Methods of complex analysis in knot theory

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Three geometries of constant curvature

1. Euclidean geometry $\mathbb{E}^3$, $K = 0$ (Euclid)
2. Spherical geometry $\mathbb{S}^3$, $K > 0$ (Riemann)
3. Hyperbolic geometry $\mathbb{H}^3$, $K < 0$ (N.I. Lobachevsky and Janos Bolyai)
4. Eight geometries by William Thurston:

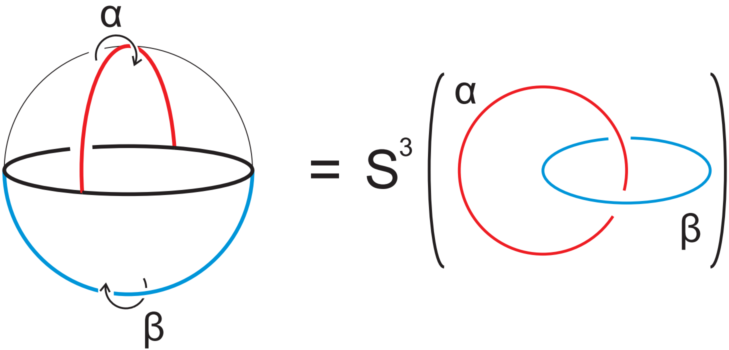
$$\mathbb{E}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Nil}, \mathrm{Solv} \text{ and } \widetilde{\mathrm{PSL}}(2, \mathbb{R}).$$

Thurston's geometrization conjecture: Any three dimensional manifold can be decomposed into pieces, each modeled in one of the eight above mentioned geometries.

This conjecture was proved by Grigori Perelman in 2003. As as consequence, he proved the famous **Poincaré conjecture**:
Any closed three dimensional manifold with the trivial fundamental group is the sphere.

Cone–manifold and geometry of knots

In the present lecture, cone-manifold is a pair $M = (\mathbb{S}^3, \Sigma)$, where $\mathbb{S}^3$ is the three dimensional sphere singular set Σ is a knot or link in $\mathbb{S}^3$. We endow the compliment $\mathbb{S}^3 \setminus \Sigma$ by Riemannian metric of constant sectional curvature $K = 0, \pm 1$ and call M to be hyperbolic, Euclidean or spherical, respectively, in the cases $K = -1$, $K = 0$ or $K = 1$.
By completion of the metric, we recognize Σ as a collection of singular geodesics with prescribed cone angles around each of its components.



Hopf link cone-manifold $2_1^2(\alpha, \beta)$.

Schläfli formula

The main tool for volume calculation is the following Schläfli formula. Let M be a 3-dimensional cone–manifold of constant curvature $K = \pm 1$. Then its volume V is a solution of the differential equation

$$KdV = \frac{1}{2} \sum_i \ell_{\alpha_i} d\alpha_i,$$

where the sum is taken over all components of the singular set Σ with lengths l_{α_i} and cone-angles α_i.

- ⋆ In the above case of Hopf link we have $K = +1$, $\ell_\alpha = \beta, \ell_\beta = \alpha$. Hence $dV = \frac{1}{2}(\beta d\alpha + \alpha d\beta)$ and $V = \frac{\alpha\beta}{2}$.

Let $\mathcal{T}(\alpha) = 3_1(\alpha)$ be a cone–manifold whose underlying space is the three-dimensional sphere $\mathcal{S}^3$ and singular set is trefoil knot $\mathcal{T}$ with cone angle α. See Figure below.

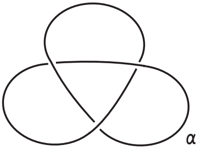


Рис.: Cone-manifold $3_1(\alpha)$

Since $\mathcal{T}$ is a toric knot by the Thurston theorem its complement $\mathcal{T}(0) = \mathcal{S}^3 \setminus \mathcal{T}$ in the $\mathcal{S}^3$ does not admit a hyperbolic structure. However, the trefoil knot admits othrer geometric structures. By H. Seifert and C. Weber (1933) the spherical space of dodecahedron (also known as the Poincaré homology 3-sphere) is a cyclic 5-fold covering of $\mathcal{S}^3$ branched over $\mathcal{T}$. This means that cone–manifold $3_1(\frac{2\pi}{5})$ has a spherical structure.

Geometry of knots. Trefoil knot.

The following theorem describes a spherical structure on the trefoil cone-manifold.

Theorem

The trefoil cone-manifold $\mathcal{T}(\alpha)$ is spherical for $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$. The spherical volume of $\mathcal{T}(\alpha)$ is given by the formula

$$\mathrm{Vol}\,(\mathcal{T}(\alpha)) = \frac{(3\alpha - \pi)^2}{12}.$$

For the proof consider $\mathbb{S}^3$ as the unite sphere in the complex space $\mathbb{C}^2$ endowed by the Riemannian metric

$$\mathrm{d}s_\lambda^2 = |\mathrm{dz}_1|^2 + |\mathrm{dz}_2|^2 + \lambda(\mathrm{dz}_1\mathrm{d}\bar{\mathrm{z}}_2 + \mathrm{d}\bar{\mathrm{z}}_1\mathrm{dz}_2),$$

where $\lambda = (2\sin\frac{\alpha}{2})^{-1}$. Then $\mathbb{S}^3 = (\mathbb{S}^3,\, \mathrm{d}s_\lambda^2)$ is the spherical space of constant curvature $+1$.

The fundamental set for $\mathcal{T}(\alpha)$ is given by the following polyhedron

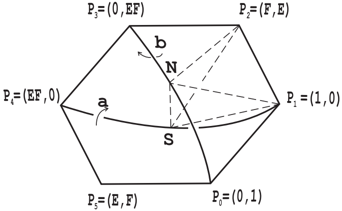


Рис.: Fundamental set for $\mathcal{T}(\alpha)$

where $E = e^{i\,\alpha}$ and $F = e^{i\,\frac{\alpha-\pi}{2}}$ (see Figure 2). The length ℓ_α of singular geodesic of $\mathcal{T}(\alpha)$ is given by $\ell_\alpha = |P_0P_3| + |P_1P_4| = 3\alpha - \pi$.
By the Schläfli formula $\mathrm{dVol}\,\mathcal{T}(\alpha) = \frac{\ell_\alpha}{2}\mathrm{d}\alpha = \frac{3\alpha-\pi}{2}\mathrm{d}\alpha$. So,
$\mathrm{Vol}\,\mathcal{T}(\alpha) = \frac{(3\alpha-\pi)^2}{12} + C$, where C is a constant of integration.
Recall that a 2-fold cover of orbifold $3_1(\pi)$ is the lens space $L(3,1)$ which, in turn, is thrice covered by the three dimensional sphere $\mathbb{S}^3$. Since the spherical volume of $\mathbb{S}^3$ is $2\pi^2$, we have $\mathrm{Vol}\,\mathcal{T}(\pi) = 2\pi^2 : 6 = \frac{\pi^2}{3}$.
Therefore, $C = 0$ and $\mathrm{Vol}\,\mathcal{T}(\alpha) = \frac{(3\alpha-\pi)^2}{12}$.

The figure eight knot or 4_1 knot is the unique prime knot of four crossings.

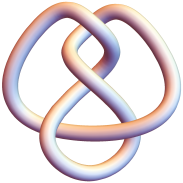


Рис.: Figure eight knot 4_1

The volume of the figure eight cone-manifold in the spaces of constant curvature is given by the following theorem.

Theorem 6 (A. Rasskazov and M., 2006)

Let $\mathrm{V}(\alpha) = \mathrm{Vol}\ 4_1(\alpha)$ *and* ℓ_α *is the length of singular geodesic of* $4_1(\alpha)$. *Then*

$(\mathbb{H}^3)$ $\mathrm{V}(\alpha) = \int_\alpha^{\alpha_0} \mathrm{arccosh}\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ 0 \le \alpha < \alpha_0 = \frac{2\pi}{3},$

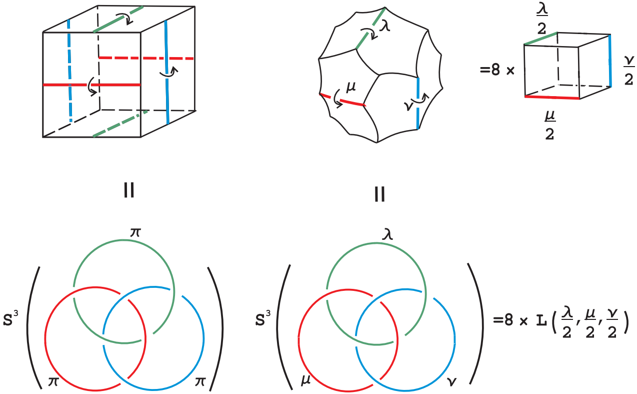
$(\mathbb{E}^3)$ $\mathrm{V}(\alpha_0) = \frac{\sqrt{3}}{108}\,\ell_{\alpha_0}^3,$

$(\mathbb{S}^3)$ $\mathrm{V}(\alpha) = \int_{\alpha_0}^{\alpha} \arccos\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ \alpha_0 < \alpha \le \pi, \quad \mathrm{V}(\pi) = \frac{\pi^2}{5},$

$\mathrm{V}(\alpha) = 2\mathrm{V}(\pi) + \pi(\alpha - \pi) - \mathrm{V}(2\pi - \alpha),\ \pi \le \alpha < 2\pi - \alpha_0.$

- Borromean Rings cone–manifold and Lambert cube

We start with a simple geometrical construction done by W. Thurston, D. Sullivan and J. M. Montesinos.



- Volume calculation for $L(\alpha, \beta, \gamma)$. The main idea.

0. Existence

$$L(\alpha, \beta, \gamma) : \begin{cases} 0 < \alpha, \beta, \gamma < \pi/2, & H^3 \\ \alpha = \beta = \gamma = \pi/2, & E^3 \\ \pi/2 < \alpha, \beta, \gamma < \pi, & S^3. \end{cases}$$

1. Schläfli formula for $V = Vol\, L(\alpha, \beta, \gamma)$

$$k\mathrm{d}V = \frac{1}{2}(\ell_\alpha d\alpha + \ell_\beta d\beta + \ell_\gamma d\gamma), \quad k = \pm 1, 0$$

In particular in hyperbolic case:

$$\begin{cases} \frac{\partial V}{\partial \alpha} = -\frac{\ell_\alpha}{2}, \frac{\partial V}{\partial \beta} = -\frac{\ell_\beta}{2}, \frac{\partial V}{\partial \gamma} = -\frac{\ell_\gamma}{2} & (*) \\ V(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0. & (**) \end{cases}$$

2. Trigonometrical and algebraic identities

(i) Tangent Rule

$$\frac{\tan\alpha}{\tanh\ell_\alpha}=\frac{\tan\beta}{\tanh\ell_\beta}=\frac{\tan\gamma}{\tanh\ell_\gamma}=T\quad(\textbf{R.Kellerhals})$$

(ii) Sine-Cosine Rule (3 different cases)

$$\frac{\sin\alpha}{\sinh\ell_\alpha}\frac{\sin\beta}{\sinh\ell_\beta}\frac{\cos\gamma}{\cosh\ell_\gamma}=1\quad(\textbf{Derevnin}-\textbf{Mednykh})$$

(iii)

$$\frac{T^2-A^2}{1+A^2}\,\frac{T^2-B^2}{1+B^2}\,\frac{T^2-C^2}{1+C^2}\,\frac{1}{T^2}=1,\quad(\textbf{HLM},\textbf{Topology}'\textbf{90})$$

where

$A=\tan\alpha, B=\tan\beta, C=\tan\gamma$. Equivalently,

$(T^2+1)(T^4-(A^2+B^2+C^2+1)T^2+A^2B^2C^2)=0.$

Remark. (ii) $\Rightarrow$(i) and (i) & (ii) $\Rightarrow$ (iii).

3. Integral formula for volume

Hyperbolic volume of $L(\alpha, \beta, \gamma)$ is given by

$$W = \frac{1}{4}\int_{T}^{\infty} \log\left(\frac{t^2 - A^2}{1 + A^2}\,\frac{t^2 - B^2}{1 + B^2}\,\frac{t^2 - C^2}{1 + C^2}\,\frac{1}{t^2}\right)\frac{\mathrm{d}t}{1 + t^2},$$

where T is a positive root of the integrant equation (iii).

Proof. By direct calculation and Tangent Rule (i) we have:

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A}\frac{\partial A}{\partial \alpha} = -\frac{1}{2}\mathrm{arctanh}\,\frac{A}{T} = -\frac{\ell_\alpha}{2}.$$

In a similar way

$$\frac{\partial W}{\partial \beta} = -\frac{\ell_\beta}{2} \quad \text{and} \quad \frac{\partial \mathrm{W}}{\partial \gamma} = -\frac{\ell_\gamma}{2}.$$

By convergence of the integral $W(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$. Hence, $W = V = Vol\, L(\alpha, \beta, \gamma)$.

How predict the volume formula? Differential of volume dV, up to multiple $-\frac{1}{2}$, consides with the following differential form

$$\omega = \ell_\alpha \, d\alpha + \ell_\beta \, d\beta + \ell_\gamma \, d\gamma.$$

Equivalently, in the above notation

$$\omega = \frac{\mathrm{ArcTanh}(\frac{A}{T})dA}{1+A^2} + \frac{\mathrm{ArcTanh}(\frac{B}{T})dB}{1+B^2} + \frac{\mathrm{ArcTanh}(\frac{C}{T})dC}{1+C^2}.$$

Now, for a moment, we suppose that T is an independent variable and extend ω to

$$\Omega = \omega + \frac{F(A,\, B,\, C,\, T)dT}{1+T^2},$$

where F is some unknown function. Since we have a freedom for choice, we suppose that $d\,\Omega = 0$.

Geometry of knots and links. The main trick

The condition $d\,\Omega = 0$ is equivalent to the following equations

1° $\frac{\partial}{\partial A}\left(\frac{F(A,B,C,T)}{1+T^2}\right) = \frac{\partial}{\partial T}\left(\frac{\mathrm{ArcTanh}(\frac{A}{T})}{1+A^2}\right),$

2° $\frac{\partial}{\partial B}\left(\frac{F(A,B,C,T)}{1+T^2}\right) = \frac{\partial}{\partial T}\left(\frac{\mathrm{ArcTanh}(\frac{B}{T})}{1+B^2}\right),$

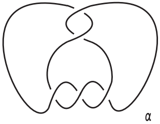
3° $\frac{\partial}{\partial C}\left(\frac{F(A,B,C,T)}{1+T^2}\right) = \frac{\partial}{\partial T}\left(\frac{\mathrm{ArcTanh}(\frac{C}{T})}{1+C^2}\right).$

Hence, up to constant of integration

$$F(A,\ B,\ C,\ T) = \frac{1}{2}\log\left(\frac{T^2 - A^2}{1 + A^2}\frac{T^2 - B^2}{1 + B^2}\frac{T^2 - C^2}{1 + C^2}\frac{1}{T^2}\right).$$

The three twist knot 5_2

The knot 5_2 is a rational knot of a slope $7/2$.



Historically, it was the first knot which was related with hyperbolic geometry. Indeed, it has appeared as a singular set of the hyperbolic orbifold constructed by L.A. Best (1971) from a few copies of Lannér tetrahedra with Coxeter scheme $\circ \equiv \circ - \circ = \circ$. The fundamental set of this orbifold is a regular hyperbolic cube with dihedral angle $2\pi/5$. Later, R. Riley (1979) discovered the existence of a complete hyperbolic structure on the complement of 5_2. In his time, it was one of the nine known examples of knots with hyperbolic complement.

Geometry of two bridge knots. 5_2– knot.

A few years later, it has been proved by W. Thurston that all non-satellite, non-toric prime knots possess this property. Just recently it became known (2007) that the Weeks-Fomenko-Matveev manifold $\mathcal{M}_1$ of volume 0.9427... is the smallest among all closed orientable hyperbolic three manifolds. We note that $\mathcal{M}_1$ was independently found by J. Przytycki and his collaborators (1986). It was proved by A. Vesnin and M. (1998) that manifold $\mathcal{M}_1$ is a cyclic three fold covering of the sphere $\mathbb{S}^3$ branched over the knot 5_2. It was shown by J. Weeks computer program Snappea and proved by Moto-O Takahahsi (1989) that the complement $\mathbb{S}^3 \setminus 5_2$ is a union of three congruent ideal hyperbolic tetrahedra.

The next theorem has been proved by A. Rasskazov and M. (2002), R. Shmatkov (2003) and J. Porti (2004) for hyperbolic, Euclidean and spherical cases, respectively.

Theorem

A cone manifold $5_2(\alpha)$ *is hyperbolic for* $0 \leq \alpha < \alpha_0$*, Euclidean for* $\alpha = \alpha_0$*, and spherical for* $\alpha_0 < \alpha < 2\pi - \alpha_0$*, where* $\alpha_0 \simeq 2.40717...$ *and* $A_0 = \cot(\frac{\alpha_0}{2})$ *is given by the formula*
$A_0 = \sqrt{1/23(-17 - 8\sqrt{2} + 2\sqrt{-235 + 344\sqrt{2}})}$.

Geometry of two bridge knots. 5_2– knot.

Theorem 8 (A. Mednykh, 2009)

Let $5_2(\alpha),\ 0 \le \alpha < \alpha_0$ *be a hyperbolic cone-manifold. Then the volume of* $5_2(\alpha)$ *is given by the formula*

$$\mathrm{Vol}\,(5_2(\alpha)) = i \int_{\bar{z}}^{z} \log\left[\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(1 + \zeta)^2}\right] \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot\frac{\alpha}{2}$ *and* $z,\ \Im z > 0$ *is a root of equation*

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

A completely different approach to find volume of the above cone-manifold is contained in our resent paper (Ji-Young Ham, Alexander Mednykh, Vladimir Petrov, 2014).

Geometry of two bridge knots. 5_2- knot.

Spherical volume of the 5_2- knot is given by the following theorem (M., 2021).

Theorem

Let $5_2(\alpha),\ \alpha_0 < \alpha < 2\pi - \alpha_0$ *be a spherical cone-manifold. Then for any* $\alpha,\ \alpha_0 < \alpha < \pi,$ *the volume* $V(\alpha)$ *of* $5_2(\alpha)$ *is given by the formula*

$$V(\alpha) = \int_{z_1}^{z_2} \log\left(\frac{8(\zeta^2 + A^2)}{(1+A^2)(1-\zeta)(1+\zeta)^2}\right) \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot\frac{\alpha}{2}$ *and* $z_1, z_2, (-1 < z_1 < z_2)$ *are roots of the cubic equation*

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

Also, $V(\pi) = \pi^2/7$ *and*

$$V(\alpha) = 2\,V(\pi) + \pi(\alpha - \pi) - V(2\pi - \alpha) \text{ for } \pi < \alpha < 2\pi - \alpha_0.$$

A-polynomial equation

In this report, we contribute a notion of A-polynomial for $M \setminus \Sigma_\alpha$ given by (D. Cooper, M. Culler, H. Gillet, D.D. Long and P.B. Shalen, 1994).
In the hyperbolic case, cone angle α and complex length $\gamma_\alpha = \ell_\alpha + i\,\varphi_\alpha$ of knot Σ_α are related by the equation

$$A(L, M) = 0, \text{ where } L = e^{\frac{\gamma_\alpha}{2}} \text{ and } M = e^{i\frac{\alpha}{2}}. \tag{1}$$

Also, by the basic properties of A-polynomial we have
$A(L, M) = A(L^{-1}, M)$ and $A(L, M) = A(L, -M)$.
Up to our knowledge, A-polynomials never used before in spherical geometry.
In the spherical geometry, A-polynomial equation has the form

$$A(L, M) = 0, \text{ where } L = e^{\frac{i}{2}(\varphi_\alpha \pm \ell_\alpha)}, \text{ and } M = e^{i\frac{\alpha}{2}}. \tag{2}$$

The proof of the spherical volume formula is based on the following Cotangent Rule. Indeed, this is a trigonometrical version of the *A*-polynomial equation.

Theorem

Let $5_2(\alpha),\ \alpha_0 < \alpha < 2\pi - \alpha_0$ *be a spherical cone-manifold. Denote by* ℓ_α *the length of the longitude of* $5_2(\alpha)$ *and by* φ_α *the angle of its lifted holonomy. Then*

$$\cot(\frac{4\alpha + \varphi_\alpha \pm \ell_\alpha}{4})\cot(\frac{\alpha}{2}) = z_{1,2},$$

where z_1 *and is* z_2 *are roots of the equation* $8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$ *and* $A = \cot(\frac{\alpha}{2})$.

Proof of the spherical volume formula

Suppose that $\alpha_0 < \alpha < \pi$. Let ℓ_α be the length of the longitude for $5_2(\alpha)$ and φ_α be the angle of its lifted holonomy. By the Cotangent Rule, there are real roots z_1 and z_2 of the equation $8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$ such that $z_1 = \cot(\frac{4\alpha+\varphi_\alpha-\ell_\alpha}{4})\cot(\frac{\alpha}{2})$ and $z_2 = \cot(\frac{4\alpha+\varphi_\alpha+\ell_\alpha}{4})\cot(\frac{\alpha}{2})$. Consider the function

$$V(\alpha) = \int_{z_1}^{z_2} \frac{\log F(A,\zeta)}{\zeta^2 - 1}\, d\zeta,$$

where $F(A,\zeta) = \frac{8(\zeta^2+A^2)}{(1+A^2)(1-\zeta)(1+\zeta)^2}$ and $A = \cot(\frac{\alpha}{2})$. To prove the integral volume formula, one has to show that $V(\alpha)$ satisfies the Schläfli equation $V'(\alpha) = \frac{\ell_\alpha}{2}$ with initial data $V(\alpha_0) = 0$. Taking into account that z_1 and z_2 are roots of the integrant, we obtain

Proof of the spherical volume formula

$$\frac{dV(\alpha)}{d\alpha} = \frac{\log F(A, z_2)}{z_2^2 - 1}\frac{dz_2}{d\alpha} - \frac{\log F(A, z_1)}{z_1^2 - 1}\frac{dz_1}{d\alpha}$$

$$+ \int_{z_1}^{z_2} \frac{\partial}{\partial A}\left(\frac{\log F(A, \zeta)}{\zeta^2 - 1}\right)\frac{dA}{d\alpha} d\zeta = \int_{z_1}^{z_2} \frac{A}{A^2 + \zeta^2} d\zeta$$

$$= \operatorname{arccot}(z_2/A) - \operatorname{arccot}(z_1/A)$$

$$= \Big(\frac{4\alpha + \varphi_\alpha + \ell_\alpha}{4}\Big) - \Big(\frac{4\alpha + \varphi_\alpha - \ell_\alpha}{4}\Big) = \frac{\ell_\alpha}{2}.$$

Specific Euclidean volume of $5_2(\alpha)$

The following theorem gives the specific volume of cone-manifold $5_2(\alpha)$ in the Euclidean case. Numerically, this result was obtained earlier by R. N. Shmatkov in his Ph.D. thesis (2003).

Theorem (M., 2021)

Let $5_2(\alpha_0)$, *where* $\alpha_0 = 2.40717\ldots$ *be an Euclidean cone-manifold. Then its specific volume* $v_0 = \frac{\mathrm{Vol}\,(5_2(\alpha_0))}{\ell^3_{\alpha_0}}$ *is given by the formula*

$$\mathrm{vol}\,(5_2(\alpha_0)) = 1/\left(6\sqrt{-6+68\sqrt{2}+4\sqrt{983+946\sqrt{2}}}\right) = 0.00990963\ldots$$

To prove the theorem, we note that $v_0 = \lim_{\alpha\to\alpha_0} \frac{\mathrm{Vol}(5_2(\alpha))}{\ell^3_\alpha}$ and $\mathrm{Vol}(5_2(\alpha)) \to 0$ and $\ell_\alpha \to 0$ as $\alpha \to \alpha_0$. Assume $0 < \alpha < \alpha_0$. Then, by making use of the Schläfli formula and L'Hôpital's rule we obtain $v_0 = \lim_{\alpha\to\alpha_0} \frac{(\mathrm{Vol}(5_2(\alpha)))'_\alpha}{(\ell^3_\alpha)'_\alpha} = \lim_{\alpha\to\alpha_0} \frac{-\ell_\alpha/2}{3\ell^2_\alpha(\ell_\alpha)'_\alpha} = \lim_{\alpha\to\alpha_0} \frac{1}{-3(\ell^2_\alpha)'_\alpha}$.

Tables

We resume the results of numerical calculation for limit of hyperbolicity α_0 and specific Euclidean volume v_0 in the following table. The table contains all hyperbolic knots up to 7 crossings.

Knot	Slope	Limit of hyperboliciy α_0	Euclidean volume v_0
4_1	$5/2, 5/3$	2.094395	0.01603750
5_2	$7/2, 7/3$	2.407169	0.00990963
6_1	$9/2, 9/5$	2.574141	0.00732926
6_2	$11/3, 11/4$	2.684035	0.00538066
6_3	$13/5, 13/8$	2.757265	0.00431666
7_2	$11/2, 11/6$	2.678787	0.00585537
7_3	$13/3, 13/9$	2.755110	0.00449424
7_4	$15/4, 15/11$	2.808209	0.00376538
7_5	$17/5, 17/7$	2.848733	0.00321842
7_6	$19/8, 19/12$	2.880078	0.00283945
7_7	$21/8, 21/13$	2.905300	0.00254482

Rational Volume Conjecture

Consider the volume function for knots and links modeled in spherical geometry. For Hopf link and Trefoil knot it is given by

$$\mathrm{Vol}\,(2_1^2(\alpha,\beta)) = \frac{\alpha\,\beta}{2} \text{ and } \mathrm{Vol}\,(\mathcal{T}(\alpha)) = \frac{(3\alpha-\pi)^2}{12}.$$

Suppose that $\alpha,\ \beta \in \pi\,\mathbb{Q}$ are rational multiples of π then the above volumes are rational multiples of π^2. For Borromean rings we have $\mathrm{Vol}\,(B(\frac{4\pi}{3},\frac{4\pi}{3},\frac{3\pi}{2})) = \frac{31}{72}\pi^2$.

Rational Volume Problem. Suppose that knot or link is modeled in spherical geometry and its cone angles are rational multiples of π. Then its spherical volume is a rational multiple of π^2.