Minimum supports of eigenfunctions in the Hamming graph

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Basic definitions

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. Let $G = (V, E)$ be a graph and let λ be an eigenvalue of G. The set of neighbors of a vertex x is denoted by $N(x)$. A function $f : V \longrightarrow \mathbb{R}$ is called a λ-eigenfunction of G if $f \not\equiv 0$ and the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any vertex $x \in V$. The support of a function $f : V \longrightarrow \mathbb{R}$ is the set $S(f) = \{x \in V \mid f(x) \neq 0\}$.

MS-problem

MS-problem

Let G be a graph and let λ be an eigenvalue of G. Find the minimum cardinality of the support of a λ-eigenfunction of G.

MS-problem is closely related to the intersection problem of two combinatorial objects and to the problem of finding the minimum cardinality of combinatorial trades and null designs.

Distance-regular graphs

A connected graph with diameter D is called distance-regular if there are constants c_i, a_i, b_i such that for all $i = 0, 1, \ldots, D$, and all vertices x and y at distance i, among the neighbors of y, there are c_i at distance $i - 1$ from x, a_i at distance i, and b_i at distance $i + 1$.

Let G be a graph of diameter D. For a vertex x of G and $0 \leq i \leq D$ denote $N_i(x) = \{y \in V \mid d_G(y, x) = i\}$.

Distance-regular graphs

Suppose that G is a graph with vertex set V and diameter D. The distance-i graph G_i of G is defined as follows:

- the vertex set of G_i is V
- two vertices are adjacent in G_i if and only if they are at distance i in G.

By A_i we denote the adjacency matrix of G_i.

Distance-regular graphs

Let $G = (V, E)$ be a distance-regular graph of diameter D with intersection numbers a_i, b_i, c_i for $0 \leq i \leq D$.
Considering the combinatorial definition of distance-regularity from the matrices point of view, we obtain the following recurrence:

$$A_i A = a_i A_i + b_{i-1} A_{i-1} + c_{i+1} A_{i+1}, \tag{1}$$

for $i = 0, 1, \ldots, D$ where $b_{-1} A_{-1} = c_{D+1} A_{D+1} = 0$.
Using (1), one can show that there exist polynomials P_i of degree i such that:

$$A_i = P_i(A), \quad i = 0, 1, \ldots, D$$

The weight distribution bound

Lemma ([1, Corollary 1])

Let f be a λ-eigenfunction of a distance-regular graph with diameter D. Then the following bound takes place:

$$|S(f)| \geq \sum_{i=0}^{D} |P_i(\lambda)|.$$

The numbers $P_i(\lambda)$ can be calculated by the following recurrence:

$$P_0(\lambda) = 1,$$
$$P_1(\lambda) = \lambda,$$
$$P_i(\lambda) = \frac{\lambda P_{i-1}(\lambda) - b_{i-2}P_{i-2}(\lambda) - a_{i-1}P_{i-1}(\lambda)}{c_i}, \text{ where } i = 2, \ldots, D.$$

[1] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, To the theory of q-ary Steiner and other-type trades, Discrete Mathematics 339(3) (2016) 1150–1157.

The weight distribution bound

The weight distribution bound is achieved for the following cases:

- the smallest eigenvalue of the Hamming graph
- the smallest eigenvalue of the Johnson graph
- the smallest eigenvalue of the Grassmann graph
- Paley graph of square order
- strongly regular bilinear forms graph over a prime field
- n-dimensional hypercube, where n is odd, and its eigenvalue -1
- n-dimensional hypercube, where n is even, and its eigenvalue 0

Selected results on MS-problem

MS-problem has been studied for the following families of graphs:

- bilinear forms graphs (Sotnikova, 2019)
- cubical distance-regular graphs (Sotnikova, 2018)
- Doob graphs (Bespalov, 2018)
- Grassmann graphs (Krotov, Mogilnykh, Potapov, 2016)
- Hamming graphs (Vorob'ev, Krotov, 2014; Krotov 2016; V., Vorobev, 2019; V., 2021)
- Johnson graphs (Vorob'ev, Mogilnykh, V., 2018)
- Paley graphs (Goryainov, Kabanov, Shalaginov, V., 2018)
- Star graphs (Goryainov, Kabanov, Konstantinova, Shalaginov, V., 2020)

Hamming graph

Let $\Sigma_q = \{0, 1, \ldots, q-1\}$. The Hamming graph $H(n, q)$ is defined as follows:

- the vertex set of $H(n, q)$ is Σ_q^n
- two vertices are adjacent if they differ in exactly one coordinate

The Hamming graph $H(n, q)$ has $n+1$ distinct eigenvalues $\lambda_i(n, q) = n(q-1) - q \cdot i$, where $0 \leq i \leq n$. Denote by $U_i(n, q)$ the $\lambda_i(n, q)$-eigenspace of $H(n, q)$. The direct sum of subspaces

$$U_i(n, q) \oplus U_{i+1}(n, q) \oplus \ldots \oplus U_j(n, q)$$

for $0 \leq i \leq j \leq n$ is denoted by $U_{[i,j]}(n, q)$.

Some results

Theorem (Krotov, Vorob'ev, 2014)

Let f be a $\lambda_i(n,q)$-eigenfunction of $H(n,q)$. Then

$$|S(f)| \geq 2^i \cdot (q-2)^{n-i}$$

for $\frac{iq^2}{2n(q-1)} > 2$ and

$$|S(f)| \geq q^n \cdot (\frac{1}{q-1})^{i/2} \cdot (\frac{i}{n-i})^{i/2} \cdot (1-\frac{i}{n})^{n/2}$$

for $\frac{iq^2}{2n(q-1)} \leq 2$.

These bounds are sharp only for the following special cases:
$i = n$ and arbitrary q;
$n = 2m$, $q = 2$ and $i = m$;

MS-problem for the Hamming graph $H(n, 2)$

Theorem (Krotov, 2016)

The minimum cardinality of the support of a $\lambda_i(n, 2)$-eigenfunction of $H(n, 2)$ is $\max(2^i, 2^{n-i})$.

Problem 2

Let $n \geq 1$, $q \geq 2$ and $0 \leq i \leq j \leq n$. Find the minimum cardinality of the support of functions from the space $U_{[i,j]}(n, q)$.

Results on Problem 2

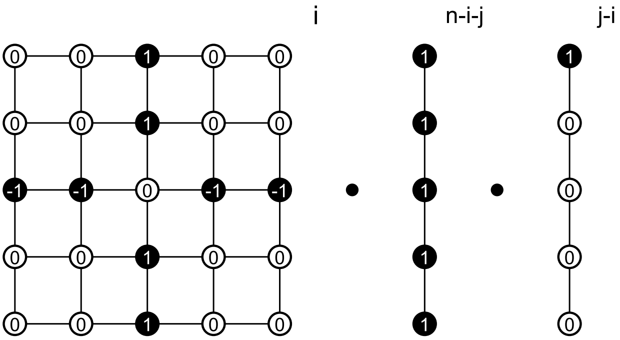
Theorem (V., Vorobev, 2019)

- Let $f \in U_{[i,j]}(n,q)$, where $q \geq 3$, $i+j \leq n$ and $f \not\equiv 0$. Then
$$|S(f)| \geq 2^i \cdot (q-1)^i \cdot q^{n-i-j}$$
and this bound is sharp.
- Let $f \in U_{[i,j]}(n,q)$, where $q \geq 4$, $i+j > n$ and $f \not\equiv 0$. Then
$$|S(f)| \geq 2^i \cdot (q-1)^{n-j}$$
and this bound is sharp.

Moreover, a characterization of functions that are optimal in the space $U_{[i,j]}(n,q)$ was obtained for $q \geq 3$, $i+j \leq n$ and $q \geq 5$, $i=j$, $i > \frac{n}{2}$.

Problem 2 for $q \geq 3$ and $i + j \leq n$

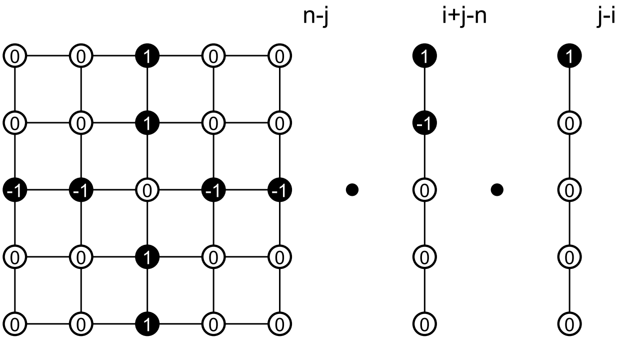
Optimal functions for $q \geq 3$ and $i + j \leq n$ can be constructed as follows:



$$H(n,q) = H(2,q)^i \square H(1,q)^{n-i-j} \square H(1,q)^{j-i}$$

Problem 2 for $q \geq 4$ and $i + j > n$

Optimal functions for $q \geq 4$ and $i + j > n$ can be constructed as follows:



$$H(n, q) = H(2, q)^{n-j} \square H(1, q)^{i+j-n} \square H(1, q)^{j-i}$$

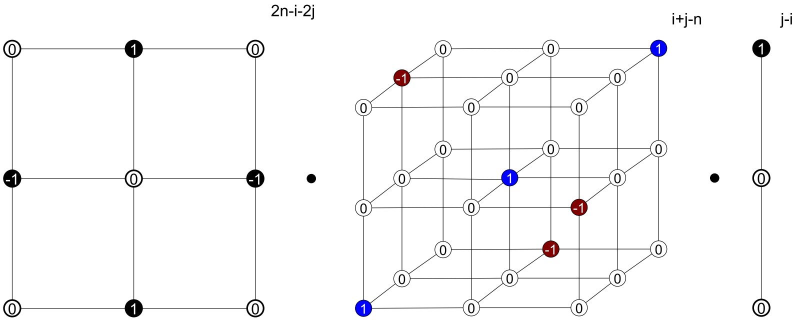
Case $q = 3$ and $i + j > n$

Theorem (V., 2021)

- Let $f \in U_{[i,j]}(n, 3)$, where $\frac{i}{2} + j \leq n$, $i + j > n$ and $f \not\equiv 0$. Then
$$|S(f)| \geq 2^{3(n-j)-i} \cdot 3^{i+j-n}$$
and this bound is sharp.
- Let $f \in U_{[i,j]}(n, 3)$, where $\frac{i}{2} + j > n$ and $f \not\equiv 0$. Then
$$|S(f)| \geq 2^{i+j-n} \cdot 3^{n-j}$$
and this bound is sharp.

Problem 2 for $q = 3$, $i + j > n$ and $\frac{i}{2} + j \leq n$

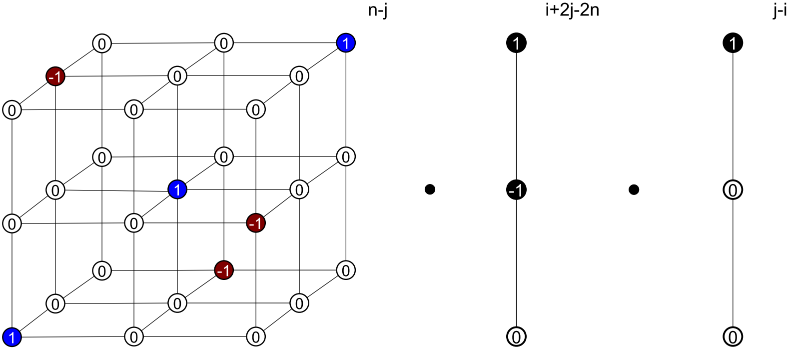
Optimal functions for $q = 3$, $i + j > n$ и $\frac{i}{2} + j \leq n$ can be constructed as follows:



$$H(n,3) = H(2,3)^{2n-i-2j} \square H(3,3)^{i+j-n} \square H(1,3)^{j-i}$$

Problem 2 for $q = 3$ and $\frac{i}{2} + j > n$

Optimal functions for $q = 3$ и $\frac{i}{2} + j > n$ can be constructed as follows:



$$H(n,3) = H(3,3)^{n-j} \square H(1,3)^{i+2j-2n} \square H(1,3)^{j-i}$$

Problem 2

So, the problem of a characterization of functions that are optimal in $U_{[i,j]}(n,q)$ is open for the following cases:

- $q = 2$
- $q \geq 3$ and $i + j > n$ $(i \neq j)$

Elementary optimal functions

We define a function φ_k on the vertices of the Hamming graph $H(k, 2)$ by the following rule:

$$\varphi_k(x) = \begin{cases} 1, & \text{if } x = 0^k; \\ -1, & \text{if } x = 1^k; \\ 0, & \text{otherwise.} \end{cases}$$

We define a function ψ_k on the vertices of the Hamming graph $H(k, 2)$ by the following rule:

$$\psi_k(x) = \begin{cases} 1, & \text{if } x = 0^k; \\ 1, & \text{if } x = 1^k; \\ 0, & \text{otherwise.} \end{cases}$$

We define a function I_k on the vertices of the Hamming graph $H(k, 2)$ by the following rule:

$$I_k(x) = \begin{cases} 1, & \text{if } x = 0^k; \\ 0, & \text{otherwise.} \end{cases}$$

Case $q = 2$ and $i + j \geq n$

Theorem (V., 2022)

Let $f \in U_{[i,j]}(n, 2)$, where $i + j \geq n$. The equality $|S(f)| = 2^i$ holds if and only if f is equivalent to

$$\varphi_{n_1} \otimes \cdots \otimes \varphi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

where $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, k, ℓ and r are nonnegative integers, $k + \ell = i$ and $\ell \geq n - j$.

Case $q = 2$ and $i + j \leq n$

Theorem (V., 2022)

Let $f \in U_{[i,j]}(n, 2)$, where $i + j \leq n$. The equality $|S(f)| = 2^{n-j}$ holds if and only if f is equivalent to

$$\psi_{n_1} \otimes \cdots \otimes \psi_{n_k} \otimes \varphi_{m_1} \otimes \cdots \otimes \varphi_{m_\ell} \otimes I_r,$$

where $n = n_1 + \ldots + n_k + m_1 + \ldots + m_\ell + r$, $n_1, \ldots, n_k$ are odd positive integers, $m_1, \ldots, m_\ell$ are even positive integers, k, ℓ and r are nonnegative integers, $k + \ell = n - j$ and $\ell \geq i$.