The Darboux theory of integrability for polynomial differential systems in the plane

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The integrability problem

- Polynomial systems of ordinary differential equations

$$x_t = P(x, y), \quad y_t = Q(x, y)$$

- Polynomial vector fields $V \subset \mathbb{C}^{(m+2)(m+1)-l} \times (\mathbb{C} \setminus \{0\})^l$

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}, \quad P(x, y), Q(x, y) \in \mathbb{C}[x, y]$$

Problems

1. Find the functional classes of first integrals that vector fields from V can have.
2. Find all the vector fields from V having a first integral from some functional class.

The Darboux theory of integrability

Functional classes of first integrals

- rational;
- meromorphic;
- Darboux;
- Liouvillian

Darboux functions

$$G(x,y) = \prod_{j=1}^{K} F_j^{d_j}(x,y) \exp\{R(x,y)\}, \quad R(x,y) \in \mathbb{C}(x,y),$$

$$F_1(x,y), \ldots, F_K(x,y) \in \mathbb{C}[x,y], \quad d_1, \ldots, d_K \in \mathbb{C}$$

Liouvillian functions

belong to the following differential field extension of the field of rational functions $\mathbb{C}(x,y)$:

$$\mathbb{C}(x,y) = K_0 \subset K_1 \subset \ldots \subset K_M = L, \quad K_{j+1} = K_j(s), \quad \Delta = \{\partial_x, \partial_y\}$$

- s is an algebraic element over K_j;
- s is a transcendental element over K_j such that $\forall \delta \in \Delta \Rightarrow \delta s \in K_j$;
- s is a transcendental element over K_j such that $\forall \delta \in \Delta \Rightarrow \dfrac{\delta s}{s} \in K_j$.

The Darboux theory of integrability

Differential form: $\omega = Q(x,y)dx - P(x,y)dy$

Integrating factor: $M(x,y) : D \subset \mathbb{C}^2 \to \mathbb{C}$

- $M(x,y)\{Q(x,y)dx - P(x,y)dy\} = dI(x,y)$;
- $M(x,y) \in \mathbb{C}^1(D) \;\Rightarrow\; \mathcal{X}M = -\mathrm{div}(\mathcal{X})\mathrm{M}, \quad \mathrm{div}(\mathcal{X}) = P_x + Q_y$;
- symplectic form: $\Omega = M(x,y)dx \wedge dy, \quad (x,y) \in D$.

The Darboux theory of integrability

Theorem (J. Chavarriga et al., 2003; C. Christopher et al., 2019)

A polynomial vector field $\mathcal{X}$ is Darboux integrable if and only if it has a rational integrating factor.

Theorem (M. F. Singer, 1992)

A polynomial vector field $\mathcal{X}$ is Liouvillian integrable if and only if it has a Darboux integrating factor.

The Darboux theory of integrability

Darboux functions

$$M(x,y) = \prod_{j=1}^{K} F_j^{d_j}(x,y)\exp\{R(x,y)\}, \quad R(x,y) \in \mathbb{C}(x,y),$$

$$F_1(x,y), \ldots, F_K(x,y) \in \mathbb{C}[x,y], \quad d_1, \ldots, d_K \in \mathbb{C}$$

Theorem (C. Christopher, 1999)

If a Darboux function $M(x,y)$ is an integrating factor of a polynomial vector field $\mathcal{X}$, then $F_1(x,y)$,..., $F_K(x,y)$, $\exp\{R(x,y)\}$ are invariants of the vector field $\mathcal{X}$.

Invariants of a polynomial vector field $\mathcal{X}$

- Algebraic invariants (Darboux polynomials)

$$F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C} : \mathcal{X}F = \lambda(x,y)F, \quad \lambda \in \mathbb{C}[x,y]$$

$\lambda(x,y)$ is called the cofactor of $F(x,y)$

- Exponential invariants (multiple algebraic invariants)

$$E(x,y) = \exp\left\{\frac{S(x,y)}{T(x,y)}\right\} : \mathcal{X}E = \varrho(x,y)E, \; S, T, \varrho \in \mathbb{C}[x,y]$$

$\varrho(x,y)$ is called the cofactor of $E(x,y)$

Integrability conditions

- Darboux first integrals: $$I = \prod_{j=1}^{K} F_j^{d_j}(x,y) \exp\left\{\frac{S(x,y)}{T(x,y)}\right\}$$

$$\sum_{j=1}^{K} d_j \lambda_j(x,y) + \varrho(x,y) = 0;$$

- Darboux integrating factors: $$M = \prod_{j=1}^{K} F_j^{d_j}(x,y) \exp\left\{\frac{S(x,y)}{T(x,y)}\right\}$$

$$\sum_{j=1}^{K} d_j \lambda_j(x,y) + \varrho(x,y) = -\mathrm{div}\mathcal{X}$$

Finding algebraic invariants

The Poincaré problem

For a given polynomial vector field $\mathcal{X}$ find an upper bound on the degrees of its irreducible algebraic invariants: $\mathcal{P}(\mathcal{X})$.

Partial solution 1. (D. Cerveau, A. Lins Neto, 1991)

If all the singularities of irreducible invariant algebraic curves of $\mathcal{X}$ are of nodal type, then the following estimate is valid: $\mathcal{P}(\mathcal{X}) \leq \deg \mathcal{X} + 2$.

Partial solution 2. (M. M. Carnicer, 1994)

If there are no dicritical singularities of the vector field $\mathcal{X}$ on irreducible invariant algebraic curves, then the following estimate is valid: $\mathcal{P}(\mathcal{X}) \leq \deg \mathcal{X} + 2$.

Finding algebraic invariants

The methods of finding algebraic invariants

- The method of undetermined coefficients (the method of Prelle and Singer)
- The Lagutinskii's method (the method of the extactic polynomial)
- Decomposition into weight-homogeneous components: $\mathcal{X}^{(0)} F^{(0)} = \lambda^{(0)}(x, y) F^{(0)}$
- Methods, based on symmetries
- The method of fractional power series (Puiseux series)

Finding algebraic invariants

- Fields of Puiseux series

$$\mathbb{C}_\infty\{x\} = \left\{ y(x) = \sum_{k=0}^{+\infty} b_k x^{\frac{l_0}{n} - \frac{k}{n}}, \quad x_0 = \infty \right\},$$

$$\mathbb{C}_{x_0}\{x\} = \left\{ y(x) = \sum_{k=0}^{+\infty} c_k (x - x_0)^{\frac{l_0}{n} + \frac{k}{n}}, \quad x_0 \in \mathbb{C} \right\}$$

- Rings of polynomials over the fields of Puiseux series

$$\mathbb{C}_\infty\{x\}[y], \quad \mathbb{C}_{x_0}\{x\}[y]$$

Finding algebraic invariants

Projection operators:

$\{W(x,y)\}_+$ yields the polynomial part of $W(x,y) \in \mathbb{C}_\infty\{x\}[y]$;
$\{W(x,y)\}_-$ yields the non-polynomial part of $W(x,y) \in \mathbb{C}_\infty\{x\}[y]$.

The Newton–Puiseux theorem

Any solution $y(x)$ of the equation $F(x,y) = 0$, $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ can be locally represented by a convergent Puiseux series.

We are interested in Puiseux series satisfying the equation

$$(H): \quad P(x,y)y_x - Q(x,y) = 0$$

Finding algebraic invariants

Theorem (M. V. Demina, 2018)

Let $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ *be an irreducible algebraic invariant of a polynomial vector field* $\mathcal{X}$*. Then* $F(x,y)$ *takes the form*

$$F(x,y) = \left\{ \mu(x) \prod_{j=1}^{N} \{y - y_{j,\infty}(x)\} \right\}_{+}, \quad \mu(x) \in \mathbb{C}[x],$$

where $y_{1,\infty}(x)$, …, $y_{N,\infty}(x)$ *are pairwise distinct Puiseux series from the field* $\mathbb{C}_{\infty}\{x\}$ *that satisfy equation* (H)*.*

Finding the polynomial $\mu(x)$

Theorem (M. V. Demina, 2021)

Let $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ be an irreducible algebraic invariant of a polynomial vector field $\mathcal{X}$. If $x_0 \in \mathbb{C}$ is a zero of the polynomial $\mu(x)$, then the following statements are valid:

- At least one Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that has a negative exponent in the leading-order term solves equation (H).

Finding the polynomial $\mu(x)$

- If the number of distinct Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that solve equation (H) and have negative exponents in leading-order terms

$$y_{j,\,x_0}(x) = c_0^{(j)}(x-x_0)^{-q_j} + o\left((x-x_0)^{-q_j}\right), \quad c_0^{(j)} \neq 0, \\ q_j \in \mathbb{Q}, \quad q_j > 0, \quad 1 \leq j \leq L \in \mathbb{N} \tag{1}$$

  is finite, then the following inequality $m_0 \leq \sum_{j=1}^{L} q_j$ holds, where $m_0 \in \mathbb{N}$ is the multiplicity of the polynomial $\mu(x)$ at its zero x_0.

Finding the cofactor of an invariant

Theorem (M. V. Demina, 2021)

$$\lambda(x,y) = \left\{ \sum_{m=0}^{+\infty} \sum_{j=1}^{N} \frac{\{Q(x,y) - P(x,y)(y_{j,\infty})_x\}(y_{j,\infty})^m}{y^{m+1}} + P(x,y) \sum_{m=0}^{+\infty} \sum_{l=1}^{L} \frac{\nu_l x_l^m}{x^{m+1}} \right\}_+ ,$$

where $x_1, \ldots, x_L$ are pairwise distinct zeros of the polynomial $\mu(x) \in \mathbb{C}[x]$ with multiplicities $\nu_1, \ldots, \nu_L \in \mathbb{N}$ and $L \in \mathbb{N} \cup \{0\}$.

The uniqueness properties

Theorem 1. (M. V. Demina, 2021)

Suppose for some $x_0 \in \overline{\mathbb{C}}$ a Puiseux series $y(x)$ from the field $\mathbb{C}_{x_0}\{x\}$ satisfies equation (H) and possesses uniquely determined exponents and coefficients. Then there exists at most one irreducible algebraic invariant $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x]$ of the related vector field $\mathcal{X}$ such that this series is annihilated by $F(x, y)$, i.e. the series $y(x)$ solves the equation $F(x, y) = 0$.

The uniqueness properties

Theorem 2. (M. V. Demina, 2021)

If for some $x_0 \in \overline{\mathbb{C}}$ the number of distinct Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that satisfy equation (H) is finite, then the related vector field $\mathcal{X}$ possesses a finite number (possibly none) of irreducible algebraic invariants. Moreover, the number of pairwise distinct irreducible algebraic invariants does not exceed the number of distinct Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that satisfy equation (H).

The Poincaré problem

The finiteness property $(A_{f,f})$

1. There exists only a finite number of Puiseux series from the field $\mathbb{C}_\infty\{x\}$ that satisfy equation (H).
2. There exists only a finite number of complex numbers $x_0 \in \mathbb{C}$ and a only finite number of Puiseux series belonging to each of the fields $\mathbb{C}_{x_0}\{x\}$ that have negative exponents in the leading-order terms and satisfy equation (H).

Theorem (Partial solution 3)

Let (H) belong to the set $A_{f,f}$, then the Poincaré problem for the related vector field $\mathcal{X}$ has a solution: $\mathcal{P}(\mathcal{X}) \leq \deg^* \mathcal{X}$.

Finding algebraic invariants

The method of Puiseux series

1. Find all Puiseux series (centered at finite points and infinity) that satisfy equation (H).
2. Consider all possible factorizations of algebraic invariants in the ring $\mathbb{C}_\infty\{x\}[y]$.
3. Construct and solve the algebraic system resulting from the condition

$$\left\{\mu(x)\prod_{j=1}^{N}\{y - y_{\infty,j}(x)\}\right\}_{-} = 0.$$

Finding exponential invariants

Multiplicity of algebraic invariants

Two algebraic invariants: $f(x,y)$ and $f(x,y)+\varepsilon g(x,y)$, $\varepsilon \in \mathbb{C}$

$$\lim_{\varepsilon\to 0}\left[\frac{f(x,y)+\varepsilon g(x,y)}{f(x,y)}\right]^{\frac{1}{\varepsilon}} = \exp\left[\frac{g(x,y)}{f(x,y)}\right].$$

- V. N. Gorbuzov, 1998
- C. Christopher, J. Llibre, J. V. Pereira, 2007

The Puiseux integrability

Local invariants of a polynomial vector field $\mathcal{X}$

- Elementary algebraic invariants

$$F(x,y)=y-y_{j,x_0}(x)\in\mathbb{C}_{x_0}\{x\}[y],\ F(x,y)=y_{j,x_0}(x)\in\mathbb{C}_{x_0}\{x\},$$

$$\mathcal{X}F=\lambda(x,y)F,\quad \lambda(x,y)\in\mathbb{C}_{x_0}\{x\}[y]$$

- Elementary exponential invariants

$$E(x,y)=\exp\left[g_l(x)y^l\right],\quad g_l(x)\in\mathbb{C}_{x_0}\{x\},\quad l\in\mathbb{N}\cup\{0\};$$

$$E(x,y)=\exp\left[\frac{u(x,y)}{\{y-y_{j,x_0}(x)\}^n}\right],\ y_{j,x_0}(x)\in\mathbb{C}_{x_0}\{x\},$$

$$u(x,y)\in\mathbb{C}_{x_0}\{x\}[y],\ n\in\mathbb{N};\ \mathcal{X}E=\varrho(x,y)E,\ \varrho(x,y)\in\mathbb{C}_{x_0}\{x\}[y]$$

The Puiseux integrability

Definition (M. V. Demina, J. Giné, C. Valls, 2022)

A polynomial vector field $\mathcal{X}$ is called Puiseux integrable near a line $\{x = x_0, y \in \overline{\mathbb{C}}\}$, $x_0 \in \overline{\mathbb{C}}$ if it has a formal integrating factor

$$M(x, y) = \exp\left\{\frac{g(x, y)}{f(x, y)}\right\} \prod_{j=1}^{K} F_j^{d_j}(x, y), \quad K \in \mathbb{N} \cup \{0\},$$

where $F_1(x, y)$, $\ldots$, $F_K(x, y)$, $g(x, y)$, and $f(x, y)$ are Puiseux polynomials from the ring $\mathbb{C}_{x_0}\{x\}[y]$ and $d_1, \ldots, d_K \in \mathbb{C}$.

Polynomial Liénard systems

$$x_{tt} + f(x)x_t + g(x) = 0, \quad f(x), g(x) \in \mathbb{C}[x], \; f(x)g(x) \not\equiv 0;$$
$$x_t = y, \qquad y_t = -f(x)y - g(x) = 0,$$

- Polynomial vector fields

$$\mathcal{X} = y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y}$$

- Abel differential equations

the second kind : $\quad yy_x + f(x)y + g(x) = 0,$

the first kind : $\; w_x - g(x)w^3 - f(x)w^2 = 0, \; w(x) = \dfrac{1}{y(x)}$

Polynomial Liénard systems

$$L_{n,m} = \left\{ y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y} : \ \deg f = m, \deg g = n \right\}$$

$$m \geq n, \quad (m, n) \neq (0, 0)$$

1. Vector fields from $L_{n,m}$ do not have algebraic invariants provided that $g(x) \neq Cf(x)$, $C \in \mathbb{C}$; [K. Odani, 1995].
2. Vector fields from $L_{n,m}$ are not Liouvillian integrable provided that $g(x) \neq Cf(x)$, $C \in \mathbb{C}$; [J. Llibre, C. Valls, 2013].

Polynomial Liénard systems

$$L_{n,m} = \left\{ y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y} : \ \deg f = m, \deg g = n \right\}$$

$$m < n, \quad (m, n) \neq (0, 1)$$

1. A generic vector field from $L_{n,m}$ is not Liouvillian integrable.
2. Vector fields from $L_{n,m}$ are not Darboux integrable provided that $n \neq 2m + 1$.
3. For any n and m there exist vector fields from $L_{n,m}$ that are Liouvillian integrable.
4. The problem of Liouvillian integrability is solved completely provided that $n \neq 2m + 1$. In the case $n = 2m + 1$ our results are complete in the non-resonant case.

Polynomial Liénard systems

Example: a family of Liouvillian integrable vector fields from $L_{n,m}$

$$f(x) = \frac{(k+2l)}{4} w^{l-1} w_x,\ g(x) = \frac{k}{8}\left(w^{2l-1} + 4\beta w^{k-1}\right) w_x,\ w(x) \in \mathbb{C}[x]$$

$$\beta \in \mathbb{C},\ \deg w = \frac{m+1}{l},\ \frac{n+1}{m+1} = \frac{k}{l},\ (l,k) = 1$$

- Liouvillian first integral:

$$I(x,y) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{l}{k}; \frac{3}{2}; -\frac{(2y+w^l)^2}{4\beta w^k}\right) \frac{(2l-k)(2y+w^l)}{4kw^{\frac{k}{2}}\beta^{\frac{1}{2}+\frac{l}{k}}} + z^{\frac{1}{2}-\frac{l}{k}}$$

- Darboux integrating factor: $M(x,y) = z^{-\left(\frac{1}{2}+\frac{l}{k}\right)}$, $z = \left(y + \frac{w^l}{2}\right)^2 + \beta w^k$

Non-autonomous systems

- Non-autonomous systems of ordinary differential equations

$$x_t = P(x,y,t), \quad y_t = Q(x,y,t)$$

- Non-autonomous vector fields

$$\mathcal{X}_t = \frac{\partial}{\partial t} + P(x,y,t)\frac{\partial}{\partial x} + Q(x,y,t)\frac{\partial}{\partial y}, \quad P,\, Q \in \mathbb{M}(D)[x,y]$$

- Non-autonomous invariants

$$F(x,y,t) \in \mathbb{M}_a(D)[x,y] \setminus \mathbb{M}_a(D) : \quad \mathcal{X}_t F = \lambda(x,y,t) F$$

Non-autonomous systems

- Functional Puiseux series

$$\mathbb{C}^t_\infty\{x\} = \left\{ y(x) = \sum_{k=0}^{+\infty} b_k(t) x^{\frac{l_0}{n} - \frac{k}{n}}, \quad x_0 = \infty \right\};$$

$$\mathbb{C}^t_{x_0}\{x\} = \left\{ y(x) = \sum_{k=0}^{+\infty} c_k(t) (x - x_0)^{\frac{l_0}{n} + \frac{k}{n}}, \quad x_0 \in \mathbb{C} \right\}$$

- Non-autonomous Darboux first integrals

$$I(x,y,t) = \prod_{j=1}^{K} F_j^{d_j}(x,y,t) \exp\{R(x,y,t)\},\ R(x,y,t) \in \mathbb{C}_a(D)(x,y),$$

$$F_1(x,y,t), \ldots, F_K(x,y,t) \in \mathbb{C}_a(D)[x,y], \quad d_1, \ldots, d_K \in \mathbb{C}$$

Summary

1. The method of Puiseux series is a power and visual method of finding algebraic invariants and solving the Poincaré problem.
2. The Darboux theory of integrability combined with the method of Puiseux series provides the necessary and sufficient conditions of Liouvillian integrability for polynomial systems in the plane.
3. The problem of Liouvillian integrability is completely solved for polynomial non-resonant Liénard systems.