sub-bundle and stable/unstable sub-bundles can vary even for a torus of a fixed dimension. For instance, for symplectic automorphisms on $\mathbb{T}^6$, we study here, there are symplectic partially hyperbolic integer matrices with

- $\dim W^c = 4, \quad \dim W^s,\ W^u = 1;$
- $\dim W^c = 2, \quad \dim W^s,\ W^u = 2.$

Recall how stable, unstable and center foliation are generated for the case of symplectic automorphisms of $\mathbb{T}^6$. A symplectic 6×6 matrix A has a decomposition of its spectrum into three parts: $Sp(A) = \sigma_s \cup \sigma_c \cup \sigma_u$ where eigenvalues in σ_s lie within the unit circle, of σ_c on the unit circle and those in σ_u are out of the unit circle. We assume below that all eigenvalues are simple. Therefore in the first case above the projection of the subspace $W^u \subset \mathbb{R}^6$ onto the torus $\mathbb{T}^6$ gives an embedded line. The subspace W^u is also a one-parametric subgroup of $\mathbb{R}^6$, hence its shifts by vectors of $\mathbb{R}^6$ gives other affine lines representing other classes of the factor-group $\mathbb{R}^6/W^u$. Their projections on the torus give unstable foliation of the automorphism f_A, its leaves are embedded infinite lines. Similar construction gives the stable foliation and the center foliation.

Recall the well-known statement about the characteristic polynomial of a symplectic matrix

Proposition

The characteristic polynomial of a symplectic matrix is self-reciprocal: $\chi(\lambda) = \lambda^{2n}\chi(1/\lambda)$. *If* λ *is an eigenvalue of a real symplectic matrix* A*, then the numbers* $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ *are also its eigenvalues, all they have the same multiplicity and the same structure of elementary divisors.* *Eigenvalues of* $\lambda = \pm 1$ *have an even multiplicity, their elementary divisors of an odd order, if exist, meet in pairs.*

Other (non-standard) symplectic structures on the torus $\mathbb{T}^{2m}$ can be also defined. To this end, choose an integer skew-symmetric non-degenerate $2m \times 2m$ matrix J, $J^\top = -J$. Such matrix defines a bilinear 2-form $[x, y] = (Jx, y)$ in $\mathbb{R}^{2m}$, where $(\cdot, \cdot)$ is the standard coordinate inner product. Then a linear map $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is called symplectic, if the identity $[Sx, Sy] = [x, y]$ holds for any $x, y \in \mathbb{R}^{2m}$.

Using the representation of skew inner product via the matrix J and properties of the inner product, the following identity for the matrix S of the symplectic map is derived: $S^{\top}JS = J$. Then a symplectic 2-form is given on the torus and map S defines a symplectic automorphism with respect to this symplectic form. For example, let B be any non-degenerate integer matrix. Having the standard skew inner product (Ix, y) in $\mathbb{R}^{2m}$ we can define new skew inner product as $[x, y] = (IBx, By) = (B^{\top}IBx, y)$. Since the matrix $J = B^{\top}IB$ is skew symmetric, integer and non-degenerate, this skew inner product generates symplectic 2-form on the torus. We shall exploit this construction later on.

One of the principal theorems for automorphisms of an abelian group is the following classical Halmos theorem

Theorem (Halmos)

A continuous automorphism f of a compact abelian group G is ergodic (and mixing) if and only if the induced automorphism on the character group G^* has not finite orbits.

In the case of the abelian group $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, characters are complex-valued functions $\chi_m(x) = \exp[2\pi i(m, x)]$ with an integer vector $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and the standard inner product (m, x). If L_A is the automorphism of the torus generated by the matrix A, then the induced automorphism on the character group acts as follows

$$[L_A^*(\chi_m)](x) = \exp[2\pi i(m, Ax)] = \exp[2\pi i(A^\top m, x)],$$

where $A^\top$ means the transposed matrix.

The existence of a finite orbit of the induced automorphism means that for some vector $m_0 \in \mathbb{Z}^n$ and natural k the identity holds

$$\exp[2\pi i((A^\top)^k m_0, x)] \equiv \exp[2\pi i(m_0, x)]$$

for all $x \in \mathbb{T}^n$. This means the integer vector m_0 be an eigen-vector of matrix $(A^\top)^k$ with the eigenvalue 1. Thus, among eigenvalues of $A^\top$ there exists λ such that $\lambda^k = 1$. But eigenvalues of A and $A^\top$ are the same. Conversely, if an integral uni-modular matrix A has an eigenvalue λ being a root of unity, $\lambda^k = 1$, then there exists a non-zero vector x such that $(A^k - E)x = 0$. But matrix $A^k - E$ has integer entries and all its minors be integer numbers. Therefore, vector x can be chosen with rational entries and hence can be made an integer vector. Thus, the transposed matrix $A^\top$ has eigenvalue λ with an integer eigenvector m_0, $(A^\top)^k m_0 = m_0$, and the related induced action on the character group has a finite orbit. Finally, by the Halmos theorem, *automorphism L_A is ergodic if and only if the matrix A has not eigenvalues being roots of unity*.

Let $P(x) = \lambda^6 + a\lambda^5 + b\lambda^4 + c\lambda^3 + b\lambda^2 + a\lambda + 1$ be a self-reciprocal polynomial with integer coefficients (a, b, c) being irreducible over field $\mathbb{Q}$. The companion matrix of this polynomial P has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -a & -b & -c & -b & -a \end{pmatrix}$$

This matrix is unimodular ($\det A = 1$) but not always symplectic with respect to the standard symplectic 2-form $[x, y] = (Ix, y)$ for $x, y \in \mathbb{R}^6$. Let us show, however, that A is symplectic with respect to the nonstandard symplectic structure on $\mathbb{R}^6$ defined by $[x, y] = (Jx, y)$ for $x, y \in \mathbb{R}^6$ for some integer non-degenerate skew-symmetric matrix J.

Rewrite the matrix identity $A^\top JA = J$ (for unknown J) as $A^\top J - JA^{-1} = 0$. A solution J of this matrix equation is

$$J = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & a & a+1 & a+1 & 1 \\ -1 & -a & 0 & a+b-c & a+1 & 1 \\ -1 & -a-1 & -a-b+c & 0 & a & 1 \\ -1 & -a-1 & -a-1 & -a & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{pmatrix}. \qquad (1)$$

Its determinant $\det J = (a+b-c-2)^2$ is non-zero, if $a+b-c \neq 2$, then A is symplectic with respect to this nonstandard symplectic structure. This will be used later on.

The topological classification of such automorphisms is determined in the first turn by the topology of a foliation generated by unstable (stable) leaves of the automorphism, since this foliation is invariant with respect to the action of f, and also by the action of f on the center submanifold and its extension. Hence, the structure of stable and unstable foliations has to be investigated in the first turn for the classification problem.

We assume henceforth that all eigenvalues are simple. Therefore in the first case above the projection of the subspace $W^u \subset \mathbb{R}^6$ onto the torus $\mathbb{T}^6$ gives an embedded line. The subspace W^u is also a one-parametric subgroup of $\mathbb{R}^6$, hence its shifts by vectors of $\mathbb{R}^6$ gives other affine lines representing other classes of the factor-group $\mathbb{R}^6/W^u$. Their projections on the torus give unstable foliation of the automorphism f_A, its leaves are embedded infinite lines. Similar construction gives the stable foliation and the center foliation.

For the second case the subspace W^u is two-dimensional and this commutative subgroup has two generators. Here there are two different cases: 1) σ_u consists of two simple real eigenvalues with the related independent real eigen-spaces; 2) σ_u consists of two complex conjugate eigenvalues and W^u is the invariant subspace w.r.t. the action of L_A, as generators of this subgroup one can choose real and imaginary parts of the complex eigenvector of the matrix A.

The projection of W^u onto the torus gives an embedded plane (it is the unstable manifold of the fixed point $\hat{O}$), other classes of $\mathbb{R}^6/W^u$ are affine 2-planes, their projections give the unstable foliation of f_A. Analogously we get the stable foliation and the center foliation.

The topological classification of automorphisms of the group $\mathbb{T}^n$ is determined by the Arov's theorem.

Theorem (Arov)

Two automorphisms T and P of a compact abelian group G are topologically conjugate if and only if they are isomorphic, that is, there is an isomorphism $Q : G \to G$ such that $Q \circ T = P \circ Q$.

It's not an easy problem to check two integer matrices be similar via an integer unimodular matrix. For the group $\mathbb{T}^n$ the necessity to be unimodular for a conjugation matrix leads to the possibility that two integer matrices can be similar over $\mathbb{Q}$ but not over $\mathbb{Z}$. This holds true even for the case of Anosov automorphisms on the 2-torus. For instance, consider two automorphisms of $\mathbb{T}^2$ generated by matrices

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 8 \end{pmatrix},$$

where C is the companion matrix of the characteristic polynomial of A. Two matrices A, C are rationally similar $TA = CT$ but not integrally similar. A general assertion is as follows

Proposition

Suppose A be an integer real unimodular matrix with simple eigenvalues. Then this matrix is rationally similar to the companion matrix C of its characteristic polynomial. Automorphisms f_A, f_C of the torus $\mathbb{T}^n$ generated by these matrices are semi-conjugate, i.e. there is an integer non-degenerate matrix T such that the relation $TA = CT$ holds and $f_T f_A = f_C f_T$. The mapping f_T is the k-fold covering map $h : \mathbb{T}^n \to \mathbb{T}^n$

Let us investigate a possible behavior of projections onto the torus of eigen-lines l^s, l^u. The projection of the eigenlines l^s, l^u onto the torus can lead to different situations. One-dimensional subspace l^u (or l^s) coincides with the one-parameter subgroup $t\gamma^u$ generated by the vector γ^u, and its projection is the image under the exponential mapping of the algebra into the group. This subgroup is included into orbits of the constant vector field on $\mathbb{T}^6$ invariant under group shifts. In angular coordinates θ on $\mathbb{T}^6$ induced by coordinates in $\mathbb{R}^6$ we get the vector field $\dot{\theta} = \gamma^u$.

The closure of any trajectory of the constant vector field on the torus is a smooth torus of some dimension, its dimension is equal to $6 - r$ where r is the number of rationally independent linear relations for the components of the vector $\gamma^u : (m, \gamma^u) = 0, m \in \mathbb{Z}^6$.

Proposition

The closure of the unstable leaf of the fixed point $\hat{O}$ for f_A is either the whole $\mathbb{T}^6$, or a four-dimensional torus, or two-dimensional torus, in the first case the leaf is transitive.

The case of two rationally independent relations for partially hyperbolic matrix is indeed possible. It is sufficient to choose, for example, a block-diagonal integer matrix A composed of two integer blocks, one of which (4×4)-block has eigenvalues $\lambda, \lambda^{-1}, \exp[i\alpha_1], \exp[-i\alpha_1]$, $\lambda > 1$, not being roots of unity, and the second (2×2)-block has two complex conjugate eigenvalues $\exp[i\alpha_2], \exp[-i\alpha_2]$ (then $\alpha_2/2\pi = 1/3$, $1/4, 1/6$). The first block generates a partially hyperbolic ergodic automorphism of $\mathbb{T}^4$ whose unstable foliation is transitive on $\mathbb{T}^4$.

The case of four rationally independent relations for partially hyperbolic matrix is also possible. It is sufficient to choose, for example, a block-diagonal integer matrix A_1 composed of two integer blocks, one of which (2×2)-block generates an Anosov automorphism, and the second (4×4)-block has two pairs complex conjugate eigenvalues $\exp[i\alpha_1], \exp[-i\alpha_1], \exp[i\alpha_2], \exp[-i\alpha_2]$ on the unit circle. Since the characteristic polynomial of this (4×4)-matrix has integer coefficients and monic, its eigenvalues are roots of unity (Kronecker).

Now we construct partially hyperbolic automorphisms of $\mathbb{T}^6$. We start with a irreducible over rational numbers cubic polynomial $P(z)$ with integer coefficients having three roots, one greater than 2 and two less than 2 in modulus. After a change of variable $z = x + x^{-1}$ we obtain a polynomial of the sixth degree which serves the characteristic polynomial of the matrix with the desired properties. For example, take with the cubic polynomial $P = z^3 - 2z^2 - z + 1$ that gives the polynomial $Q = x^6 - 2x^5 + 2x^4 - 3x^3 + 2x^2 - 2x + 1$ which is irreducible over the field $\mathbb{Q}$. Then, the companion matrix of this polynomial Q has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 3 & -2 & 2 \end{pmatrix}$$

This matrix possesses the required properties of its eigenvalues. However, this matrix is not symplectic with respect to the standard symplectic 2-form $[x, y] = (Ix, y), \quad x, y \in \mathbb{R}^6 : A^\top I A \neq I$. We show, however, A to be symplectic with respect to non-standard symplectic structure on the torus $\mathbb{T}^6$, i.e. the equality $A^\top J A = J$ holds with matrix J

$$J = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \\ -1 & 2 & 0 & 3 & -1 & 1 \\ -1 & 1 & -3 & 0 & 1 & 1 \\ -1 & 1 & 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

Again, the greater than one eigenvalue λ has the eigenvector $\gamma^u = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5)^\top$ and the same considerations show the transitivity of the related unstable foliation on $\mathbb{T}^6$. Here we use algebraic properties of λ.

Recall a number $\xi \in \mathbb{C}$ is called an *algebraic number* if it is a root of a nonzero polynomial with rational coefficients, and it is called *an algebraic integer* if it is the root of a polynomial with integral coefficients and leading coefficient 1 (a monic polynomial). The *degree* of an algebraic integer ξ is the degree of the *minimal* polynomial of ξ (a monic polynomial of the lowest degree with integer coefficients and ξ as a root). For instance, rational numbers are algebraic of degree one.
The polynomial $Q(x)$ above has degree six due to the following theorem.

Theorem

Let ξ be an algebraic number with minimal polynomial $p(x)$. Then

1. *the polynomial $p(x)$ is irreducible over $\mathbb{Q}$;*
2. *the polynomial $p(x)$ is unique;*
3. *if ξ is a root of a polynomial f over $\mathbb{Q}$, then p divides f.*

The last statement of the theorem implies that the number λ, being a root of the irreducible polynomial $Q(x)$ of degree six, is not a root of a polynomial of smaller degree with rational (integer) coefficients. So, vector γ is incommensurate.

One more useful striuctural assertion is as follows

Proposition

If f_A is a symplectic partially hyperbolic automorphism of $\mathbb{T}^6$ with the transitive unstable one-dimensional foliation, then its stable and center foliations have also dense leaves.

In this section we study the case, when the dimension of W^u (and W^s) equals two. Corresponding λ_1, λ_2 can be either two different real ones or a pair of complex conjugate numbers. Respectively, W^u is spanned either by two independent real eigenvectors γ_1^u, γ_2^u of real eigenvalues, or it is an invariant 2-dimensional real subspace corresponding to complex conjugate eigenvalues. In the latter case the invariant subspace is generated by real and complex parts γ_r^u, γ_i^u of the complex eigenvector $\gamma_r^u + i\gamma_i^u$, these are two independent real vectors.

Projection by p of W^u on $\mathbb{T}^6$ generates related unstable foliation in $\mathbb{T}^6$. In order to determine how leaves of this foliation behave in $\mathbb{T}^6$, we observe that this foliation is formed by orbits of the action of the group $\mathbb{R}^2$ on $\mathbb{T}^6$ generated by two commuting constant vector fields on $\mathbb{T}^6$ given as $\gamma_1^u \cdot \partial/\partial\theta$ and $\gamma_2^u \cdot \partial/\partial\theta$, $\theta = (\theta_1, \ldots, \theta_6)$. The related orbits are given as

$$\theta(t_1, t_2) = t_1\gamma_1^u + t_2\gamma_2^u + \theta_0 \text{ (modd 1)}.$$

In particular, for $\theta_0 = \hat{O}$ the orbit passes through the fixed point $\hat{O}$ of the automorphism. The action and the shift on the torus commute.

To understand the behavior of foliation leaves, recall the Kronecker theorem.

Theorem (Kronecker)

Let m *vectors* $\mathbf{a_i}$, $1 \leq i \leq m$, *and a vector* $\mathbf{b}$ *in* $\mathbb{R}^n$ *be given. In order for any* $\varepsilon > 0$ *there exist* m *numbers* $t_i \in \mathbb{R}$ *and an integer vector* $\mathbf{p} \in \mathbb{Z}^{\mathbf{n}}$ *such that*

$$||\sum_{i=1}^{m} t_i \mathbf{a_i} - \mathbf{p} - \mathbf{b}|| \leq \varepsilon,$$

it is necessary and sufficient that for any integer vector $\mathbf{r} \in \mathbb{Z}^{\mathbf{n}}$, *such that all* m *equalities* $(\mathbf{r}, \mathbf{a_i}) = \mathbf{0}$ *hold, the relation* $(\mathbf{r}, \mathbf{b}) = \mathbf{0}$ *also holds.*

So, if none such relations exist, one can take any $\mathbf{b} \in \mathbb{R}^\mathbf{n}$ and the following statement is valid

Corollary

In order for any $\mathbf{x} \in \mathbb{R}^\mathbf{n}$ and any $\varepsilon > 0$ there exist m real numbers $t_1, \ldots, t_m$ and an integer vector $\mathbf{p}$ such that

$$\left\| \sum_{i=1}^{m} t_i \mathbf{a_i} - \mathbf{p} - \mathbf{x} \right\| \le \varepsilon,$$

it is necessary and sufficient that there is no nonzero integer vector $\mathbf{r} \in \mathbb{Z}^\mathbf{n}$ such that $(\mathbf{r}, \mathbf{a_i}) = \mathbf{0}$ for all i.

This corollary simultaneously gives the criterion when the leaves of the unstable foliation are transitive in $\mathbb{T}^6$. Indeed, suppose the conditions of the corollary to hold for two independent vectors $\mathbf{a_1}, \mathbf{a_2}$ in W^u. Then the subspace in $\mathbb{R}^6$ generated by these two vectors, along with all its shifts by $\mathbb{Z}^6$, is dense in $\mathbb{R}^6$ and hence $p(W^u)$ is dense when projecting onto $\mathbb{T}^6$.

For the classification problem we have to prove an analog of the Proposition about closure of an unstable foliation leaf.

Proposition

The closure of the 2-dimensional unstable leaf of the fixed point $\hat{O}$ for f_A is either the whole $\mathbb{T}^6$, or a four-dimensional torus, in the first case the leaf is transitive.

For the case of an automorphism with transitive unstable foliation the following assertion is valid which shows that an automorphism with transitive unstable foliation is indeed transitive.

Proposition

An automorphism f_A with transitive unstable foliation is transitive as a diffeomorphism of $\mathbb{T}^6$.

To construct examples of automorphisms of $\mathbb{T}^6$ with two-dimensional transitive unstable foliation, we start again with a degree three monic real polynomial with integer coefficients a, b, c being irreducible over field $\mathbb{Q}$: $P = z^3 + az^2 + bz + c$. Two different cases are considered. In the first case this polynomial should have one real root z_1 lesser than 2 in absolute value and a pair of different real roots z_2, z_3 being greater than 2 in absolute values. In the second case this polynomial should have one real root z_1 lesser than 2 in absolute value and a pair of complex conjugate roots z_2, z_2^* with $|z_2| > 2$. When such polynomial has chosen, the change $z = x + x^{-1}$ and multiplication at x^3 gives the needed 6th-order irreducible polynomial.

As an example, we take the irreducible third order polynomial $P = z^3 - 2z^2 - 8z + 1$ [3]. It has two real roots of absolute value larger than two, and a real root of absolute value lesser than two. Its self-reciprocal polynomial is

$$Q(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1, \qquad (2)$$

which is irreducible over $\mathbb{Q}$, has four real roots λ_1, λ_2, $\lambda_1^{-1}, \lambda_2^{-1}$, $|\lambda_{1,2}| > 1$, and a pair of complex conjugate roots of absolute value 1. All these numbers are algebraic and Q is their minimal polynomial. Its companion matrix A has two-dimensional subspaces W^u, W^s and generates a partially hyperbolic action on $\mathbb{T}^6$ with two-dimensional unstable and stable foliations, and two-dimensional center foliation. Matrix A can be made symplectic w.r.t. the nonstandard symplectic structure generated skew symmetric nondegenerate matrix J in (1) with $a = -2, b = -5, c = -3$, then $\det J = 36$.

For the second case complex root $z_2 = u + iv$, $uv \neq 0$, can be represented as $u = (\rho + \rho^{-1})\cos\alpha$, $u = (\rho - \rho^{-1})\sin\alpha$, where $x_{3,4} = \rho\exp[\pm i\alpha]$, $\rho > 1$, $x_{3,4}^{-1} = \rho^{-1}\exp[\mp i\alpha]$. From these equalities we get the system for finding ρ, α at the given u, v:

$$\frac{u^2}{\rho^2 + \rho^{-2} + 2} + \frac{u^2}{\rho^2 + \rho^{-2} - 2} = 1,\ \tan\alpha = \frac{u(\rho^2+1)}{v(\rho^2-1)}.$$

Introducing variable $s = \rho^2 + \rho^{-2} > 2$, we come to the quadratic equation for s

$$s^2 - (u^2 + v^2)s - 4 + 2(u^2 - v^2) = 0.$$

The monic polynomial $P = z^3 + az^2 + bz + c$ has a unique real root of absolute value less than 2, two other roots have to be complex conjugate. As an example, one can take the polynomial $P = z^3 - 3z^2 + 6z - 1$ with $z_1 \approx 0.182$, $z_{2,3} \approx 1.409 \pm 1.871i$ or $P = z^3 - z - 1$ with $z_1 \approx 1.325, z_{2,3} \approx -0.662 \pm 0.562i$. This gives two reciprocal sixth degree integer irreducible polynomial

$$x^6 - 3x^5 + 9x^4 - 7x^3 + 9x^2 - 3x + 1 \text{ and } x^6 + 2x^4 - x^3 + 2x^2 + 1 \quad (3)$$

Consider now the automorphism in $\mathbb{R}^6$ generated by the companion matrix of the polynomial (2). Its eigenvectors of real roots have the form

$$\mathbf{f}_\lambda = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5)^\top, \ \lambda = \lambda_{1,2}.$$

Thus, no integers vectors $\mathbf{n}$ can exist such that $(\mathbf{n}, \mathbf{f}_{\lambda_\mathbf{k}}) = \mathbf{0}$, $k = 1, 2$, otherwise, at least one of λ_k is a root of a monic integer polynomial of degree five or lesser.

For the case of the polynomial (3) we have for its companion matrix complex conjugate eigenvalues λ, λ^* with $|\lambda| > 1$ with two complex conjugate eigenvectors

$$\mathbf{f}_\lambda = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5)^\top, \ \mathbf{f}_\lambda^*.$$

Their real and imaginary parts provide two real independent vectors $\mathbf{g_r}, \mathbf{g_i}$, $\mathbf{f}_\lambda = \mathbf{g_r} + \mathbf{i}\mathbf{g_i}$. Suppose there is an integer nonzero vector $\mathbf{n} \in \mathbb{Z}^\mathbf{6}$ such that $(\mathbf{n}, \mathbf{g_r}) = \mathbf{0}$ and $(\mathbf{n}, \mathbf{g_i}) = \mathbf{0}$. This implies the equality

$$(\mathbf{n}, \mathbf{g_r} + \mathbf{i}\mathbf{g_i}) = \mathbf{n_1} + \mathbf{n_2}\lambda + \mathbf{n_3}\lambda^\mathbf{2} + \mathbf{n_4}\lambda^\mathbf{3} + \mathbf{n_5}\lambda^\mathbf{4} + \mathbf{n_6}\lambda^\mathbf{5} = \mathbf{0},$$

i.e. λ is the algebraic number of the degree five or lesser. We come to the contradiction. So, in both cases the leaves of the unstable foliation are dense in $\mathbb{T}^6$.

We assume later on that λ is positive, otherwise we may consider A^2. In the space $\mathbb{R}^6$ we have the following orbit structure for the linear map L_A generated by the matrix A. Recall that in the case of a symplectic linear map with a 2-elliptic fixed point O (having two pairs of simple eigenvalues on the unit circle and a pair of reals), there is a four-dimensional center invariant subspace W^c corresponding to two pairs of eigenvalues on the unit circle $\nu_1, \nu_2, \bar{\nu_1}, \bar{\nu_2}, \nu_1 = \exp[i\alpha_1], \nu_2 = \exp[i\alpha_2]$. If $\alpha_i/2\pi \neq \mathbb{Q}, i = 1, 2$ (this is equivalent to $\nu_i^n \neq 1$ for any $n \in \mathbb{Z}$), the subspace W^c is foliated into invariant 2-tori everywhere except two invariant 2-planes corresponding to pairs $\nu_1, \bar{\nu_1}$ and $\nu_2, \bar{\nu_2}$ which are foliated into closed invariant curves. This follows from the fact that the restriction of the map L_A on W^c has two quadratic positive definite invariant functions (integrals) whose joint levels are invariant tori. The restriction of the map L_A to any such torus is conjugated to the shift $\theta_1 = \theta + \alpha_1 \pmod{2\pi}$, $\theta_2 = \theta + \alpha_2 \pmod{2\pi}$. If $m\alpha_1 + n\alpha_2 \neq k$ for any integer (m, n, k), then the shift on the torus is transitive, but if there is the only integer triple (up to a factor) that gives the equality, then the torus itself is foliated into closed invariant curves.

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