Non-Standard Interpolations in $\mathbb{C}^n$

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Story plan

- Standard interpolation: n=1
- Non-Standard interpolation: n=1
- Multidimensional interpolations: n>1
  (Role of Noetherian operators)
- Multidimensional residues:
  a Grothendieck and Laurent residues;
  b Gelfond-Khovanskii Theorem
  (Achieving Toric Geometry)
- Generalization of Gelfond-Khovanskii Theorem
  (Achieving Tropical Geometry)
- Example of non-standard interpolation

Standard interpolations (Lagrange)

The basic, classical (standard) interpolations include interpolations of Lagrange, Hermite, Newton, etc.
Lagrange: Given the points $\{w_j\}_{j=1}^{m} \subset \mathbb{C}$ and the values $c_j \in \mathbb{C}$, find the polynomial $f(z)$ of degree $m-1$ with the property

$$f(w_j) = c_j, \forall j = 1, \ldots, m.$$

Note that the interpolation polynomial is defined in terms of the polynomial $p(z) = (z - w_1) \cdot \ldots \cdot (z - w_m)$ by the formula

$$f(z) = p(z) \sum_{j=1}^{m} \frac{c_j}{z - w_j} \operatorname{res}_{w_j} \left(\frac{1}{p} \right).$$

Thus, specifying the interpolation nodes as the null set of the ideal $\langle p \rangle$ gives a toolkit for constructing an interpolation polynomial.

Standard interpolations (Hermite)

Hermite: Given the points $\{w_j\}_{j=1}^m \subset \mathbb{C}$ and the values $c_{j,k} \in \mathbb{C}$, where $j = 1, \ldots, m,\ \ k = 0, \ldots, \mu_j - 1$ find a polynomial $f(z)$ having at given points given values of derivatives up to orders of $\mu_j - 1$, that is,

$$f^{(k)}(w_j) = c_{j,k}, \forall j = 1, \ldots, m, \forall k = 0, \ldots, \mu_j - 1.$$

In this problem, the corresponding ideal is taken by the generated polynomial

$$p(z) = (z - w_1)^{\mu_1} \cdots (z - w_m)^{\mu_m}$$

Non-standard 1-dimensional

Problem: *Given the complex numbers* $a_{j,k}$ *(*$j = 1, \ldots, m$*;* $k = 0, \ldots, \mu_j - 1$*) and* c*. It is necessary to describe the set of all functions* f *which are analytic in the neighborhood of* $\Omega \subset \mathbb{C}$ *points* $w_1, \ldots, w_m$ *and satisfy the equation:*

$$\sum_{j=1}^{m} \sum_{k=0}^{\mu_j - 1} a_{j,k} f^{(k)}(w_j) = c. \tag{1}$$

(D. Alpay, etc., 2016). Note that if f is a solution of (1), then $f + ph$ is also a solution, where

$$p(z) = \prod_{j=1}^{m} (z - w_j)^{\mu_j}, \quad h \in \mathcal{O}(\Omega).$$

In other words, we can work in the factor ring $\mathcal{O}(\Omega)/\langle p \rangle$ by the ideal generated by the polynomial p.

Noetherian opeators

Definition (Ehrenpreis, Palamodov)

Let $I \subset \mathbb{C}[s_1, \ldots, s_n]$ be a primary ideal. A family of linear differential operators with polynomial coefficients $\partial_\ell(\boldsymbol{s}, D)$, $\ell = 1, \ldots, t$ is called a noetherian operator for I, if the conditions

$$\partial_\ell(\boldsymbol{s}, D)\varphi(\boldsymbol{s})|_{V(I)} = 0, \quad \forall \ell = 1, \ldots, t$$

are necessary and sufficient for the function $\varphi(\boldsymbol{s})$ to belong to ideal I.

Noetherian operators in the one-dimensional case

In the one-dimensional case an arbitrary polynomial has the form:

$$p(s) = (s - w_1)^{\mu_1} \cdot \ldots \cdot (s - w_k)^{\mu_k},$$

and its generated ideal is decomposed into the intersection of primal ones

$$\rho_j = \langle (s - w_j)^{\mu_j} \rangle, \quad j = 1, \ldots, k.$$

A necessary and sufficient condition for a given function φ to belong to the primary component ρ_j is vanishing of φ by the following operators with constant coefficients:

$$\mathcal{L}_{j,0}, \mathcal{L}_{j,1}, \ldots, \mathcal{L}_{j,\mu_j - 1},$$

where $\mathcal{L}_{i,j}[\varphi(s)] = \frac{d^j \varphi}{ds^j}\Big|_{s=w_i}$.

Non-standard n-dimensional

Problem (Alpay, Yger; 2019)

Let $\boldsymbol{p}^{-1}(0) = \{w_1, \ldots, w_m\}$ and U be an open subset of $\mathbb{C}^n$ containing $\boldsymbol{p}^{-1}(0)$. Fix $a_{j,l}, j = 1, \ldots, m, l \in A_{w_j}$ and c; all of them are complex numbers. We need to describe the space of holomorphic functions $f\colon U \to \mathbb{C}$ with the following property:

$$\sum_{j=1}^{m} \sum_{\ell \in A_{w_j}} a_{j,\ell} \mathcal{L}_{w_j,\ell}[f](w_j) = c. \tag{2}$$

Basis in $\mathbb{C}[s]/\langle p\rangle$

The following monomial basis

$$\mathcal{B} = \{\boldsymbol{s}^{\beta_k}; k = 0, \ldots, N(\boldsymbol{p}) - 1\}$$

in the quotient space $\mathbb{C}[\boldsymbol{z}]/\langle p\rangle$ is one of ingredients for solving the interpolation problem. In fact, this factor is the space of reminders when dividing polynomials by the ideal $\langle \boldsymbol{p}\rangle$. The basis $\mathcal{B}$ is constructed using the Gröbner basis for the ideal $\langle \boldsymbol{p}\rangle$.

Solution of the multidimensional Problem

Let $\{w_1, \dots, w_m\} = \boldsymbol{p}^{-1}(0)$, U be an open subset in $\mathbb{C}^n$ containing $\boldsymbol{p}^{-1}(0)$. Let the sequence

$$\boldsymbol{a} = \{a_{j,\boldsymbol{\ell}}, j = 1, \dots, m, \boldsymbol{\ell} \in A_{w_j}\}$$

and the complex number c be given. Let us denote the polynomials

$$h^{\boldsymbol{a}}_{w_j}(\boldsymbol{s}) = \sum_{\boldsymbol{\ell} \in A_{w_j}} a_{j,\boldsymbol{\ell}}(\boldsymbol{s} - w_j)^{\boldsymbol{\ell}} / \boldsymbol{\ell}!,$$

making up the sequence $\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}} = [h^{\boldsymbol{a}}_{w_1}, \dots, h^{\boldsymbol{a}}_{w_m}]$, and let

$$\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] = (\alpha_0[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}], \dots, \alpha_{N(\boldsymbol{p})-1}[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}])$$

be the projection of this sequence onto the quotient space $\mathbb{C}[\boldsymbol{z}]/\langle p \rangle$.

Solution of the multidimensional problem

Theorem (Alpay, Yger 2019)

- *If* $\alpha[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}] = 0$*, then the problem has no solution in the case* $c \neq 0$*, and any function* $f \in \mathcal{O}(U)$ *is a solution in the case* $c = 0$*;*
- *If* $\alpha[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}] \neq 0$*, then* $f \in \mathcal{O}(U)$ *satisfies the condition* (2) *iff*

$$\alpha[\boldsymbol{f}] \cdot \boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] \cdot \alpha[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}]^T = c,$$

where T *is the transposition sign, and* $\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}]$ *is the Grothendieck global residues matrix:*

$$\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] = \mathrm{Res}\left[\frac{\boldsymbol{s}^{\beta_{k_1}+\beta_{k_2}}\,d\boldsymbol{s}}{p_1(\boldsymbol{s})\ldots p_n(\boldsymbol{s})}\right]_{0\leq k_1,k_2\leq N\langle\boldsymbol{p}\rangle-1}$$

1-dimensional residues

In one variable there are two notions of the residue: by integral over small circle

$$\operatorname*{res}_a g = \frac{1}{2\pi i} \int\limits_{|z-a|=\varepsilon} g(z)dz$$

and by coefficient c_{-1} of the Laurent decomposition

$$g(z) = \sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

If a multidimentional analogue of a holomorphic function is understood as a mapping $\mathbb{C}^n \to \mathbb{C}^n$, it is convienient to use the so called *local (pointed) Grothendieck residue* as an integral definition.

Grothendieck residue

Grothendieck residue is a cornerstone of complex analysis and algebraic geometry and it plays a key roles in the theory of singularity and foliations.
Assume that the sequence of germs

$$f_1, \ldots, f_n \in \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$$

have isolated common zero at $a \in \mathbb{C}^n$. Consider a meromorphic differential n-form

$$\omega = \frac{1}{(2\pi i)^n} \frac{h(z)\, dz}{f_1(z) \ldots f_n(z)}, \quad (\text{with } dz = dz_1 \wedge \cdots \wedge dz_n)$$

Grothendieck residue

Definition

The Grothendieck residue, associated with $f = (f_1, \dots, f_n)$ and h, is determined as an integral

$$\underset{a}{\operatorname{res}}_f(h) = \int_{\Gamma_a} \omega$$

of the form ω over a very special cycle

$$\Gamma_a = \{z \in U_a \colon |f_j(z) = \varepsilon_j, j = 1, \dots, n\}$$

where the neighborhood U_a of a and ε_j are chosen such that the closure $\overline{U}_a$ does not contain roots different from a and $\Gamma_a \subset\subset U_a$.

Amoeba and its complement

Definition

Given a Laurent polynomial f its amoeba A_f is the image of the hypersurface $V = f^{-1}(0)$ under the map

$$\mathrm{Log}\colon (z_1, \dots, z_n) \to (\log|z_1|, \dots, \log|z_n|).$$

For the amoeba we will also use notation A_V.

Amoeba reflects the distribution of the algebraic set V. But more precisely, the amoeba defines «emptness» for V.

Amoeba of a curve in $(\mathbb{C}\backslash 0)^2$

Newton polytope of f

The shape of the amoeba is closely related to the Newton polytope Δ_f of the polynomial f. Recall that Δ_f is defined as the convex hull in $\mathbb{R}^n$ of the index set A in the experession

$$f(z_1, \ldots, z_n) = \sum_{\alpha \in A} a_\alpha z^\alpha$$

The set of integer points in Δ_f admits a natural partition $\Delta_f \cap \mathbb{Z}^n = \bigcup_\Gamma A_\Gamma$, where Γ is any face on Δ_f and A_Γ denotes the intersection of $\mathbb{Z}^n$ with the reflective interior of Γ. We shall consider the dual cone C_ν of Δ_f at ν defined as

$$C_\nu = \left\{ s \in \mathbb{R}^n \colon \langle s, \nu \rangle = \max_{\alpha \in \Delta_f} \langle s, \alpha \rangle \right\}$$

Notice that $\dim C_\nu = n - \dim \Gamma$ when $\nu \in A_\Gamma$. In particular, C_ν has nonempty interior if ν is a vertex of Δ_f, and it equals $\{0\}$ whenever ν is an interior point of Δ_f.

The order map on the complement ${}^c\!A_f$

Theorem (Forsberg, Passare, Tsikh)

On the set $\{E\}$ of connected components of ${}^c\!A_f$ there is an injective map (the order map)

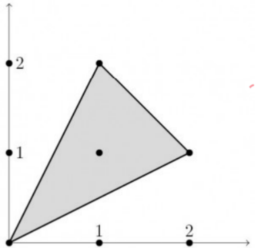
$$\nu\colon \{E\} \to \Delta_f \cap \mathbb{Z}^n$$

with the property that the dual cone $C_{\nu(E)}$ is equal to the recession cone of E. That is, for any $u \in E$ one has $u + C_\nu \in E$ and no strictly larger cone is contained in E. (Notice that if ν is the k-skeleton of Δ_f the C_ν has dimension $n-k$).
Thus, connected components can be numbered as E_ν with integer $\nu \in \Delta_f$.

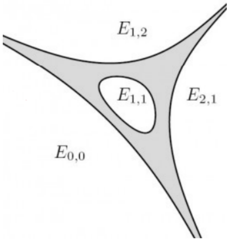
Corollary

The cardinality of the set $\{E\}$ of connected components satisfies the inequalities

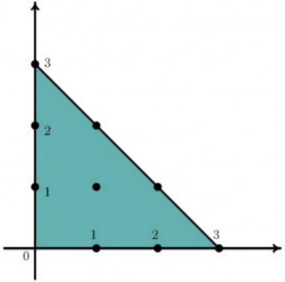
$$\#\,\mathrm{Vert}\,\Delta_f \leqslant \#\{E\} \leqslant \#\Delta_f \cap \mathbb{Z}^n$$



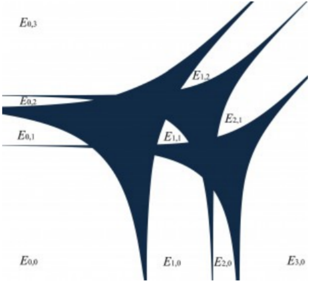
2
1
1
2



$E_{1,2}$
$E_{1,1}$
$E_{2,1}$
$E_{0,0}$



3
2
1
0
1
2
3



$E_{0,3}$
$E_{1,2}$
$E_{0,2}$
$E_{2,1}$
$E_{0,1}$
$E_{1,1}$
$E_{0,0}$
$E_{1,0}$
$E_{2,0}$
$E_{3,0}$

Gelfond-Khovanskii formula

Theorem (Gelfond-Khovanskii formula)

Assume that the Newton polytopes $\Delta_1, \ldots, \Delta_n$ *of polynomials* $f_1, \ldots, f_n$ *are unfolded. Then the sum of all local residues in* $(\mathbb{C} \setminus 0)^n$ *is calculated by the formula:*

$$\sum_{\{a\}} \operatorname{res}_{a} f\,(h) = \sum_{\nu \in \operatorname{Vert} \Delta} k_\nu \operatorname{Res}_{E_\nu} \left(\frac{h}{f_1 \ldots f_n} \right)$$

where $\operatorname{Res}_{E_\nu}$ *is the coefficient* c_{-I} *of the Laurent decomposition for* $\frac{h}{f_1 \ldots f_n}$ *in the connected component* E_ν*.*

In fact one can prove that the sum $\sum_{\{a\}} \Gamma_a$ of local Grothendieck cycles Γ_a is homologically equivalent to the sum

$$\sum_{\nu \in \operatorname{Vert} \Delta} k_\nu \operatorname{Log}^{-1}(u_\nu), \quad u_\nu \in E_\nu$$

Combinatorial coefficients for vertices

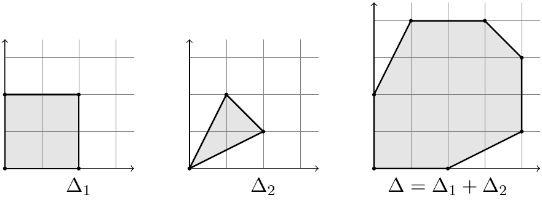
We ascribe the combinatorial coefficient to each vertex A of the sum $\Delta = \Delta_1 + \ldots + \Delta_n$ of unfolded polytopes. Each face $\Gamma \subset \Delta$ is a sum $\Gamma_1 + \ldots + \Gamma_n$ of faces $\Gamma_i \subset \Delta_i$.

Definition

Combinatorial coefficient k_A is the local degree of the germ

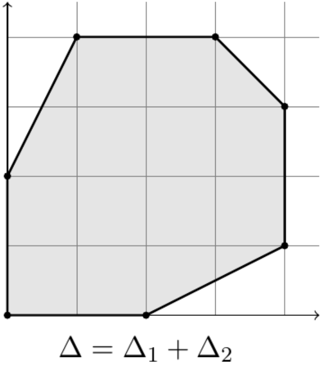
$$(\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$$

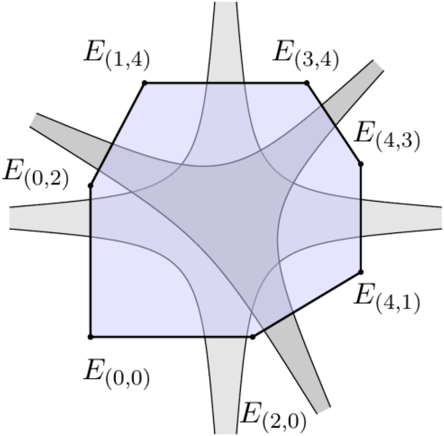
of the characteristic map $(h_1, \ldots, h_n)\colon \partial\Delta \to \partial\mathbb{R}^n_+$, where each component h_i is zero precisely on that face of Γ, for which the term Γ_i is a vertex of Δ_i.



Distribution between amoebas and polyhedron

Let $f_1 = 1 + z^2 + w^2 + z^2 w^2$, $f_2 = 1 + z^2 w + z w^2$.





Colouring book

Let Δ be an arbitrary n-dimensional polyhedron in $\mathbb{R}^n$, and Φ be the set of all faces with a fixed orientation. Consider the chain complex $C = (\Phi; \partial)$ freely generated by Φ.
Let $I = \{1, \ldots, n\}$ be the set of indices which are interpreted as different colors with corresponding numbers.

Definition

The map $\chi\colon I \to 2^{\Phi \setminus \Delta}$ we call a *colouring book*. Denote by $H(i)$ the subgroup in C generated by faces with color i.

Combinatorial coefficients for faces

Definition

The sequence of chains $\xi_0, \xi_1, \ldots, \xi_{n-1}$ is called a *Resolvent* of the cycle $\partial\Delta$ if:

- $\xi_p \colon I^{p+1} \to C$ — an alternated map with image $\xi_p(i_0, i_1, \ldots, i_p) \in H(i_0) \cap H(i_1) \cap \cdots \cap H(i_p)$;
- $\partial\Delta = \xi_0(1) + \cdots + \xi_0(n)$;
- $\partial\xi_p(i_0, i_1, \ldots, i_p) = \sum_{i \in I} \xi_{p+1}(i, i_0, i_1, \ldots, i_p)$.

Definition

For the proper face $\Gamma \subset \Delta$ of dimension m and a subset $i_1 < i_2 < \cdots < i_{n-m}$ of colors we define *the combinatorial coefficient*

$$k_\Gamma(i_1, i_2, \ldots, i_{n-m})$$

as an integer coefficient with which Γ appears in the chain $\xi_{n-m-1}(i_1, i_2, \ldots, i_{n-m})$.

Main result

Theorem (Durakov, Tsikh, Ulvert)

Assume that $f = (f_1, \dots, f_n)$ *has a finite number of solutions in* $(\mathbb{C} \setminus 0)^n$. *Then*

$$\sum_a \operatorname{res}_{a\ f}(h) = \sum_{\nu \in \partial\Delta \cap \mathbb{Z}^n} k_\nu \operatorname{Res}_{E_\nu} \left(\frac{h}{f_1 \dots f_n} \right)$$

In the homological sense it means that

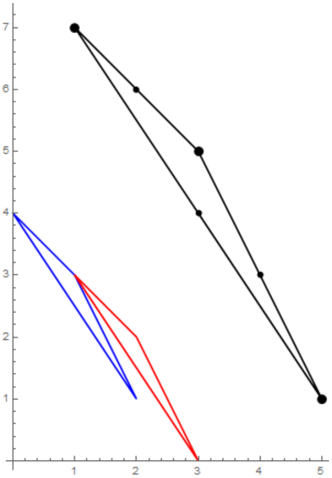
$$\sum_{\{a\}} \Gamma_a = \sum_{\nu \in \partial\Delta \cap \mathbb{Z}^n} k_\nu \operatorname{Log}^{-1}(u_\nu), \quad u_\nu \in E_\nu.$$

Nongeneral position of $\Delta_1, \ldots, \Delta_n$

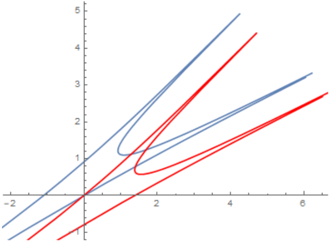
Let us consider the system of polynomials in two variables:

$$f_1 = 3z_1^2 z_2 + z_2^4 + 2z_1 z_2^3,$$
$$f_2 = z_1^3 + 4z_1 z_2^3 + 3z_1^2 z_2^2$$

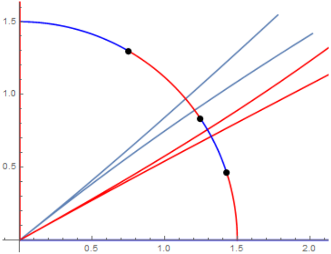
with the following Newton polytopes in nongeneral position.



Nongeneral position of $\Delta_1, \ldots, \Delta_n$



Amoebas A_{f_1} and A_{f_2}



The local distribution at $z = 0$ on the Reinhardt diagram

Here

$$\sum_{\{a\}} \Gamma_a = \Gamma_{51} - \Gamma_{34} + \Gamma_{17} - \Gamma_{26} + \Gamma_{35} - \Gamma_{43}$$

Locally

$$\Gamma_0 = \Gamma_{51} - \Gamma_{34} + \Gamma_{17}.$$

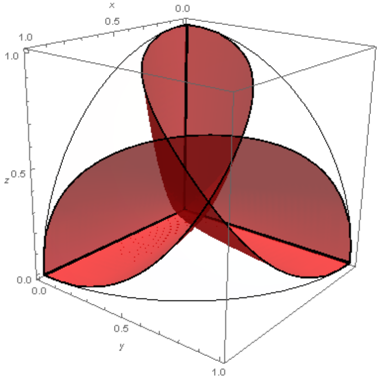
Example of non-standard interpolation

Let us consider an example when the single point $a = 0$ is defined as an isolated zero of the polynomial system

$$P_1 = z_1^3 - z_2 z_3 = 0$$
$$P_2 = z_2^3 - z_1 z_3 = 0$$
$$P_3 = z_3^3 - z_1 z_2 = 0$$

The multiplicity at 0 equals 11.
The Grothendieck cycle

$$\Gamma_0 = \Gamma_{511} + \Gamma_{151} + \Gamma_{115} - \Gamma_{222}$$



Collection of Noether operators for the ideal $I_0\langle \boldsymbol{P}\rangle$

$$\{\mathcal{L}_{0,\ell}\} = \Big\{\mathcal{L}_{0,000} = \left(-\partial^0 - \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} - \frac{1}{4!}\frac{\partial^4}{\partial z_1^4} - \frac{1}{4!}\frac{\partial^4}{\partial z_2^4} - \frac{1}{4!}\frac{\partial^4}{\partial z_3^4}\right);$$

$$\mathcal{L}_{0,100} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_1^3} - \frac{\partial^2}{\partial z_2 \partial z_3}\right); \mathcal{L}_{0,010} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_2^3} - \frac{\partial^2}{\partial z_1 \partial z_3}\right);$$

$$\mathcal{L}_{0,001} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_3^3} - \frac{\partial^2}{\partial z_1 \partial z_2}\right); \mathcal{L}_{0,110} = \left(-\frac{\partial}{\partial z_3}\right); \mathcal{L}_{0,101} = \left(-\frac{\partial}{\partial z_2}\right);$$

$$\mathcal{L}_{0,011} = \left(-\frac{\partial}{\partial z_1}\right); \mathcal{L}_{0,200} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_1^2}\right); \mathcal{L}_{0,020} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_2^2}\right);$$

$$\mathcal{L}_{0,002} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_3^2}\right); \mathcal{L}_{0,111} = \left(-\partial^0\right); \mathcal{L}_{0,300} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_1}\right);$$

$$\mathcal{L}_{0,030} \left(-\frac{1}{3!}\frac{\partial}{\partial z_2}\right); \mathcal{L}_{0,003} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_3}\right); \mathcal{L}_{0,400} = \left(-\frac{1}{4!}\partial^0\right);$$

$$\mathcal{L}_{0,040} = \left(-\frac{1}{4!}\partial^0\right); \mathcal{L}_{0,004} = \left(-\frac{1}{4!}\partial^0\right)\Big\}.$$

Theorem

If $\alpha[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}] \neq 0$, then the holomorphic function $f(\boldsymbol{s})$ satisfies the Alpay-Yger problem for single point ($m = 1$) iff the coordinatization of f satisfies the following condition:

$$\left(a_{000} + a_{111} - \frac{a_{400} + a_{040} + a_{004}}{24}\right)\alpha_1[f] + \left(a_{011} + \frac{a_{300}}{6}\right)\alpha_2[f] +$$
$$+ \left(a_{101} + \frac{a_{030}}{6}\right)\alpha_3[f] + \left(a_{110} + \frac{a_{003}}{6}\right)\alpha_4[f] + \frac{a_{200}}{2}\alpha_5[f] +$$
$$\frac{a_{020}}{2}\alpha_6[f] + \frac{a_{002}}{2}\alpha_7[f] + a_{001}\alpha_8[f] + a_{010}\alpha_9[f] +$$
$$+ a_{001}\alpha_{10}[f] + a_{000}\alpha_{11}[f] = -c.$$

This means that the coordinate vector of f in the local algebra lies in the prescribed affine hyperplane $\Pi_a \subset \mathbb{C}^{11}$.

Thank you for your attention!

$$\sum_{w\in \boldsymbol{p}^{-1}(0)} \det H_w(z) \left[\sum_{\substack{\ell\le d_w - I\\ k\le d_w - I - \ell}} \frac{c_{w,\ell}}{\ell!}(z-w)^{\ell+k}\, \mathrm{res}_w \left(\frac{(z-w)^{d_w - I - k}}{\boldsymbol{p}^I} \right) \right],$$

where $H_w(z) = ||h_{ik}||$ is a matrix from the representation

$$\begin{pmatrix} p_1(z) \\ \vdots \\ p_n(z) \end{pmatrix} = ||h_{ik}|| \begin{pmatrix} (z_1 - w_1)^{d_1} \\ \vdots \\ (z_n - w_n)^{d_n} \end{pmatrix}.$$