On the significance of parameters in the comprehension schema in second-order arithmetic

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Table of contents

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This talk is devoted to the effect of parameters in the schema of comprehension in second-order arithmetic.

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1. Peano's axioms for numbers.
2. **Induction** $\Phi(0) \wedge \forall\, k \left(\Phi(k) \Longrightarrow \Phi(k+1)\right) \Longrightarrow \forall\, k\, \Phi(k)$, for every formula $\Phi(k)$ in $\mathcal{L}(\mathbf{PA}_2)$, and we allow parameters in $\Phi(k)$, *i. e.*, free variables other than k.

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$\mathbf{PA}_2^*$ is the subsystem of $\mathbf{PA}_2$ with **CA** replaced by **CA***.

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Theorem 1

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The proof of $\mathbf{CA}^*$ in $\langle \omega\, ; X \rangle$ is based on the standard homogeneity and permutation-related properties of the forcing notion $\mathbb{C}^\omega$ involved, which is the finite-support product of the Cohen forcing $\mathbb{C}$. □

Thus $\mathbf{PA}_2^*$ does not prove some simple consequences of the full **CA**.

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Second theorem

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$\mathbf{CA}(\Sigma^1_2)$ is **CA** (with parameters) restricted to Σ^1_2 formulas.

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This is made via a product/iterated Sacks forcing with countable support and with I as the "length" of the iteration.

Theorem 2 proof, part II

TOC

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- the sets a_ξ do not belong to X by construction,
- yet each a_ξ is definable in X, with $a_{\langle \xi \rangle}$ as the only parameter, by means of the structure of the constructibility degrees over $a_{\langle \xi \rangle}$ — thus $\mathbf{CA}$ fails in $\langle \omega\,; X \rangle$.

□

A finite axiomatizability problem

TOC

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Work in progress.

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