On the significance of parameters in the comprehension schema in second-order arithmetic

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Parameters

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This talk is devoted to the effect of parameters in the schema of comprehension in second-order arithmetic.

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1. Peano's axioms for numbers.
2. **Induction** $\Phi(0) \wedge \forall k \left(\Phi(k) \Longrightarrow \Phi(k+1)\right) \Longrightarrow \forall k \, \Phi(k)$, for every formula $\Phi(k)$ in $\mathcal{L}(\mathbf{PA}_2)$, and we allow parameters in $\Phi(k)$, *i. e.*, free variables other than k.

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$\mathbf{CA}^*$ is the parameter-free sub-schema of $\mathbf{CA}$ (that is, $\Phi(k)$ contains no free variables other than k).

$\mathbf{PA}_2^*$ is the subsystem of $\mathbf{PA}_2$ with $\mathbf{CA}$ replaced by $\mathbf{CA}^*$.

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We study the problem in the context of **deductive strength**, and we obtain the opposite answer.

Theorem 1

$\mathbf{PA}_2^*$ *does not prove that every set has its complement.*

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Proof (sketch)

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Then $\langle \omega \,; X \rangle$ is a model of $\mathbf{PA}_2^*$ in which sets a_n do not have their complements, and hence the full $\mathbf{CA}$ fails.

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The proof of $\mathbf{CA}^*$ in $\langle \omega\,; X \rangle$ is based on the standard homogeneity and permutation-related properties of the forcing notion $\mathbb{C}^\omega$ involved, which is the finite-support product of the Cohen forcing $\mathbb{C}$. □

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Thus $\mathbf{PA}_2^*$ does not prove a simple consequence of the full $\mathbf{CA}$.

Second theorem

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Theorem 2

$\mathbf{PA}_2^* + \mathbf{CA}(\Sigma^1_2)$ *does not prove a certain instance of the full* $\mathbf{CA}$.

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$\mathbf{CA}(\Sigma^1_2)$ is $\mathbf{CA}$ (with parameters) restricted to Σ^1_2 formulas.

Theorem 2 proof sketch

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Then $\langle \omega ; X \rangle$ proves Theorem 2.

Model 2: picture

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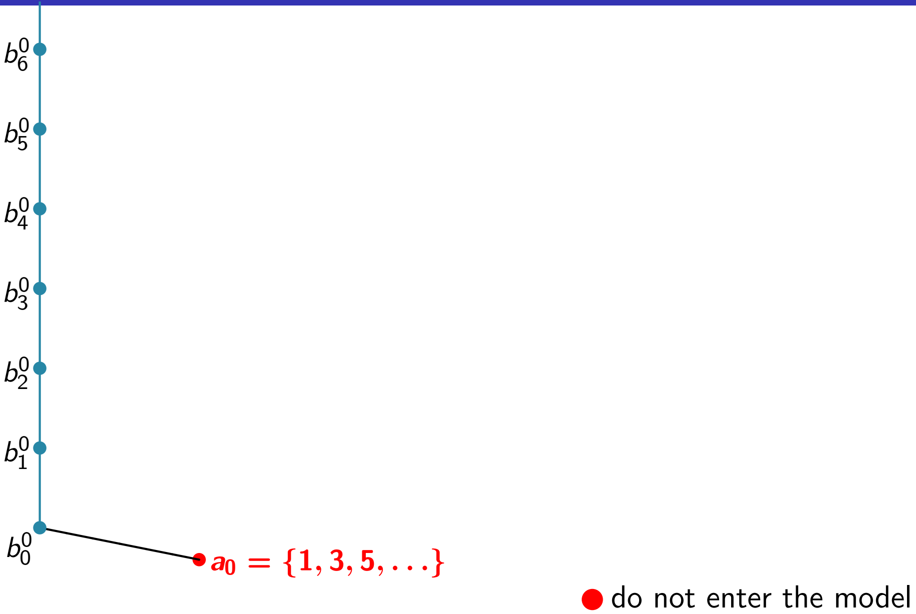
$\bullet\, a_0 = \{1, 3, 5, \dots\}$

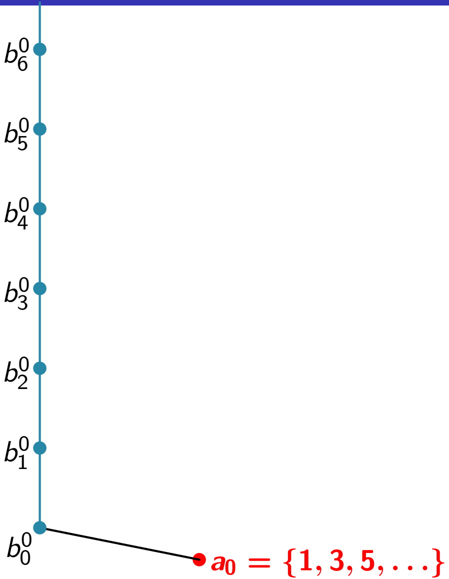
• $a_0 = \{1, 3, 5, \ldots\}$

● do not enter the model


b_6^0
b_5^0
b_4^0
b_3^0
b_2^0
b_1^0
b_0^0
$a_0 = \{1, 3, 5, \ldots\}$
do not enter the model

Model 2: picture

TOC



the structure of Sacksness

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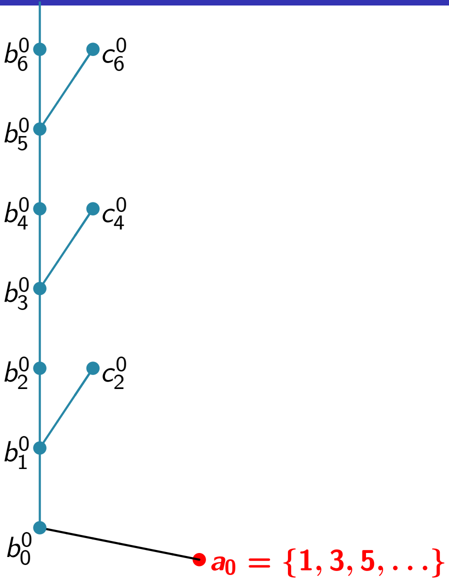

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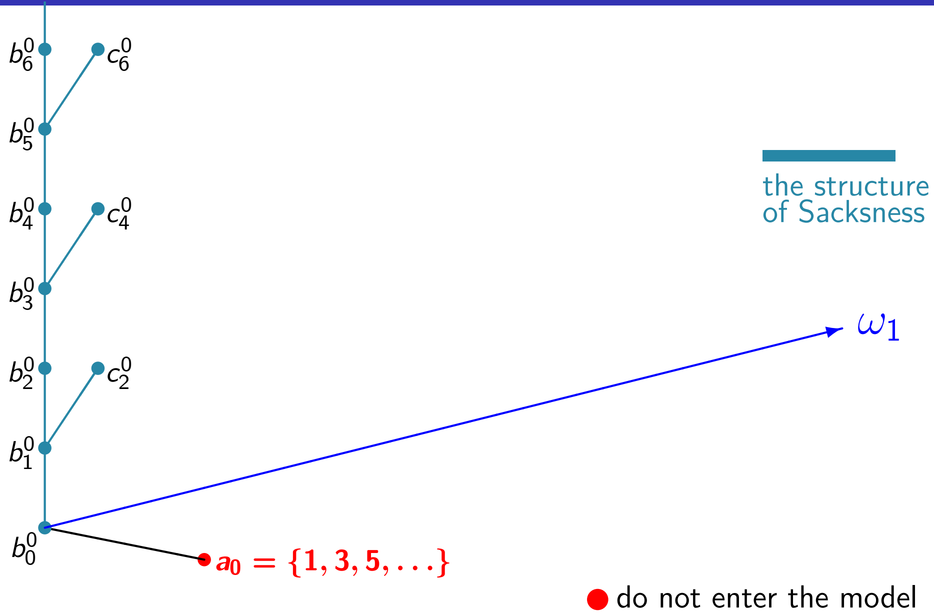
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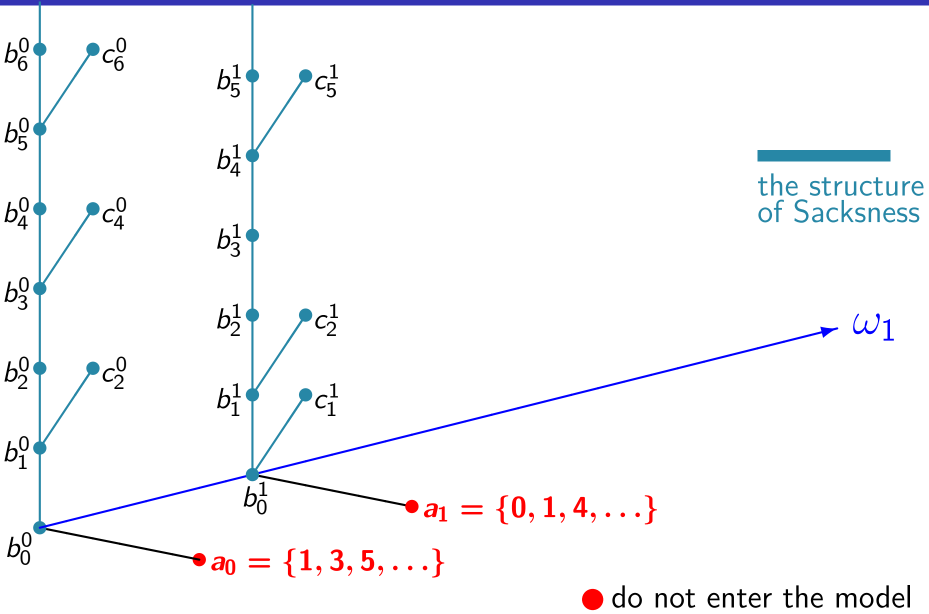

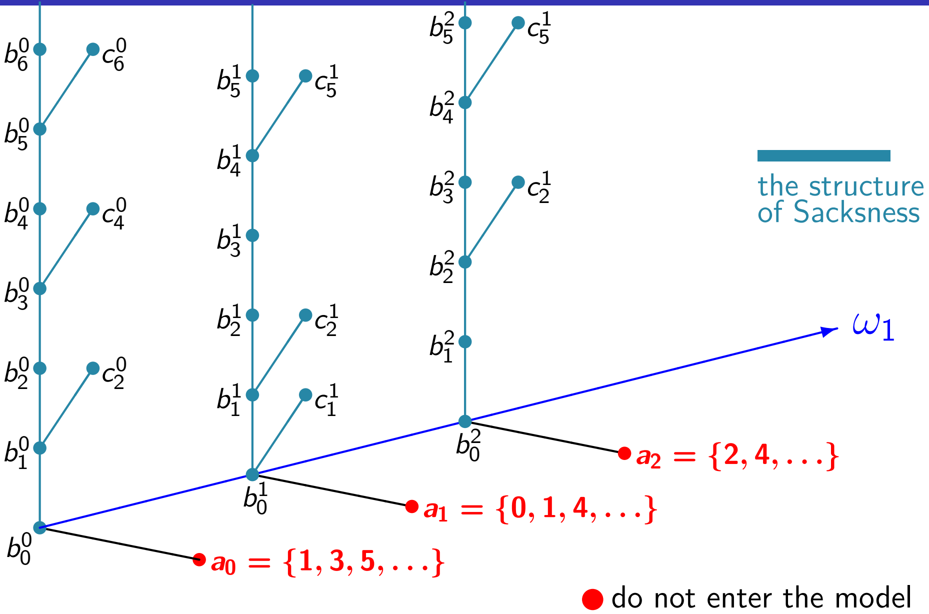
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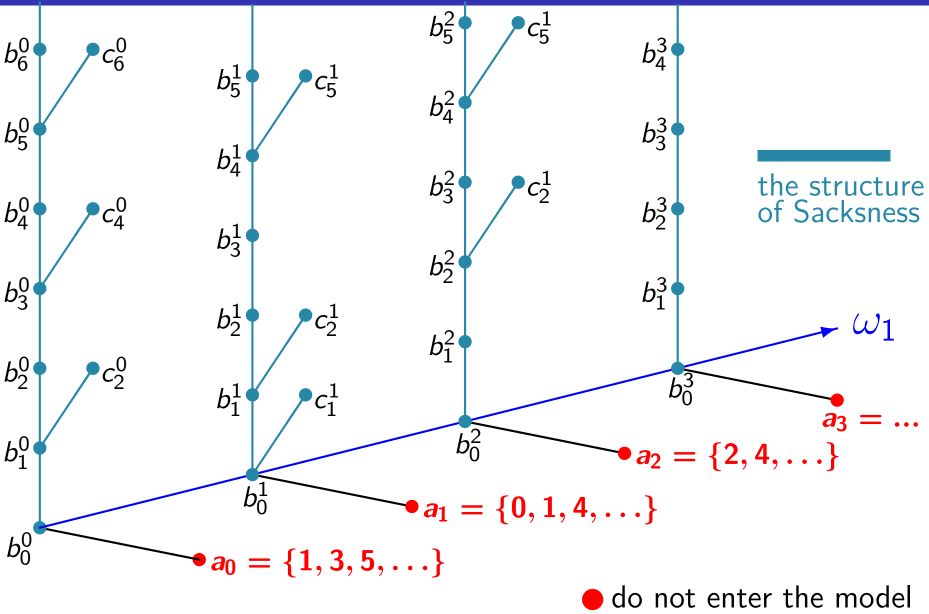

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Work in progress.

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The speaker thanks **everybody** for interest and patience