Влияние второго запаздывания на локальную динамику

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Problem Statement

Equation with One Delay

$$\dot{x} + x = ax(t - T_1) + f(x, x(t - T_1)), \quad T_1 > 0$$

Equation with Two Delays

$$\dot{x} + x = ax(t - T_1) + bx(t - T_2) + f(x, x(t - T_1), x(t - T_2)), \quad T_1, T_2 > 0$$

1. T_1 is some constant; T_2 is large
2. T_1 and T_2 are large, same order
3. T_1 and T_2 are large, and $T_2 \gg T_1$

Goals

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1. To study local dynamics in neighbourhood of equilibrium for sufficiently small ε.
2. To find asymptotic approximation of solutions

Method

With help of special substitution we'll built new system (without small parameters) – *Quasinormal form*, which solutions give main parts for solutions of principal.

One Delay

Equation with one delay

$$\varepsilon\dot{x} + x = ax(t-1) + f(x, x(t-1))$$

Critical case

$a = -(1 + \varepsilon^\beta a_1), \quad 0 < \beta < 2$

Quasinormal form

$$\frac{\partial u}{\partial \tau} = z^2 \frac{\partial^2 u}{\partial r^2} + a_1 u + f u^3$$

$$u(\tau, r) = -u(\tau, r+1)$$

$$x(t) = \varepsilon^{\beta/2} u\left(\varepsilon^\beta t, \left(\frac{z}{\varepsilon^{1-\beta/2}} + \theta(\varepsilon) + o(1)\right)t\right) + o(\varepsilon^{\beta/2})$$

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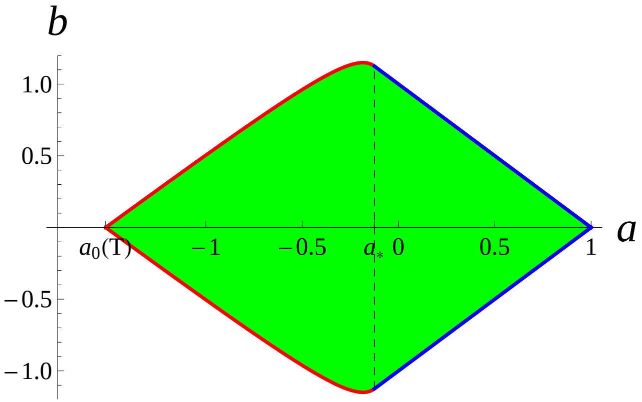
$$u(\tau, r) = -u(\tau, r+1)$$

$$x(t) = \varepsilon^{\beta/2}u\left(\varepsilon^{\beta}t, (\frac{z}{\varepsilon^{1-\beta/2}} + \theta(\varepsilon) + o(1))t\right) + o(\varepsilon^{\beta/2})$$

Large and fixed delays

$$\varepsilon \dot{x} + x = ax(t - \varepsilon T) + bx(t-1) + f(x, x(t-1), x(t-\varepsilon h))$$



Red point

$$\lambda_k = \frac{i\delta}{\varepsilon} + (\theta(\varepsilon)i + \Omega i + 2\pi k)ic_1(\varepsilon) +$$
$$+ \varepsilon^2(c_2(\theta(\varepsilon) + \Omega + 2\pi k)^2 + c_3) + o(1)), \quad c_2 < 0.$$

Function $\theta = \theta(\varepsilon)$

Function $\theta = \theta(\varepsilon)$ takes values in the interval $[0, 2\pi)$ such that $\delta\varepsilon^{-1} + \theta$ is proportional to 2π.

Function $\theta(\varepsilon)$ is

- bounded,
- piecewise continuous,
- takes all its values infinite number of times when $\varepsilon \to 0$.

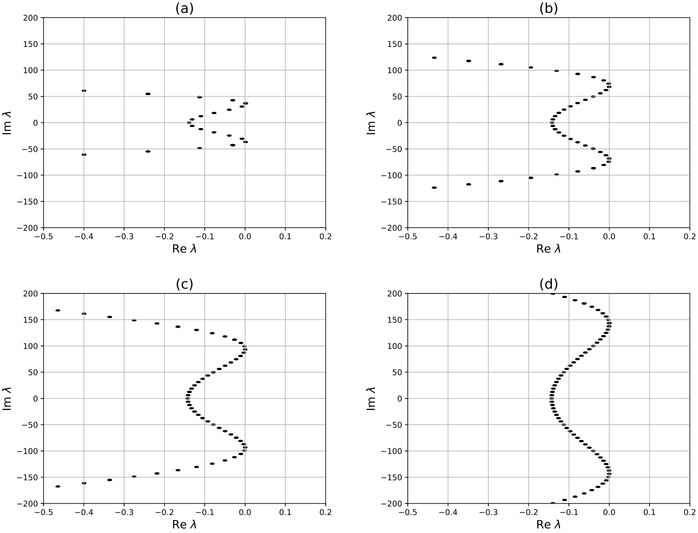


Рис.: Spectrum for a) $\varepsilon = 0.02$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.0075$; d) $\varepsilon = 0.005$.

Quasinormal form

Asymptotic series

$$x = \varepsilon\left(be^{is}u(\tau,r) + \overline{b}e^{-is}\overline{u}(\tau,r)\right) + \varepsilon^2 u_2(\tau,r,s) + \varepsilon^3 u_3(\tau,r,s) + \dots$$

Here we use three scales: "slow" time $\tau = \varepsilon^2 t$, "normal" time $r = c_1(\varepsilon)t$ and "fast" time $s = (\delta\varepsilon^{-1} + (\theta+\Omega)c_1)t$. Unknown functions $u(\tau,r)$, $u_2(\tau,r,s)$ and $u_3(\tau,r,s)$ are periodic by r

$$u(\tau, r+1) \equiv u(\tau, r) \tag{1}$$

and by s.

Quasinormal form

$$\frac{\partial u}{\partial \tau} = -c_2\frac{\partial^2 u}{\partial r^2} + d_1(\theta+\Omega))i\frac{\partial u}{\partial r} + (d_2(\theta+\Omega)^2 + d_3)u + d_4 u|u|^2 \tag{2}$$

Theorem 1

Let $u_0(\tau, r)$ be periodic solution of boundary problem (2), (1) for $\theta = \theta_* \in [0, 2\pi)$. Consider sequence $\varepsilon_n \to 0$ such that $\theta(\varepsilon_n) = \theta_*$. Then

$$x(t) = \varepsilon\Big[bu_0(\varepsilon^2 t, c_1(\varepsilon)t)\exp(i(\delta\varepsilon^{-1} + \theta_* + \Omega)t) + \\ + \overline{b}\overline{u_0}(\varepsilon^2 t, c_1(\varepsilon)t)\exp(-i(\delta\varepsilon^{-1} + \theta_* + \Omega)t)\Big]$$

produce asymptotically small discrepancy of the order $O(\varepsilon^2)$ uniformly for $t \in [0, +\infty)$ in the initial delay system for $\varepsilon = \varepsilon_n$.

When $\varepsilon \to 0$ value $\theta = \theta(\varepsilon)$ takes each value from the interval $[0, 2\pi)$ infinite number of times, so solutions of (2), (1) and its stability may change an infinite number of times. This allows us to conclude about the possibility of an infinite process of direct and inverse bifurcations in initial delay equatiion when $\varepsilon \to 0$.

Example: Logistic equation with two delays

Problem to research

$$\varepsilon \dot{u} = [au(t-1) - u(t-\varepsilon h)](1+u)$$

Parameters

$$h = 1 \quad a = 0.4297 + \varepsilon^2,$$

$$\theta : \frac{1.31}{\varepsilon} + \theta = 2\pi n$$

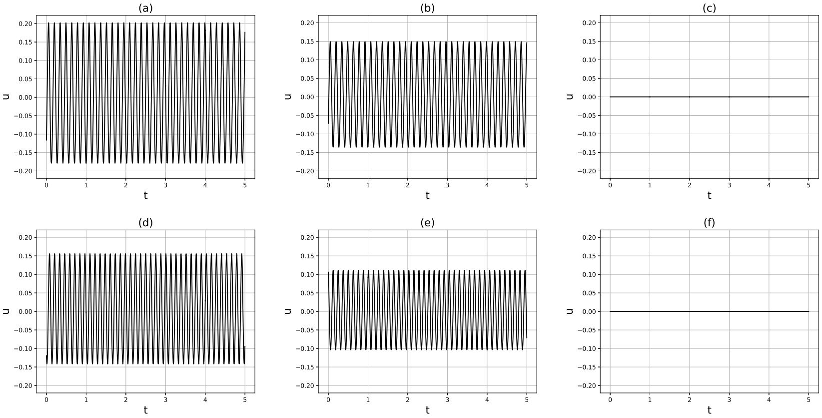


Рис.: Graphics of solutions for a) $\varepsilon = 0.03$, b) $\varepsilon = 0.0298$, c) $\varepsilon = 0.0295 - 0.027$, d) $\varepsilon = 0.0267$, e) $\varepsilon = 0.02605$, f) $\varepsilon = 0.0255$.

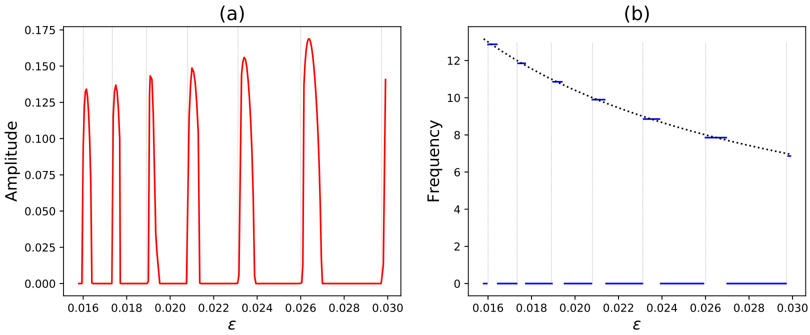


Рис.: Bifurcation diagram for small ε. (a) Dependence of amplitude of stable regime on ε; (b) dependence of frequency of solution on ε. Dotted black curve is plot of $\delta(2\pi\varepsilon)^{-1}$. Vertical dashed grey lines mark points of discontinuity of $\theta(\varepsilon)$.

Two large delays

Equation with Two Delays

$$\dot{x}+x = ax(t-T_1)+bx(t-T_2)+f(x,x(t-T_1),x(t-T_2)), \quad T_1, T_2 > 0$$

Main Assumptions

$$T_1 = \frac{1}{\varepsilon}, \qquad T_2 = \frac{k+\varepsilon^{\alpha}k_1}{\varepsilon}, \qquad k \geq 1, \alpha > 0,$$

$$0 < \varepsilon \ll 1.$$

$$\varepsilon\dot{x} + x = ax(t-1) + bx(t-k-\varepsilon^{\alpha}k_1) + f\left(x, x(t-1), x(t-k-\varepsilon^{\alpha}k_1)\right)$$

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Linear Analysis

$$\varepsilon\dot{x} + x = ax(t-1) + bx\big(t - k - \varepsilon^{\alpha}k_1\big)$$

Characteristic Equation

$$\varepsilon\lambda + 1 = a\exp(-\lambda) + b\exp(-\lambda(k + \varepsilon^{\alpha}k_1))$$

For irrational k critical case is $|a| + |b| = 1$.
Let $k = \frac{m}{n}$

- $\alpha > 1$ – stability region don't depend on k_1
- $\alpha = 1$ – stability region depend on k_1
- $\alpha < 1$ – stability region is $|a| + |b| < 1$

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Case $\alpha > 1$

$$R(\Omega) = a\cos\Omega + b\cos k\Omega.$$

$$R_0 = R_0(a, b) = \max R(\Omega)$$

$R_0 < 1$ – equilibrium is stable
$R_0 > 1$ – equilibrium is unstable
$R_0 = 1$ – critical case

Let $R(\Omega_0) = 1$,

$$\omega_0 = -a\sin\Omega_0 - b\sin(k\Omega_0),$$

Characteristic equation has infinite set of roots

$$\lambda_p = \frac{\omega_0}{\varepsilon} i + \theta(\varepsilon) i + \Omega_0 i + 2\pi p i + o(1), \quad p \in \mathbb{Z}$$

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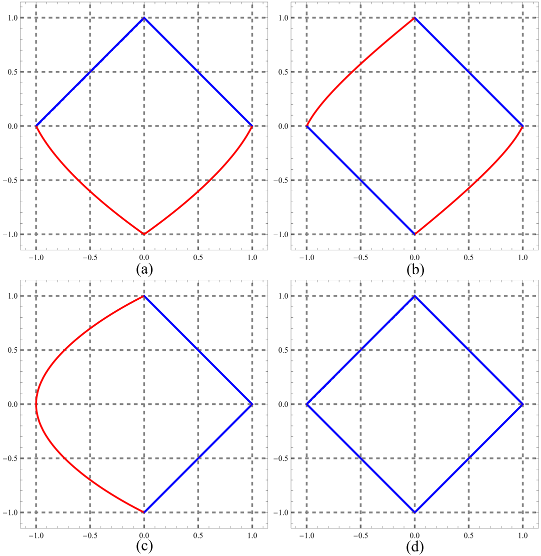
Case $\alpha > 1$



(a) $k = 4/3$, (b) $k = 5/3$, (c) $k = 3/2$, (d) irrational k

Case $\alpha > 1$. Nonlinear analysis

$$a = a_0 + \varepsilon^\beta a_1, \quad b = b_0 + \varepsilon^\beta b_1, \quad \beta > 0$$

a_0 and b_0 such that $R_0(a_0, b_0) = 1$

Slow oscillations

Quasinormal form - parabolic equation (z – arbitrary)

$$\frac{\partial u}{\partial \tau} = z^2 d_1 \frac{\partial^2 u}{\partial r^2} + d_2 u + d_3 u^3, \quad u(\tau, r+1) = -u(\tau, r)$$

Theorem

Let $u(\tau, r)$ be periodic solution of QNF. Then function

$$x_*(t, \varepsilon) = \varepsilon^{\beta/2} u_*(\varepsilon^\beta t, n\left(\frac{z}{\varepsilon^{1-\beta/2}} + \theta_z + o(1)\right)t).$$

satisfy initial equation with asymptotically small residual ($O(\varepsilon^\beta)$ uniformly with respect to all $t \geq 0$).

Case $\alpha > 1$. Nonlinear analysis

Rapid oscillations

Quasinormal form - parabolic equation (z – arbitrary)

$$\frac{\partial u}{\partial \tau} = z^2 D_1 \frac{\partial^2 u}{\partial r^2} + D_2 u + D_3 u|u|^2, \quad u(\tau, r+1) = u(\tau, r)$$

Theorem

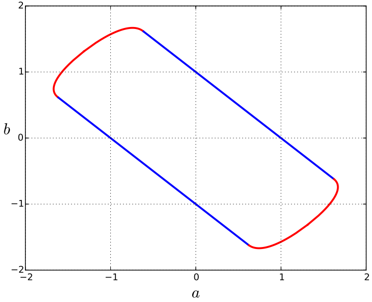
Let $u(\tau, r)$ be periodic solution of QNF. Then function

$$x_*(t,\varepsilon) = \varepsilon^{\beta} \Bigg(\exp(\frac{i\omega_0 t}{\varepsilon}) u_*(\varepsilon^{\beta} t, n\left(\frac{z}{\varepsilon^{1-\beta/2}} + \theta_z + o(1)\right) t) + \\ + \exp(-\frac{i\omega_0 t}{\varepsilon}) \overline{u}_*(\varepsilon^{\beta} t, n\left(\frac{z}{\varepsilon^{1-\beta/2}} + \theta_z + o(1)\right) t) \Bigg)$$

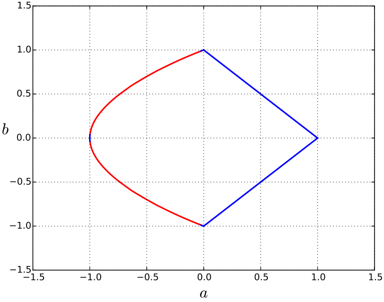
satisfy initial equation with asymptotically small residual ($O(\varepsilon^{\beta})$ uniformly with respect to all $t \geq 0$).

Results for $\alpha = 1$

$$\begin{cases} 1 = a\cos\Omega + b\cos(k\Omega + k_1\omega), \\ -\omega = a\sin\Omega + b\sin(k\Omega + k_1\omega) \end{cases}$$



$k = 1,\ k_1 = 1$



$k = 3/2,\ k_1 = 1$

Results for $\alpha = 1$

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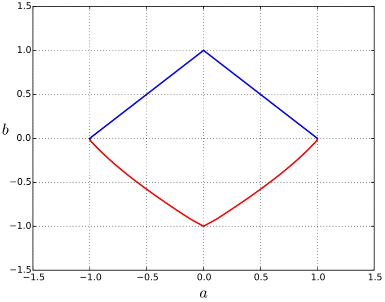
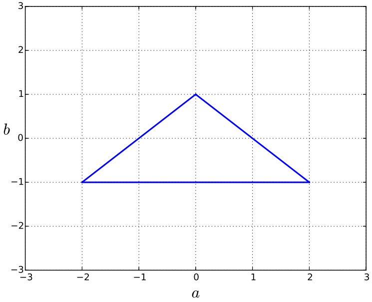


$k = 4/3,\ k_1 = -1$



$k = 2,\ k_1 = 1$

Case $\alpha = 1$. Nonlinear analysis

The results in the main repeat the results for the case $\alpha > 1$.
The only difference: the quasinormal forms always are complex parabolic equations.

Case $\alpha < 1$

Critical case

$$|a_0| + |b_0| = 1$$

Quasinormal form

$$\frac{\partial u}{\partial \tau} = z_1^2 d_1 \frac{\partial^2 u}{\partial r^2} + z_1 z_2 d_2 \frac{\partial^2 u}{\partial r \partial s} + z_2^2 d_3 \frac{\partial^2 u}{\partial s^2} + A_1 u + B_1 u^3,$$

$$u(\tau, r, s+1) = \begin{cases} u(\tau, r, s), & a_0 \geq 0, \\ -u(\tau, r, s), & a_0 < 0; \end{cases}$$

$$u(\tau, r+1, s) = \begin{cases} u(\tau, r, s), & b_0 \geq 0, \\ -u(\tau, r, s), & b_0 < 0. \end{cases}$$

Dependance on small parameter

The coefficients of the quasinormal form depends on $\theta = \theta(\varepsilon)$.
$\varepsilon_l \to 0$ as $l \to \infty$ such that $\theta(\varepsilon_l) = Q$

Theorem (case $\beta = 2\alpha < 1$)

Let for $\theta = Q$ QNF has periodic solution $u(\tau, s, r)$. Then

$$x_*(t) = \varepsilon^\alpha u(\varepsilon^{2\alpha} t, r, s)$$

satisfy initial equation with asymptotically small residual for $\varepsilon = \varepsilon_l$. Here
$s = \left(-n\left(\frac{m}{\varepsilon^\alpha k_1 n^2} + \theta_1\right) + 1 + \varepsilon^\alpha k_1 |b_0|(1 + n\theta_1)\right) t,$

$$r = \left(n\left[\left(\frac{z}{\varepsilon^\gamma} + \theta_z\right)\frac{1}{\varepsilon^\alpha k_1 n} + \theta\right] - \frac{z}{dk_1}\varepsilon^\alpha - \frac{k_1|b_0|n\theta}{d}\varepsilon^\alpha + o(\varepsilon^\alpha)\right) t,$$

Conclusion

- The local dynamics of equation with two large delays is studied.
- In critical cases the problem is reduced to a quasinormal form - a parabolic equation. It does not contain small parameters or depends on them regularly.
- Explicit formulas are obtained for the solutions of the original equation in terms of solutions for QNF.
- The dependence of parameters on $\theta(\varepsilon)$ may lead to an infinite process of direct and inverse bifurcations for $\varepsilon \to 0$.
- Dependence of results on algebraic properties of k.