On the connection between commuting differential and difference operators.

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We denote by $L_k,\ L_s$ the operators of orders $k = N_- + N_+$ and $s = M_- + M_+$

$$L_k = \sum_{j=-N_-}^{N_+} u_j(n)T^j, \qquad L_s = \sum_{j=-M_-}^{M_+} v_j(n)T^j,$$

where $n \in \mathbb{Z},\ N_\pm, M_\pm \geq 0$, T is the shift operator

$$Tf(n) = f(n+1), \qquad f : \mathbb{Z} \to \mathbb{C}.$$

If two difference operators L_k and L_s commute, then there is a nonzero polynomial $F(z,w)$ such that $F(L_k,L_s)=0$. The polynomial F defines the *spectral curve* of the pair $L_k,\ L_s$

$$\Gamma = \{(z,w) \in \mathbb{C}^2 | F(z,w)=0\}.$$

The common eigenvalues are parametrized by the spectral curve

$$L_k\psi = z\psi, \quad L_s\psi = w\psi, \quad (z,w)\in\Gamma.$$

The dimension of the space of common eigenfunctions of the pair $L_k,\ L_s$ for fixed eigenvalues is called the *rank* of $L_k,\ L_s$

$$l = \dim\{\psi : L_k\psi = z\psi, \quad L_s\psi = w\psi, \quad (z,w)\in\Gamma.\}$$

Any commutative ring of difference operators in one discrete variable is isomorphic to the ring of meromorphic functions on a spectral curve with m fixed poles (I. M. Krichever, S. P. Novikov). Such operators are said to be m–points.

Spectral data for two–point operators of rank 1 were found by I. M. Krichever; examples of such operators were found by I. M. Krichever and D. Mumford. Eigenfunctions for two–point operators of rank 1 (Baker–Akhiezer functions) can be found explicitly in terms of theta function of the spectral curves.

Spectral data for one–point operators of rank 1 were found by G. S. Mauleshova and A. E. Mironov; examples of such operators for hyperelliptic spectral curves of any genus were constructed by them.

Spectral data for one–point operators of rank $l > 1$ were obtained by I. M. Krichever and S. P Novikov. These operators play an important role in constructing algebro–geometric solutions of $1D$ and $2D$ Toda chains. One–point Krichever–Novikov operators of rank 2 were studied by G. S. Mauleshova and A. E. Mironov; in particular, examples of such operators for hyperelliptic spectral curves of any genus were constructed.

Consider the hyperelliptic spectral curve Γ defined by the equation

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0, \tag{1}$$

for the base point we take $q = \infty$. Let $\psi(n, P)$ be the corresponding to the Baker–Akhiezer function. Then there exist commuting operators $L_2,\ L_{2g+1}$ such that

$$L_2\psi = \big((T + U_n)^2 + W_n\big)\psi = z\psi, \quad L_{2g+1}\psi = w\psi.$$

Theorem 1

The relation

$$L_2 - z = (T + U_n + U_{n+1} + \chi(n,P))(T - \chi(n,P)),$$

holds, where

$$\chi = \frac{\psi(n+1,P)}{\psi(n,P)} = \frac{S_n}{Q_n} + \frac{w}{Q_n},$$

$$S_n(z) = -U_n z^g + \delta_{g-1}(n) z^{g-1} + \ldots + \delta_0(n), \quad Q_n = -\frac{S_{n-1} + S_n}{U_{n-1} + U_n}.$$

The functions U_n, W_n, S_n satisfy the equation

$$F_g(z) = S_n^2 + (z - U_n^2 - W_n) Q_n Q_{n+1}.$$

Corallary

The functions $S_n(z), U_n, W_n$ satisfy the equation

$$(U_n + U_{n+1})(S_n - S_{n+1}) - (z - U_n^2 - W_n)Q_n + (z - U_{n+1}^2 - W_{n+1})Q_{n+2} = 0.$$

Theorem 2

In the case of an elliptic spectral curve Γ, given by the equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0,$$

operator L_2 type

$$L_2 = (T + U_n)^2 + W_n,$$

where

$$U_n = -\frac{\sqrt{F_1(\gamma_n)} + \sqrt{F_1(\gamma_{n+1})}}{\gamma_n - \gamma_{n+1}}, \quad W_n = -c_2 - \gamma_n - \gamma_{n+1},$$

γ_n — arbitrary function parameter, commute with some operator L_3.

Example 1

The operator

$$L_2^{\sharp} = (T + r_1 \cos(n))^2 + \frac{r_1^2 \sin(g)\sin(g+1)}{2\cos^2(g+\frac{1}{2})}\cos(2n),$$

$r_1 \neq 0$ commutes with a operator $L_{2g+1}^{\sharp}$.

Example 2

The operator

$$L_2^{\checkmark} = (T + \alpha_2 n^2 + \alpha_0)^2 - g(g+1)\alpha_2^2 n^2, \quad \alpha_2 \neq 0$$

commutes with a operator $L_{2g+1}^{\checkmark}$.

$$[H,\ M]=0,$$

where $H=\partial_x^2+u(x)$, $M=\partial_x^{2g+1}+v_{2g}(x)\partial_x^{2g}+\ldots+v_0(x)$. The spectral curve Γ of the pair $H,\ M$ is given by an equation of the form $w^2=F_g(z)$, and if $\psi(x)$ is a common eigenfunction, i.e.,

$$H\psi(x)=z\psi(x), \qquad M\psi(x)=w\psi(x),$$

then $(z,w)\in\Gamma$. Moreover, we have

$$H-z=(\partial_x+\chi_0(x))(\partial_x-\chi_0(x)),$$

where

$$\chi_0=\frac{R_x}{2R}+\frac{w}{R}, \quad R(x,z)=z^g+\alpha_{g-1}(x)z^{g-1}+\ldots+\alpha_0(x)$$

(B. A. Dubrovin, V. B. Matveev, S. P. Novikov). The polynomial R satisfies the equation

$$F_g(z)=R^2(z-u)+\frac{R_x^2}{4}-\frac{RR_{xx}}{2}.$$

Let

$$\hat{L}_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + \big(u(x,t,\varepsilon) + u(x+\varepsilon,t,\varepsilon)\big)\frac{T_\varepsilon}{\varepsilon} - v(x,\varepsilon).$$

We consider the one–point algebraic–geometric solution of rank one

$$\partial_t u(x,t,\varepsilon) + \partial_t u(x+\varepsilon,t,\varepsilon) = \tag{2}$$

$$u^2(x,t,\varepsilon) - u^2(x+\varepsilon,t,\varepsilon) + v(x,\varepsilon) - v(x+\varepsilon,\varepsilon).$$

Equation (2) is equivalent to the commutativity condition

$$[\hat{L}_2, \partial_t - (\frac{T_\varepsilon}{\varepsilon} + u(x,t,\varepsilon))] = 0.$$

Theorem 3

For $g = 1$, the one–point algebraic-geometric solution of rank one of equation (2) has the form

$$v(x,\varepsilon) = \gamma(x,\varepsilon) + \gamma(x+\varepsilon,\varepsilon) - \left(\frac{\sqrt{F_1(\gamma(x,\varepsilon))} + \sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\gamma(x,\varepsilon) - \gamma(x+\varepsilon,\varepsilon)}\right)^2,$$

$$u(x,t,\varepsilon) = -\frac{\sqrt{F_1(\gamma(x,\varepsilon))} + \sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\gamma(x,\varepsilon) - \gamma(x+\varepsilon,\varepsilon)} -$$

$$\frac{\sqrt{F_1(\wp(t))} + \sqrt{F_1(\gamma(x,\varepsilon))}}{\wp(t) - \gamma(x,\varepsilon)} + \frac{\sqrt{F_1(\wp(t))} + \sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\wp(t) - \gamma(x+\varepsilon,\varepsilon)},$$

where $F_1(z) = z^3 + c_1 z + c_0$, $\gamma(x,\varepsilon)$ is any function parameter, $\wp(t)$ is the Weierstrass elliptic function satisfying the equation

$$(\wp'(t))^2 = 4F_1(\wp(t)). \qquad (*)$$

The operators $\hat{L}_2$, $\hat{L}_3$ satisfy the equation $\hat{L}_3^2 = F_1(\hat{L}_2)$.

If

$$\gamma(x,\varepsilon)=\wp(x-\varepsilon),$$

then

$$\hat{L}_2=\frac{T_\varepsilon^2}{\varepsilon^2}-\big(2\zeta(\varepsilon)+\zeta(x-\varepsilon+t)-\zeta(x+\varepsilon+t)\big)\frac{T_\varepsilon}{\varepsilon}+\wp(\varepsilon),$$

$$\hat{L}_3=\frac{T_\varepsilon^3}{\varepsilon^3}-\big(3\zeta(\varepsilon)+\zeta(x-\varepsilon+t)-\zeta(x+2\varepsilon+t)\big)\frac{T_\varepsilon^2}{\varepsilon^2}+$$

$$\big((\zeta(\varepsilon)+\zeta(x-\varepsilon+t)-\zeta(x+t))(\zeta(\varepsilon)+\zeta(x+t)-\zeta(x+\varepsilon+t))+$$

$$2\wp(\varepsilon)+\wp(x+t)\big)\frac{T_\varepsilon}{\varepsilon}+\frac{1}{2}\wp'(\varepsilon),$$

$$\partial_t-\big(\frac{T_\varepsilon}{\varepsilon}+u(x,t,\varepsilon)\big)=\partial_t-\big(\frac{T_\varepsilon}{\varepsilon}-\zeta(\varepsilon)-\zeta(x-\varepsilon+t)+\zeta(x+t)\big),$$

where $\zeta(z)$ is the Weierstrass elliptic function.

Moreover,

$$\hat{L}_2 = \big(\partial_x^2 - 2\wp(x+t)\big) + O(\varepsilon),$$

$$\hat{L}_3 = \big(\partial_x^3 - 3\wp(x+t)\partial_x - \frac{3}{2}\wp'(x+t)\big) + O(\varepsilon),$$

$$\partial_t - \big(\frac{T_\varepsilon}{\varepsilon} - \zeta(\varepsilon) - \zeta(x-\varepsilon+t) + \zeta(x+t)\big) = (\partial_t - \partial_x) + O(\varepsilon).$$

Herewith, the spectral curve of the pair of commuting differential operators

$$\partial_x^2 - 2\wp(x+t), \qquad \partial_x^3 - 3\wp(x+t)\partial_x - \frac{3}{2}\wp'(x+t)$$

is the same as for ε–difference operators $\hat{L}_2,\ \hat{L}_3$.

Thank you for attention!