On Hopf-type equations arising in the problem of integrable geodesic flows.

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Quasi-linear systems of PDEs

Quasi-linear systems of the form

$$A(U)U_x + B(U)U_y = 0,$$

$$U_t = A(U)U_x, \qquad U = (u_1, \ldots, u_n)^T$$

appears in such areas like

- gas-dynamics
- non-linear elasticity
- integrable geodesic flows on 2-surfaces

and many others.

Hopf equation (inviscid Burgers' equation)

Consider the following equation

$$u_t + uu_x = 0.$$

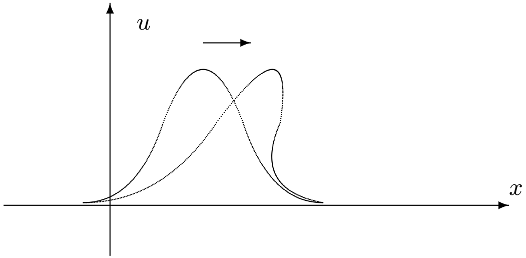
on the unknown function $u(x,t)$. The characteristic are straight lines given by the equation

$$x = ut + \xi,$$

u is constant along characteristics. The solution to the Cauchy problem $u|_{t=0} = g(x)$ is given by the implicit formula

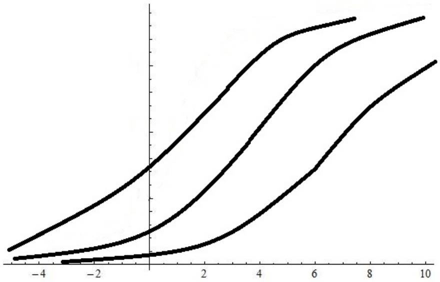
$$u(x,t) = g(x - ut).$$

It follows from this formula that the higher any point is placed, the faster it is. In a typical situation we have the following picture:



Hopf equation (inviscid Burgers' equation)

A solution in the form of **the rarefaction wave**:



Integrable geodesic flows on a 2-surface

Let

$$ds^2 = g_{ij}(x)dx^i dx^j, \qquad i, j = 1, 2$$

be a Riemannian metric on a 2-surface $\mathbb{M}^2$. **The geodesic flow** is given by the following Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad H = \frac{1}{2} g^{ij} p_i p_j.$$

The first integral of this flow is any function $F : T^*\mathbb{M}^2 \to \mathbb{R}$ such that

$$\dot{F} = \{F, H\} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right) \equiv 0.$$

Polynomial integrals of low degrees

G.D. Birkhoff (1927): [1] Local linear and quadratic integrals.

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad F_1 = p_1,$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

V.N. Kolokoltsov (1982): [2] Linear and quadratic integrals on the 2-sphere and the 2-torus.

The 2-sphere

There exist integrable cases with polynomial first integrals of degrees **1, 2, 3, 4**.

Conjecture on possible degrees (V.V. Kozlov, 1993).

The maximal possible degree of any *irreducible* polynomial in momenta first integral of the geodesic flow on a surface of a genus g seems to be not larger than $4 - 2g$.

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[1] Birkhoff G.D., *Dynamical systems.* Vol. 9. American Mathematical Society Colloquium Publications, New York (1927).

[2] Kolokol'tsov V.N., *Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial in the velocities*, Math. USSR Izv, **46**:5 (1982), 291 – 306.

Cubic integral on the 2-torus [3]

The existence of a cubic integral of a geodesic flow on the 2-torus is equivalent to existence of solutions to the following system:

$$\frac{\partial(uv)}{\partial z} + \frac{\partial(u\overline{v})}{\partial \overline{z}} = 0, \qquad \frac{\partial v}{\partial \overline{z}} = \frac{\partial u}{\partial z}. \qquad (*)$$

In the soliton theory there is the well-known Veselov-Novikov equation (VN):

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial z^3} + \frac{\partial^3 u}{\partial \overline{z}^3} + \frac{\partial(uv)}{\partial z} + \frac{\partial(u\overline{v})}{\partial \overline{z}},$$

where the function v satisfies the following equation

$$\frac{\partial v}{\partial \overline{z}} = \frac{\partial u}{\partial z},$$

which is uniquely solvable in the class of fast-decaying functions on $\mathbb{R}^2$ and in the class of functions on the 2-torus with zero average.

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[3] Taimanov I.A., *On first integrals of geodesic flows on a two-torus*, Proc. Steklov Inst. Math., **295** (2016), 225 – 242.

Reductions of the Veselov-Novikov equation [4]

1) If $u = u(y)$, $v = -u$, then any function $u(y)$ defines a stationary solution to the VN equation. This corresponds to the metric $ds^2 = \Lambda(y)(dx^2 + dy^2)$.

2) If $u = u(x)$, $v = u$, then the VN equation goes to the KdV equation:

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + 2u\frac{\partial u}{\partial x}.$$

So one can consider the VN equation as the two-dimensionalization of the KdV equation. In the dispersionless limit

$$t \to \epsilon^{-1}t, \qquad x \to \epsilon^{-1}x, \qquad \epsilon \to 0$$

the KdV equation goes to the Hopf equation $\frac{\partial u}{\partial t} = 2u\frac{\partial u}{\partial x}$.

To sum it up, one can consider the system (*) as the the equation for stationary solutions to the two-dimensionalization of the Hopf equation = the dispersionless limit of the VN equation.

[4] Taimanov I.A., *On first integrals of geodesic flows on a two-torus*, Proc. Steklov Inst. Math., **295** (2016), 225 – 242.

Natural mechanical systems on the 2-torus

Consider a Hamiltonian system with the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + V(x, y),$$

where V is an analytic function on the plane $\mathbb{R}^2$ with the period lattice $\Lambda = \mathbb{Z}^2$.

Theorem (A., 2020) [5]
Suppose that this natural mechanical system admits an additional first integral of the form

$$F = \frac{a(x, y)p_1 + b(x, y)p_2 + c(x, y)}{f(x, y)p_1 + g(x, y)p_2 + h(x, y)}$$

with analytic periodic coefficients.
Then the potential has the form $V(x, y) = V_1(\alpha x + \beta y)$ *and, consequently, there exists a linear in momenta first integral* $F_1 = \alpha p_2 - \beta p_1$.

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[5] Agapov S.V., *Rational integrals of a natural mechanical system on the 2-torus.* Sib. Math. Journ. **61**:2 (2020), 199 – 207.

Natural mechanical systems on the 2-torus

One needs to search for meromorphic solutions to the Hopf equation:

$$\left(\frac{f}{g}\right)_x - \left(\frac{f}{g}\right)\left(\frac{f}{g}\right)_y = 0.$$

There exist non-constant non-periodic meromorphic solutions: [6]

$$\frac{f}{g} = -\frac{y + c_1}{x + c_2},$$

where c_1, c_2 are arbitrary constants.

[6] Saleeby E. G. Meromorphic solutions of generalized inviscid Burgers' equations and a family of quadratic PDEs // J. Math. Anal. Appl. 2015. V. 425, N 1. P. 508–519.

Magnetic geodesic flow (systems with gyroscopic forces)

$$\frac{d}{dt}y^i = \{y^i, H(y)\}_{mg}, \qquad i = 1, \ldots, N.$$

In coordinates $(y^1, \ldots, y^N) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$, $N = 2n$ **the magnetic Poisson bracket** is given by

$$\{x^i, p_j\}_{mg} = \delta^i_j, \qquad \{x^i, x^j\}_{mg} = 0, \qquad \{p_i, p_j\}_{mg} = \Omega_{ij}(x).$$

Consider a Hamiltonian system

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2$$

on a 2-surface in a non-zero magnetic field given by the closed 2-form $\omega = \Omega(x) dx^1 \wedge dx^2$ with $H = \frac{1}{2} g^{ij}(x) p_i p_j$ and the Poisson bracket:

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial F}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial x^i} \right) + \Omega(x^1, x^2) \left(\frac{\partial F}{\partial p_1} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial p_1} \right).$$

The only known examples of integrable geodesic flows on the 2-torus on all energy levels

An integrable geodesic flow

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad F_1 = p_1;$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

An integrable magnetic geodesic flow

$$ds^2 = dx^2 + dy^2, \quad \omega = B dx \wedge dy, \quad B = const \neq 0, \quad F_1 = \cos\left(\frac{p_1}{B} - y\right);$$

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y) dx \wedge dy, \qquad F_1 = p_1 + u(y).$$

Quadratic integrals on several energy levels

Theorem (I.A. Taimanov, 2016) [7]
Consider the magnetic flow of the Riemannian metric $ds^2 = \Lambda(x,y)(dx^2+dy^2)$ *with the non-zero magnetic form* ω*. Suppose the magnetic flow admits a polynomial in momenta first integral* F_2 *(with analytic periodic coefficients) of degree* $N=2$ *at* $M=2$ *different energy levels. Then in some coordinates we have*

$$ds^2 = \Lambda(y)(dx^2+dy^2), \qquad \omega = -u'(y)dx\wedge dy$$

so there exists another integral F_1 *which is linear in momenta:* $F_1 = p_1 + u(y)$.

The following equation on the unknown function $f(x,y)$ appears in the proof:

$$\frac{\partial f}{\partial y} = Cf^2\frac{\partial f}{\partial x},$$

here C is a constant.

[7] Taimanov I.A., *On first integrals of geodesic flows on a two-torus.* Proc. Steklov Inst. Math., **295** (2016), 225 – 242.

Polynomial integrals on several energy levels

Recently this result was generalized on the case of an arbitrarily high degree integrals.

Butler, Naqvi (2020) — $N = 3,\ M = 2.$

A., Valyuzhenich (2019) [8] and **A., Valyuzhenich, Shubin (2021)** [9] — the case of an integral of an arbitrary degree N at $M = \frac{N+1}{2}$ or $M = \frac{N+2}{2}$ different energy levels (depending on the parity of N).

The proof is more or less straightforward but requires complicated calculations, certain combinatoric technique and the analysis of solutions to the system of equations on two unknown functions $f(x,y), g(x,y)$ of the following kind:

$$f_x = g_y,\ P(f,g) = 0.$$

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[8] S. Agapov and A. Valyuzhenich, *Polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels*, Disc. Cont. Dyn. Syst. - Series A, **39**:11 (2019), 6565 – 6583.

[9] S.V. Agapov, A.A. Valyuzhenich and V.V. Shubin, *Some remarks on high degree polynomial integrals of the magnetic geodesic flow on the two-dimensional torus*, Sib. Math. Journ., **62**:4 (2021), 581 – 585.

Scheme of the proof

There is a polynomial P such that $P(f,g)=0$. For instance,

$$N=3, \quad K_1 g + K_2 f + \frac{1}{3}g^3 - gf^2 - K_3 = 0;$$

$$N=4, \quad fg(f^2-g^2) - 2K_1 fg + K_2(f^2-g^2) + K_3 g + K_4 f - K_5 = 0,$$

where K_j are arbitrary constants. Also we have $f_x = g_y$.
So we obtain:

$$g_y - f'(g) g_x = 0.$$

It can be shown that either one of the functions f, g must be constant or

$$f(x,y) = \widetilde{f}(x-ky), \qquad g(x,y) = \widetilde{g}(x-ky)$$

and $\widetilde{f}(x-ky) \equiv -k\widetilde{g}(x-ky) + k_0$, where k, k_0 are constants. In both cases we managed to prove that the magnetic field $\Omega(x,y)$ and the metric $\Lambda(x,y)$ depend only on one argument $(x-ky)$.

Back to the geodesic flows

Introduce the coordinates $ds^2 = g(t,x)dt^2 + dx^2$. The Hamiltonian takes the form $H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$. Suppose that there exists a quadratic integral of the form[10]

$$F = \frac{a(t,x)}{g(t,x)^2}p_1^2 + \frac{a_1(t,x)}{g(t,x)}p_1p_2 + a_2(t,x)p_2^2.$$

Assume that $a_1(t,x) = g(t,x)$, $a_2(t,x) \equiv 1$. Then

$$a_t + gg_x = 0, \qquad g_t + ga_x + 2(1-a)g_x = 0.$$

This system is **semi-Hamiltonian**[11]. It can be written in the form of conservation laws:

$$(a_0)_t + \left(\frac{g^2}{2}\right)_x = 0 \quad \left(\frac{1}{2g^2}\right)_t + \left(\frac{1-a_0}{g^2}\right)_x = 0.$$

[10] M. Bialy, A.E. Mironov, *Rich quasi-linear system for integrable geodesic flows on 2-torus,* Discrete and Continuous Dynamical Systems - Series A, **29**:1 (2011), 81 – 90.

[11] S.P. Tsarev, *The geometry of hamiltonian systems of hydrodynamic type. The generalized hodograph method,* Izv. Math., **37**:2 (1991), 397 – 419.

Back to the geodesic flows

It also admits Riemann invariants $R_1(t,x), R_2(t,x)$:

$$a(t,x) = R_1(t,x) + R_2(t,x) - 1, \qquad g(t,x)^2 = -4(R_1(t,x) - 1)(R_2(t,x) - 1)$$

and can be diagonalized:

$$(r_1)_t + 2r_2(r_1)_x = 0,$$

$$(r_2)_t + 2r_1(r_2)_x = 0,$$

where $r_1(t,x) = 1 - R_1(t,x),\ r_2(t,x) = 1 - R_2(t,x)$.

The Euler–Poisson–Darboux equation

The generalized hodograph method[12] produces the following equation on the unknown function $\Psi(r_1, r_2)$:

$$\Psi_{r_1 r_2} + \frac{\Psi_{r_1} - \Psi_{r_2}}{r_1 - r_2} = 0.$$

The general solution[13] to this equation has the form

$$\Psi(r_1, r_2) = 2u(r_1) + 2v(r_2) + (r_1 - r_2)(v'(r_2) - u'(r_1))$$

and we obtain the general solution of the initial problem in the implicit form:

$$t = -\frac{1}{2}\left(u''(r_1) + v''(r_2)\right), \qquad x = u'(r_1) + v'(r_2) - r_1 u''(r_1) - r_2 v''(r_2).$$

[12] S.P. Tsarev, *The geometry of hamiltonian systems of hydrodynamic type. The generalized hodograph method*, Izv. Math., **37**:2 (1991), 397 – 419.

[13] F.G. Tricomi, *Differential equations*, 1961.

Thank you for your attention!