II Конференция Математических Центров

Octopuses in the Boolean cube: families with pairwise small intersections

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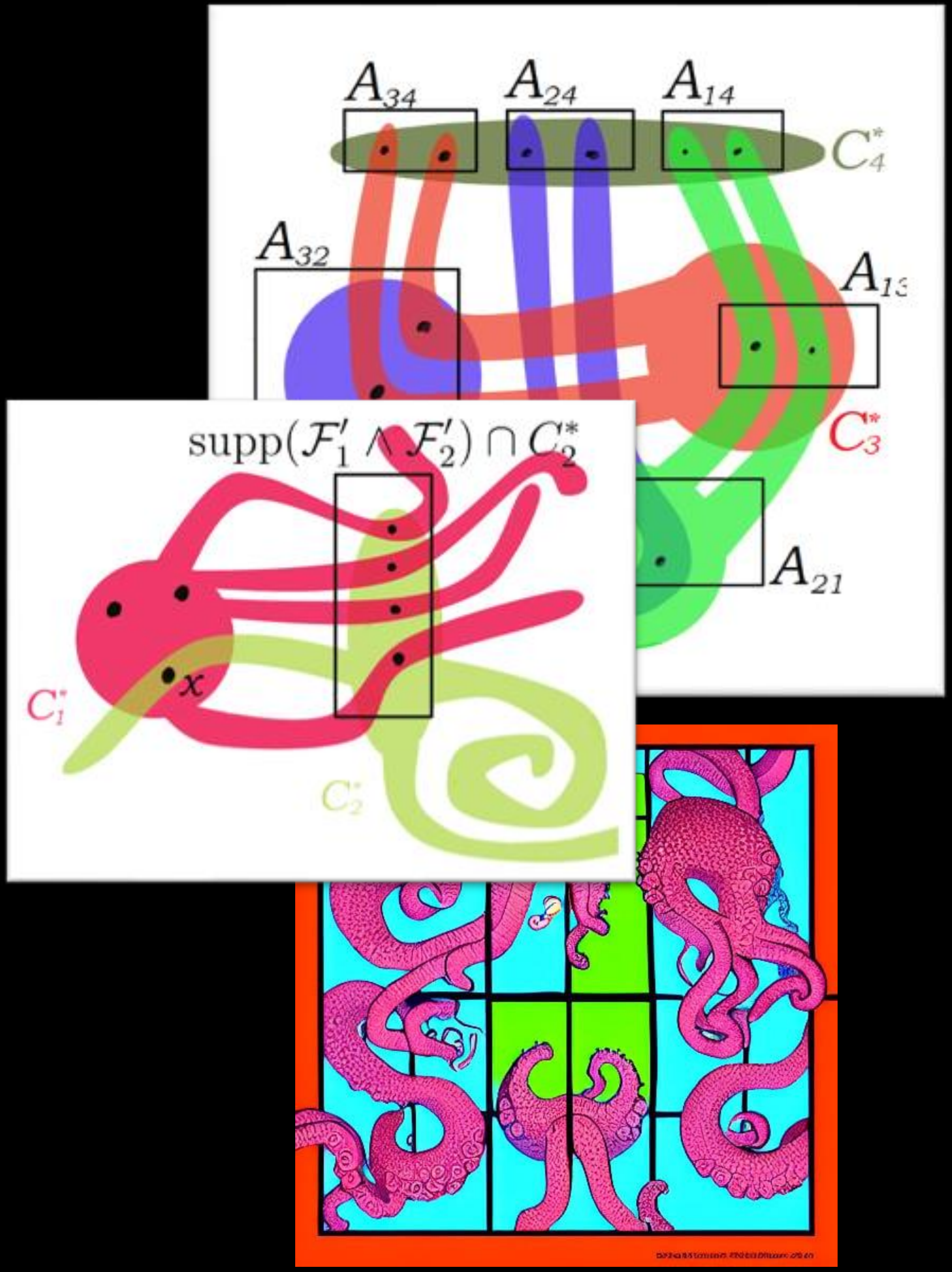

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PROBLEM STATEMENT

MAIN DEFINITION

Definition 1. *Fix a positive integer* ℓ. *Let* $\mathbf{m} = (\mathbf{m}_S)_{S\in\binom{[\ell]}{2}}$ *be a vector of non-negative integers indexed by unordered pairs* $\{k,k'\} \in \binom{[\ell]}{2}$. *For simplicity we suppress brackets in* $\mathbf{m}_{\{k,k'\}}$ *and assume that* $\mathbf{m}_{k,k'}$, $\mathbf{m}_{k',k}$, *and* $\mathbf{m}_{\{k,k'\}}$ *identify the same entry. Families* $\mathcal{F}_1,\ldots,\mathcal{F}_\ell \subset 2^{[n]}$ satisfy an $\mathbf{m}$-overlapping property *if for any distinct* $k_1,k_2 \in [\ell]$ *and any sets* $F_1 \in \mathcal{F}_{k_1}$, $F_2 \in \mathcal{F}_{k_2}$ *we have*

$$|F_1 \cap F_2| \leqslant \mathbf{m}_{k_1,k_2}.$$

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PROBLEM STATEMENT

Problem 1. *Let* n, ℓ *be positive integers,* $\mathbf{m}$ *be a vector of* $\binom{\ell}{2}$ *non-negative integers and* $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \subset 2^{[n]}$ *be families with the* $\mathbf{m}$*-overlapping property. What is the maximal value* $s^*(n, \ell, \mathbf{m})$ *of the product* $|\mathcal{F}_1| \cdot \ldots \cdot |\mathcal{F}_\ell|$*?*

MAIN THEOREM, ASYMPTOTICS

Theorem 4. *Suppose* $\ell, \mathbf{m}$ *are fixed while* n *tends to infinity. Then the asymptotic of* $s^*(n, \ell, \mathbf{m})$ *is the following:*

$$s^*(n, \ell, \mathbf{m}) = \left(1 + O\left(\frac{1}{n}\right)\right) \cdot 2^n \cdot \prod_{S \in \binom{[\ell]}{2}} \left(\frac{1}{\mathbf{m}_S!} \Big(\frac{\mathbf{m}_S \cdot n}{\sum_{S' \in \binom{[\ell]}{2}} \mathbf{m}_{S'}} \Big)^{\mathbf{m}_S} \right). \tag{2}$$

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PROBLEM STATEMENT

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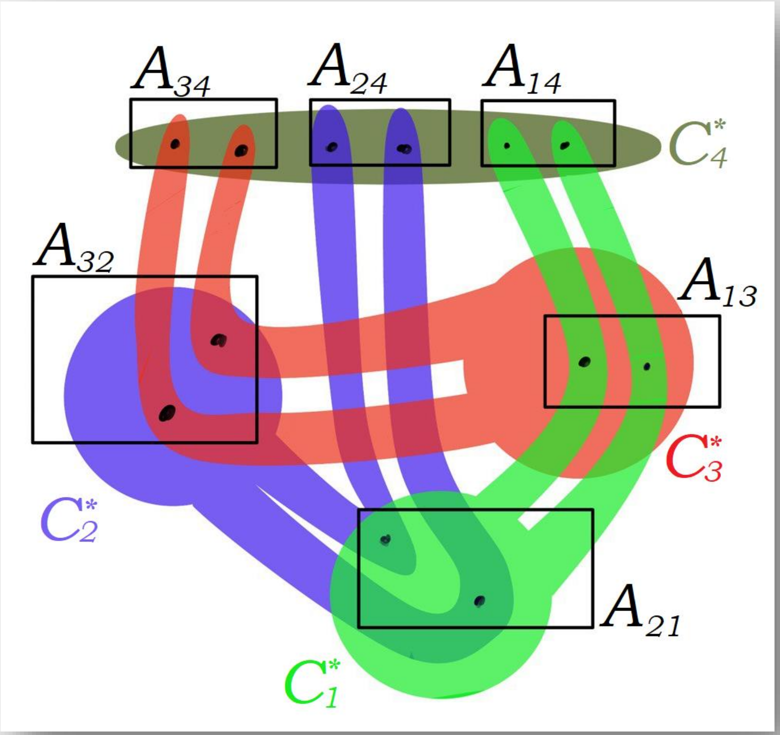
PROBLEM STATEMENT

MAIN THEOREM, STABILITY



Moreover, let $\mathcal{F}_k$, $k \in [\ell]$ *be the families from the extremal example. Then there is a tournament* $T_\ell = ([\ell], E)$ *and there are sets* A_S *indexed by* $S \in \binom{[\ell]}{2}$ *such that the following holds:*

1. *sets* A_S, $S \in \binom{[\ell]}{2}$ *form a partition of* $[n]$,
2. *for any* $k \in [\ell]$

$$2^{\bigsqcup_{k' \in In_k} A_{\{k',k\}}} \vee \bigvee_{k' \in Out_k} \binom{A_{\{k',k\}}}{\leqslant \mathbf{m}_{\{k,k'\}}} \subset \mathcal{F}_k$$

if n *is large enough.*

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SCETCH OF THE PROOF

DAYKIN'S INEQUALITY

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of sets. Then

$$|\mathcal{A}||\mathcal{B}| \leqslant |\mathcal{A} \vee \mathcal{B}||\mathcal{A} \wedge \mathcal{B}|.$$

COROLLARY

Upper bound:

$$s^*(n, \ell, \mathbf{m}) \leqslant C n^{\sigma} 2^n.$$

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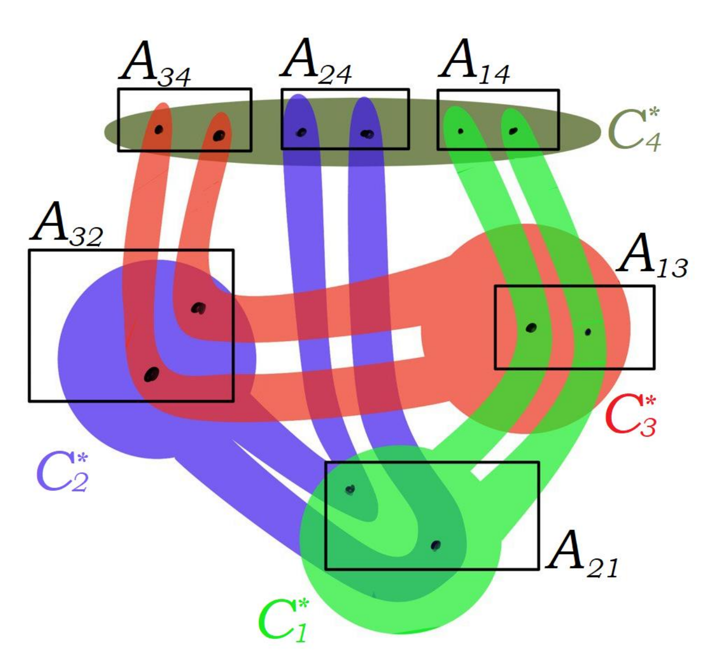
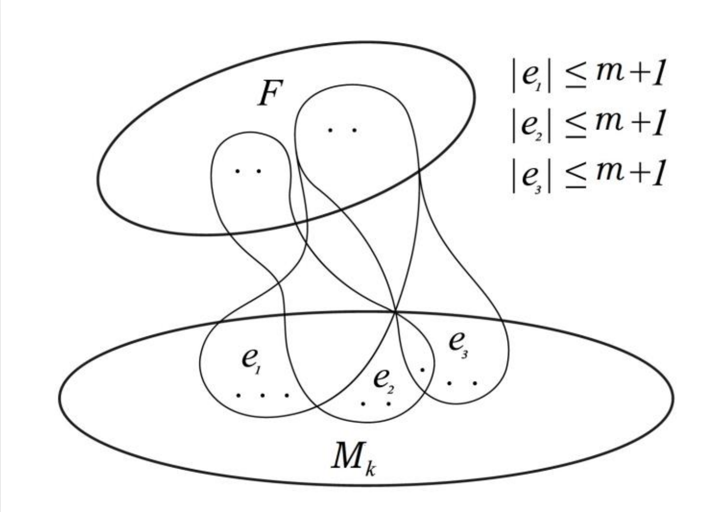
SKETCH OF THE PROOF

Lemma 17. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be a collection of* $\mathbf{m}$*-overlapping extremal families and let $M_k \in \mathcal{F}_k$, $k \in [\ell]$ be the sets of maximal cardinality in the respective families. Define $R = [n] \setminus \bigcup_{k=1}^{l} M_k$. Then there is a constant C such that*

$$|R| \leqslant C \log_2 n.$$





$$|\mathcal{F}_k| \leqslant n^{m(\ell-1)} 2^{|M_k|} \sum_{F \in \mathcal{F}|_R} (1 - 2^{-m})^{\frac{|F|}{m}}$$

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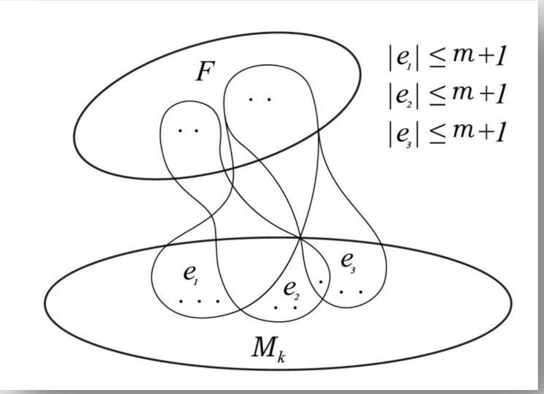
SKETCH OF THE PROOF

Corollary 10. *For any collection of families* $\mathcal{A}_1, \ldots, \mathcal{A}_\ell \subset 2^{[n]}$ *and* $0 \leqslant p \leqslant 1$ *we have*

$$\prod_{k=1}^{\ell} \mu_p(\mathcal{A}_k) \leqslant \prod_{k=1}^{\ell} \mu_p\left(\bigvee_{S \in \binom{[\ell]}{k}} \left(\bigwedge_{s \in S} \mathcal{A}_s\right)\right).$$

Moreover, the same holds for any measure that is proportional to μ_p.



$$|\mathcal{F}_k| \leqslant n^{m(\ell-1)} 2^{|M_k|} \sum_{F \in \mathcal{F}|_R} (1 - 2^{-m})^{\frac{|F|}{m}}$$

→

$$\prod_{k=1}^{\ell} |\mathcal{F}_k| \leqslant n^{m\ell(\ell-1)} 2^{\sum_{k=1}^{\ell} |M_k|} \prod_{k=1}^{\ell} \left(\sum_{F \in \mathcal{F}_k|_R} \varepsilon_m^{|F|} \right)$$

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SKETCH OF THE PROOF

Proposition 19. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be a collection of* $\mathbf{m}$*-overlapping families that is extremal. Then there is a set I of size $O(\log n)$, such that for each $x \in [n] \setminus I$ there is $k \in [\ell]$ such that $d_k(x) \geqslant \frac{1}{3}$.*

Proof is based on the entropy argument.

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SKETCH OF THE PROOF

Proposition 20. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be a collection of* **m***-overlapping families that is extremal. Then for any subset of indices of $K \subset [\ell]$ and sets $F_k \in \mathcal{F}_k$, $k \in K$*

$$\prod_{k \in K} d_k(F_k) \leqslant C_D n^{-\sum_{\{k,k'\} \in \binom{K}{2}} |F_k \cap F_{k'}|},$$

where C_D is some constant depending on **m**, *ℓ.*

Proof.

Consider families $\mathcal{F}_k$, $k \in [\ell] \setminus K$, and $\mathcal{F}_k(F_k)$, $k \in K$ as families in $2^{[n]}$. They satisfy the $\mathbf{m}'$-overlapping property, where

$$\mathbf{m}'_S = \begin{cases} \mathbf{m}_S, & S \not\subset K \\ \mathbf{m}_S - \left|\bigcap_{s \in S} F_s\right|, & S \subset K. \end{cases}$$

QED.

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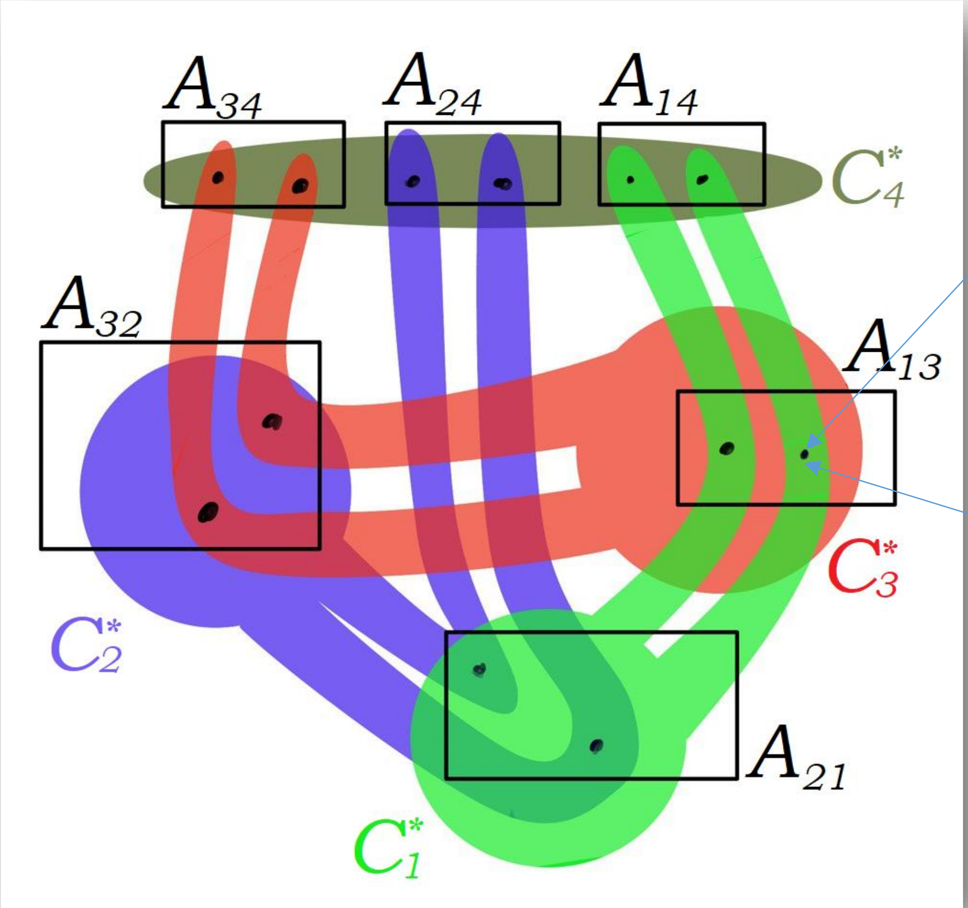
SKETCH OF THE PROOF



$$d(x) \geq \frac{1}{3}$$

$$d(x) \leq \frac{C}{n\sqrt{n}}$$

for all other colors except green

For any families of sets $\mathcal{A}_1, \ldots, \mathcal{A}_\ell$,

$$\prod_{k=1}^{\ell} |\mathcal{A}_k| \leqslant \prod_{k=1}^{\ell} \left| \bigvee_{S \in \binom{[\ell]}{k}} \left(\bigwedge_{s \in S} \mathcal{A}_s \right) \right|.$$