# Dynamics of the coupled oscillator model

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# Statement of the problem

## Model

$$\begin{cases} \dot{u}_0 + u_0 = \lambda F(u_0(t-T)) + \gamma(u_1 - u_0), \\ \dot{u}_1 + u_1 = \lambda F(u_1(t-T)) + \gamma(u_0 - u_1). \end{cases} \tag{1}$$

## Main assumptions

$$T > 0, \quad \lambda \gg 1, \quad 0 < \gamma \ll 1,$$

$$F(u) = \begin{cases} f(u), & |u| \le p, \\ 0, & |u| > p. \end{cases}$$

Function $f(u)$ satisfies the following conditions on the segment $u \in [-p, p]$

1) $f(u)$ is smooth;
2) $f(p) = f'(p) = f(-p) = f'(-p) = 0$;
3) $f''(p) \neq 0, f''(-p) \neq 0$;
4) $u \cdot f(u) > 0$ for $0 < |u| < p$.

# Strength of coupling

## Four cases

$$
\begin{array}{ll}
1) & \gamma = \frac{\gamma_1}{\ln \lambda}, \\
2) & \gamma = \frac{\gamma_1}{\lambda^{\alpha} \ln \lambda}, \quad 0 < \alpha \leq 1/2, \\
3) & \gamma = \frac{\gamma_1}{\lambda^{\alpha}}, \quad 1/2 < \alpha < 1, \\
4) & \gamma = \frac{\gamma_1}{\lambda},
\end{array}
$$

where $\gamma_1$ is positive constant, $\lambda \gg 1$.

# Method of investigation

1. We construct a set of initial conditions $S(m, k, x)$.
2. For all initial functions from set $S(m, k, x)$ we calculate asymptotics at $\lambda \to \infty$ of solutions of system (1).
3. We prove that there exists a value $(\bar{m}, \bar{k}, \bar{x})$ such that each considered solution after certain time $t_*$ belongs to the set $S(\bar{m}, \bar{k}, \bar{x})$, i.e., there exists operator of translation along trajectories $\Pi$ such that

$$\Pi S(m, k, x) \subset S(\bar{m}, \bar{k}, \bar{x}).$$

4. We get analytical dependence of $\bar{m}$, $\bar{k}$, $\bar{x}$ on $m$, $k$, $x$ in the form

$$(\bar{m}, \bar{k}, \bar{x})^T = \psi((m, k, x)^T) + o(1)$$

at $\lambda \to \infty$.

# Method of investigation

5. We prove that rough cycle of the mapping

$$(\bar{m}, \bar{k}, \bar{x})^T = \psi((m, k, x)^T) \tag{2}$$

corresponds to a relaxation cycle of initial system and these cycles have the same stability properties.
6. By rough stable cycles of mapping (2) we construct exponentially orbitally stable relaxation cycles of initial system.

# Set of initial conditions $S(m, k, x)$

**Set $S(m, k, x)$**

$$\begin{array}{lll} u_m(s), & u_{1-m}(s) & \in C_{[-T,0]}(\mathbb{R}) \\ |u_m(s)| > p, & |u_{1-m}(s)| > p & \text{for } s \in [-T, 0), \\ u_m(0) = kp, & u_{1-m}(0) = xp. & \end{array}$$

Values of parameters:
$m$ equals either 0 or 1, $k$ equals either 1 or $-1$,
$|x| > 1$, $kx > 0$.

# Idea of selecting set $S(m, k, x)$

**Model**

$$\begin{cases} \dot{u}_0 + u_0 = \lambda F(u_0(t-T)) + \gamma(u_1 - u_0), \\ \dot{u}_1 + u_1 = \lambda F(u_1(t-T)) + \gamma(u_0 - u_1), \end{cases} \tag{1}$$

Suppose that $|u_0(t)| \geq p$ and $|u_1(t)| \geq p$ on some time segment of the length $T$ ($t \in [\tau - T, \tau]$), then on the segment $t \in [\tau, \tau + T]$ $F(u_0(t-T)) \equiv F(u_1(t-T)) \equiv 0$ and system (1) has form

$$\begin{cases} \dot{u}_0 + u_0 = \gamma(u_1 - u_0), \\ \dot{u}_1 + u_1 = \gamma(u_0 - u_1). \end{cases} \tag{3}$$

# Asymptotics of solutions

## Model

$$\begin{cases} \dot{u}_m + u_m = \lambda F(u_m(t-T)) + \gamma(u_{1-m} - u_m), \\ \dot{u}_{1-m} + u_{1-m} = \lambda F(u_{1-m}(t-T)) + \gamma(u_m - u_{1-m}), \end{cases} \tag{1}$$

$$\begin{cases} \dot{u}_m + u_m = \lambda F(u_m(t-T)) + \gamma(u_{1-m} - u_m), \\ \dot{u}_{1-m} + u_{1-m} = \gamma(u_m - u_{1-m}), \end{cases} \tag{2}$$

$$\begin{cases} \dot{u}_m + u_m = \gamma(u_{1-m} - u_m), \\ \dot{u}_{1-m} + u_{1-m} = \gamma(u_m - u_{1-m}), \end{cases} \tag{3}$$

# Constructing of mapping

Let $t_m > 2T$ and $t_{1-m} > 2T$ be the first moments of time such that

$$|u_m(t_m)| = p, \quad |u_{1-m}(t_{1-m})| = p.$$

Let $t_* = \min\{t_m, t_{1-m}\}$.
Then

$$u_{\bar{m}}(t_*) = \bar{k}p, \quad u_{1-\bar{m}}(t_*) = \bar{x}p. \tag{4}$$

We obtain that

$$t_* = (1 + o(1)) \ln \lambda \text{ as } \lambda \to +\infty$$

$$|u_{\bar{m}}(s + t_*)| > p, \quad |u_{1-\bar{m}}(s + t_*)| > p \text{ if } s \in [-T, 0) \tag{5}$$

# Main result

$$\begin{cases} \bar{m} = M(k, x), \\ \bar{k} = k, \\ \bar{x} = X(k, x) + o(1). \end{cases} \tag{6}$$

$$\bar{x} = X(k, x). \tag{7}$$

**Theorem of correspondence**

Suppose mapping (6) has a cycle of period $r$ such that absolute value of multiplier of corresponding cycle of mapping (7) is less than 1. Then for all sufficiently large $\lambda > 0$ system (1) has exponentially orbitally stable cycle with asymptotically large amplitude and period.

# Case $\gamma = \frac{\gamma_1}{\ln\lambda}$. Mapping.

Let

$$g_k(t) = \int\limits_T^t \exp(s-t) f(kp\exp(T-s))ds,$$

$$R(x) = \begin{cases} \dfrac{1+\exp(-2\gamma_1)}{1-\exp(-2\gamma_1)}, & |x|e^{-T} > 1, \\ \dfrac{g_k(2T)(1+\exp(-2\gamma_1)) + g(2T,x)(1-\exp(-2\gamma_1))}{g_k(2T)(1-\exp(-2\gamma_1)) + g(2T,x)(1+\exp(-2\gamma_1))}, & |x|e^{-T} < 1. \end{cases}$$

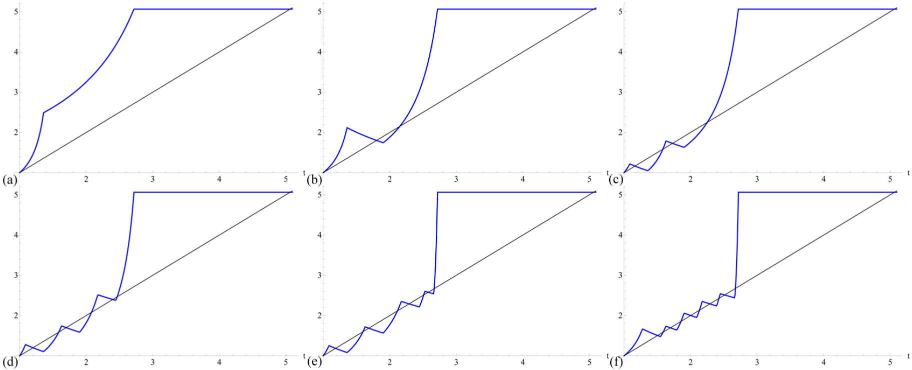
Then constructed mapping has form

$$\bar{m} = \begin{cases} m & \text{if} \quad 0 < R(x) < 1, \\ 1-m & \text{if} \quad R(x) > 1, \end{cases}$$

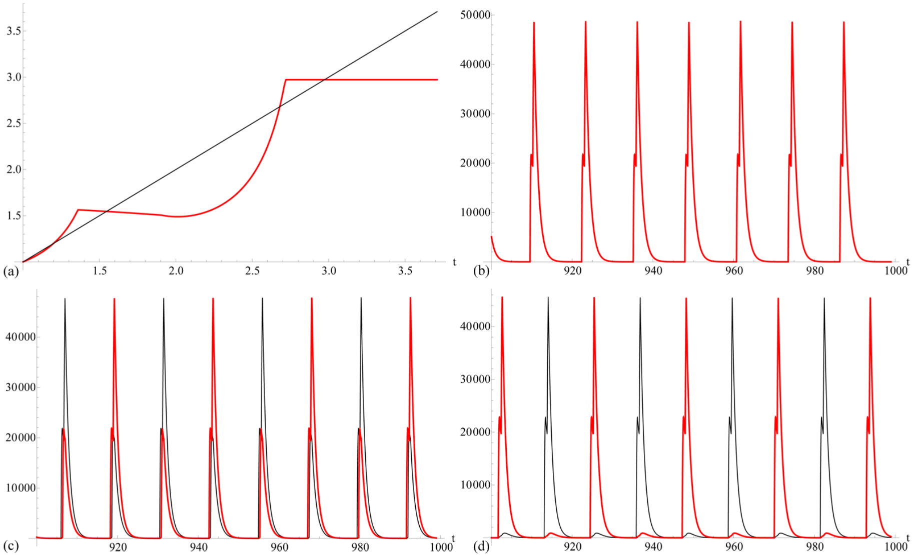
$$\bar{k} = k,$$

$$\bar{x} = k\max\{R(x), 1/R(x)\}.$$

# Case $\gamma = \frac{\gamma_1}{\ln \lambda}$. Dynamics of $x$-component of mapping



# Case $\gamma = \frac{\gamma_1}{\ln \lambda}$. Solutions of system (1)



# Case $\gamma = \frac{\gamma_1}{\lambda^\alpha \ln \lambda}$, $0 < \alpha \leq 1/2$. Mapping.

## Case $0 < \alpha < 1/2$.

$$\begin{cases} \bar{m} = 1 - m, \\ \bar{k} = k, \\ \bar{x} = \dfrac{\operatorname{sign}(g_k(2T))}{\gamma_1(1-\alpha)} + o(1). \end{cases}$$

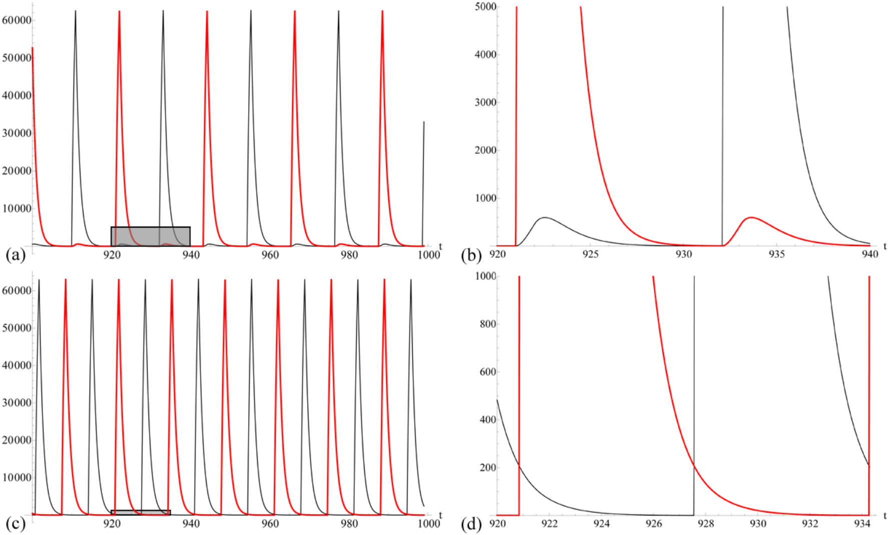
## Case $\alpha = 1/2$.

$$\begin{cases} \bar{m} = 1 - m, \\ \bar{k} = k, \\ \bar{x} = \dfrac{g_k(2T)}{|xpe^{-2T} + \frac{\gamma_1}{2} g_k(2T)|} + o(1). \end{cases}$$

where

$$g_k(t) = \int_T^t \exp(s-t) f(kp \exp(T-s)) ds.$$

# Case $\gamma = \frac{\gamma_1}{\lambda^\alpha \ln \lambda}$, $0 < \alpha \leq 1/2$. Solutions of system



# Case $\gamma = \frac{\gamma_1}{\lambda^\alpha}$, $1/2 < \alpha < 1$. Mapping.

## Set $S(m,k,x,\beta)$

$$\begin{array}{lll} u_m(s), & u_{1-m}(s) & \in C_{[-T,0]}(\mathbb{R}) \\ |u_m(s)| > p, & |u_{1-m}(s)| > p & \text{for } s \in [-T,0), \\ u_m(0) = kp, & u_{1-m}(0) = xp\lambda^\beta, & \end{array}$$

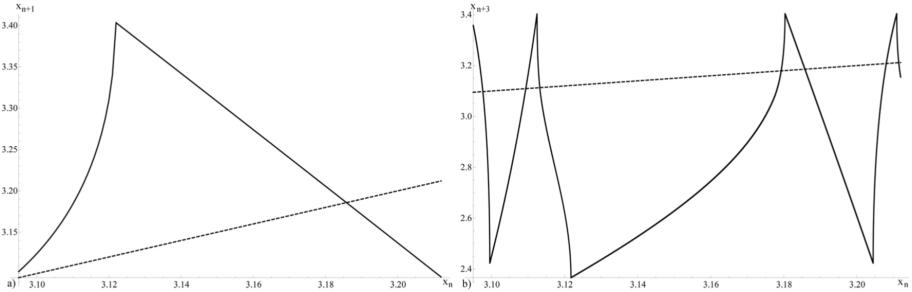
where $x \neq 0$, $1-\alpha < \beta < \alpha$.

## Mapping

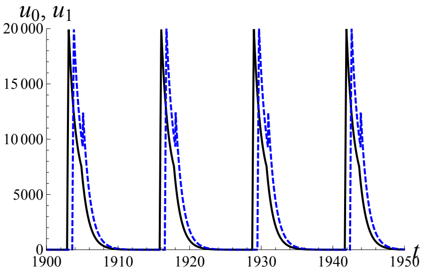
$$\bar{m} = 1 - m,$$
$$\bar{k} = k,$$
$$\bar{\beta} = 1 - \beta,$$
$$\bar{x} = \frac{g_k(2T)}{|x|pe^{-2T}}.$$

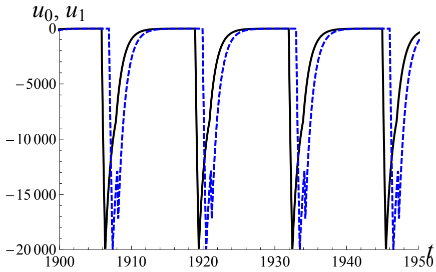
All solutions of mapping are non-rough cycles =>
no theorem of correspondence.

# Case $\gamma = \frac{\gamma_1}{\lambda}$. Dynamics of $x$-coordinate of mapping



# Case $\gamma = \frac{\gamma_1}{\lambda}$. Stable cycles of system (1)



# Dependence of dynamics of system (1) on $\gamma$

## Stable solutions for different orders of $\gamma$ at $\lambda \to +\infty$

**Case** $\gamma = O(\frac{1}{\ln \lambda})$
Homogeneneous cycle (not always stable), $\geq 1$ inhomogeneous (antiphase) cycle

**Case** $\gamma = O(\frac{1}{\lambda^\alpha \ln \lambda})$, $0 < \alpha \leq 1/2$
1 inhomogeneous (antiphase) cycle

**Case** $\gamma = O(\frac{1}{\lambda^\alpha})$, $1/2 < \alpha < 1$
2-parameter family of non-rough asymptotic (by the discrepancy) inhomogeneous (antiphase) cycles

**Case** $\gamma = O(\frac{1}{\lambda})$
$\geq 0$ inhomogeneous (not always antiphase) cycles

# Generalizations

1. Extension of class of functions $F$.
2. $N$-dimensional systems ($N > 2$).
3. System of equations, where feedback functions and coupling parameters are not the same in all equations of system.
4. This method is appropriate to study dynamics of systems with delayed control

$$\dot{u}_0 + u_0 = \lambda F(u_0(t - T)) + \gamma(u_1(t - h) - u_0),$$
$$\dot{u}_1 + u_1 = \lambda F(u_1(t - T)) + \gamma(u_0(t - h) - u_1).$$

# Conclusions

1. Nonlocal dynamics of a system of two nonlinear differential equations with delay has studied.
2. Studying of existence and stability of relaxation cycles of initial infinite dimensional system is reduced to studying dynamics of constructed three-dimensional mapping.
3. By stable rough cycles of constructed mapping we get exponentially orbitally stable relaxation cycles of initial system. Their asymptotics and period, number of coexisting stable cycles depend on strength of coupling.

# Thank you for your attention!