Dualities for categories of partially ordered structures

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Theorem (M. H. Stone)

The category of distributive $(0,1)$*-lattices with* $(0,1)$*-homomorphisms is dually equivalent to the category of spectral spaces with spectral maps.*

Theorem (M. H. Stone)

The category of Boolean algebras with homomorphisms is dually equivalent to the category of Boolean spaces with continuous maps.

One can generalize these results of M. H. Stone in two directions:

- dualities for distributive posets;
- dualities for $(0, 1)$-lattices which are close to distributive.

Dualities for distributive posets

For a poset $\langle P; \leq \rangle$ and $X \subseteq P$:

$L(X)$ is the set of all lower bounds of X;

$U(X)$ is the set of all upper bounds of X.

Distributive posets

A c-**poset** is a structure $\mathcal{P} = \langle P; \leq, \varphi \rangle$ such that:

- $\langle P; \leq \rangle$ is a poset;
- φ is an algebraic closure operator on P which defines a **completion** of $\langle P; \leq \rangle$; that is,

$$\varphi\colon P \to \operatorname{Id}\mathcal{P}, \quad \varphi\colon x \mapsto \varphi(x)$$

  is an order embedding of $\langle P; \leq \rangle$ into the complete lattice $\operatorname{Id}\mathcal{P}$ of φ-closed subsets of P.

Corollary

If $\mathcal{P} = \langle P; \leq, \varphi \rangle$ is a c-poset then $\varphi(x) = L(x)$ for all $x \in P$.

A c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$ is **distributive** or just a **distributive poset**, if the following condition is satisfied:

- the lattice $\operatorname{Id} \mathcal{P}$ is distributive.

Any φ-closed subset of P is a φ-**ideal** of $\langle P; \leq \rangle$ or just an **ideal** of $\mathcal{P}$.

A set $F \subseteq P$ is a **filter** of $\langle P; \leq \rangle$ if it is down-directed with respect to $\leq$.

Lemma (Ts. Batueva, MS)

For a c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$ and a proper ideal I of $\mathcal{P}$, TFAE.

1. $P \backslash I$ *is a filter of* $\langle P; \leq \rangle$.
2. I *is a* $\cap$*-prime element in* $\operatorname{Id} \mathcal{P}$.
3. $L(a_0, a_1) \subseteq I$ *implies that* $a_i \in I$ *for some* $i < 2$.

Let $\mathcal{P} = \langle P; \leq, \varphi \rangle$ be a c-poset. An ideal I of $\mathcal{P}$ is **prime** if it satisfies one of the equivalent statements of Lemma above.

Theorem (Ts. Batueva, MS)

Let $\mathcal{P} = \langle P; \leq, \varphi \rangle$ be a distributive c-poset, let $I \subseteq P$ be a nonempty ideal of $\mathcal{P}$, and let $F \subseteq P$ be a nonempty filter such that $I \cap F = \varnothing$. Then there is a prime ideal $Q \subseteq P$ such that $I \subseteq Q$ and $Q \cap F = \varnothing$.

The category DP

Let **DP** denote the category whose objects are distributive c-posets and whose morphisms are mappings $f\colon P_0 \to P_1$, where $\mathcal{P}_0 = \langle P_0; \leq, \varphi_0\rangle$ and $\mathcal{P}_1 = \langle P_1; \leq, \varphi_1\rangle$ are distributive c-posets, which satisfy the following condition:

- f is **proper**; that is, $f^{-1}(I)$ is a prime ideal of $\mathcal{P}_0$ for each prime ideal I of $\mathcal{P}_1$;

DP-morphisms preserve meets, joins, 0, 1. In particular, they are monotone.

Lemma

Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be distributive $(0,1)$-lattices. Then $f : \mathcal{S}_0 \longrightarrow \mathcal{S}_1$ is a **DP**-*morphism if and only if f is a $(0,1)$-lattice homomorphism.*

The following categories are full subcategories of **DP**:

$\mathbf{DP}_0$, $\mathbf{DP}_1$, $\mathbf{DP}_{01}$, $\mathbf{DSL}^{\wedge}$, $\mathbf{DSL}_0^{\wedge}$, $\mathbf{DSL}_1^{\wedge}$, $\mathbf{DSL}_{01}^{\wedge}$, $\mathbf{DSL}^{\vee}$, $\mathbf{DSL}_0^{\vee}$, $\mathbf{DSL}_1^{\vee}$, $\mathbf{DSL}_{01}^{\vee}$, $\mathbf{DL}$, $\mathbf{DL}_0$, $\mathbf{DL}_1$, $\mathbf{DL}_{01}$.

Spectra of posets [Yu. L. Ershov, MS]

Let $\mathcal{P} = \langle P; \leq, \varphi \rangle$ be a c-poset.
$\operatorname{Spec} \mathcal{P}$ is the set of all prime ideals of $\mathcal{P}$. For each $a \in P$, we put

$$V_a = \{I \in \operatorname{Spec} \mathcal{P} \mid a \notin I\}.$$

The space $\mathbb{S}\mathrm{pec}\, \mathcal{P} = \langle \operatorname{Spec} \mathcal{P}, \mathcal{T}, \mathcal{B} \rangle$, where $\mathcal{T}$ denotes the topology with the basis $\mathcal{B} = \{V_a \mid a \in P\}$, is the **spectrum of poset** $\mathcal{P}$.

The space $\mathbb{S}\mathrm{pec}\, \mathcal{S}$ is called the **spectrum of a join-semilattice** $\langle S; \vee \rangle$, where $\mathcal{S} = \langle S; \vee, \psi \rangle$ and

$$\psi(X) = \{s \in S \mid s \leq a \vee \ldots \vee a_n \text{ for some } n < \omega,\ a_0, \ldots, a_n \in X\}.$$

The **spectrum of a lattice** $\langle L; \vee, \wedge \rangle$ is the spectrum of its join-semilattice reduct $\langle L; \vee \rangle$.

Lemma

For a c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$, the following statements hold.

1. *If $a \wedge b$ exists in $\mathcal{P}$ for some $a, b \in P$ then $V_{a \wedge b} = V_a \cap V_b$. If $\mathcal{P}$ is distributive then $V_a \cap V_b = V_c$ for some $c \in P$ implies that $c = a \wedge b$ in $\mathcal{P}$.*
2. *If $\langle P; \leq \rangle$ is a join-semilattice and $\varphi = \psi$ then $V_{a \vee b} = V_a \cup V_b$ for all $a, b \in P$. If $\mathcal{P}$ is distributive then $V_a \cup V_b = V_c$ for some $c \in P$ implies that $c = a \vee b$ in $\mathcal{P}$.*

Sober spaces

Let $\mathbb{X}$ be a T_0-space.

A subset $Y \subseteq X$ is **irreducible** if $Y \subseteq F_0 \cup F_1$ for some closed sets F_0, F_1 implies that $Y \subseteq F_i$ for some $i < 2$.

$\mathbb{X}$ is **sober** if for each nonempty closed irreducible set $F \subseteq X$, there is $x \in X$ such that $F = \downarrow x$.

$\mathbb{X}$ is **almost sober** if for each proper closed irreducible set $F \subseteq X$, there is $x \in X$ such that $F = \downarrow x$.

Proposition (Yu. L. Ershov, MS)

For a c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$, the following statements hold.

1. $\mathbb{S}\mathrm{pec}\, \mathcal{P}$ is almost sober.
2. $\mathbb{S}\mathrm{pec}\, \mathcal{P}$ is sober whenever P has 0.
3. If $\mathcal{P}$ is down-directed distributive and $\mathbb{S}\mathrm{pec}\, \mathcal{P}$ is sober then $\langle P; \leq \rangle$ has 0.

Proposition (Yu. L. Ershov, MS)

For a distributive c-poset $\mathcal{P} = \langle P; \leq, \varphi\rangle$, the following statements hold.

1. The set V_a is compact in $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ for every $a \in S$. In particular, $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is compact whenever $\langle P; \leq\rangle$ has 1.
2. If $\mathcal{P}$ is up-directed and $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is compact then $\langle P; \leq\rangle$ has 1.

For a T_0-space $\mathbb{X}$ and $\mathcal{F} \subseteq \mathcal{T}(\mathbb{X})$, define a closure operator $\varphi_{\mathcal{F}}$ on $\mathcal{F}$ as follows. If $\mathcal{X} \subseteq \mathcal{F}$ then

$$\varphi_{\mathcal{F}}(\mathcal{X}) = \{U \in \mathcal{F} \mid U \subseteq \bigcup \mathcal{X}\}.$$

If $\mathcal{F}$ consists of compact sets, then $\varphi_{\mathcal{F}}$ is algebraic.

Lemma

If a family $\mathcal{B}$ *of compact open sets in* $\mathbb{X}$ *forms a base of* $\mathcal{T}(\mathbb{X})$ *then the c-poset* $\mathfrak{B} = \langle \mathcal{B}; \subseteq, \varphi_{\mathcal{B}} \rangle$ *is distributive.*

Spaces with base

Definition

A triple $\mathbb{X} = \langle X, \mathcal{T}, \mathcal{B} \rangle$ is a **topological space with base** or just a **space with base**, if

1. $\langle X, \mathcal{T} \rangle$ is a T_0-space and $\mathcal{B}$ forms a base of $\mathcal{T}$;
2. $\langle X, \mathcal{T} \rangle$ is sober if and only if $\varnothing \in \mathcal{B}$;
3. X is compact in $\langle X, \mathcal{T} \rangle$ if and only if $X \in \mathcal{B}$.

$\mathcal{B}$ is a **multiplicative base** if $\mathcal{B}$ is closed under finite nonempty intersections. $\mathcal{B}$ is an **additive base** if $\mathcal{B}$ is closed under finite nonempty unions.

$\mathcal{K}(\mathbb{X})$ is the set of all compact sets in $\mathbb{X}$.

Lemma

Let $\mathbb{X}$ be a space with additive base and let $\mathcal{B}(\mathbb{X}) \subseteq \mathcal{K}(\mathbb{X})$. Then

$$\mathcal{B}(\mathbb{X}) = \begin{cases} \mathcal{K}(\mathbb{X}), & \text{if } \varnothing \in \mathcal{B}(\mathbb{X}); \\ \mathcal{K}_0(\mathbb{X}), & \text{if } \varnothing \notin \mathcal{B}(\mathbb{X}). \end{cases}$$

Almost [semi]spectral spaces with base

Definition

A space with base $\mathbb{X}$ is an **almost semispectral space with base**, if $\langle \mathbb{X}, \mathcal{T}(\mathbb{X})\rangle$ is an almost sober space, and $\mathcal{B}(\mathbb{X})$ consists of open compact sets.

$\mathbb{X}$ is an **almost spectral space with base**, if $\langle \mathbb{X}, \mathcal{T}(\mathbb{X})\rangle$ is an almost sober space, and $\mathcal{B}(\mathbb{X})$ is a multiplicative base of $\mathcal{T}(\mathbb{X})$ consisting of open compact sets.

$\mathbb{X}$ is a **[semi]spectral space with base**, if $\mathbb{X}$ is almost [semi]spectral space with base and $\langle \mathbb{X}, \mathcal{T}(\mathbb{X})\rangle$ is a compact sober space.

The category AS

Let **AS** be the category whose objects are almost semispectral spaces with base and whose morphisms are **spectral** mappings:

> if $f\colon \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}, \mathbb{Y} \in$ **AS** then $f^{-1}(U) \in \mathcal{B}(\mathbb{X})$ for all $U \in \mathcal{B}(\mathbb{Y})$.

Lemma

If $\mathbb{X}, \mathbb{Y}$ *are almost semispectral spaces with base and* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is spectral then* f *is continuous.*

We consider the following full subcategories of the category **AS**:

- the category $\mathbf{AS}_c$ whose objects are compact almost semispectral spaces with base;
- the category $\mathbf{AS}_s$ whose objects are sober almost semispectral spaces with base;
- the category **S** whose objects are semispectral spaces with base;
- the category **ASp** whose objects are almost spectral spaces with base;
- the category $\mathbf{ASp}_c$ whose objects are compact almost spectral spaces with base;
- the category $\mathbf{ASp}_s$ whose objects are sober almost spectral spaces with base;
- the category **Sp** whose objects are spectral spaces with base;

- the category **AsSpec** whose objects are almost semispectral spaces;
- the category $\mathbf{AsSpec}_c$ whose objects are compact almost semispectral spaces;
- the category $\mathbf{AsSpec}_s$ whose objects are sober almost semispectral spaces;
- the category **sSpec** whose objects are semispectral spaces;
- the category **ASpec** whose objects are almost spectral spaces;
- the category $\mathbf{ASpec}_c$ whose objects are compact almost spectral spaces;
- the category $\mathbf{ASpec}_s$ whose objects are sober almost spectral spaces;
- the category **Spec** whose objects are spectral spaces.

Theorem (Yu. L. Ershov, MS)

For a T_0-space $\mathbb{X}$, the following holds.

1. $\mathbb{X}$ *is a semispectral space if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(0, 1, \vee)$*-semilattice.*
2. $\mathbb{X}$ *is a compact almost semispectral space if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(1, \vee)$*-semilattice.*
3. $\mathbb{X}$ *is sober almost semispectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(0, \vee)$*-semilattice.*
4. $\mathbb{X}$ *is almost semispectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $\vee$*-semilattice.*

Theorem (Yu. L. Ershov, MS)

For a T_0-space $\mathbb{X}$, the following holds.

1. *$\mathbb{X}$ is spectral if and only if $\mathbb{X}$ is homeomorphic to the spectrum of a distributive $(0,1)$-lattice.*
2. *$\mathbb{X}$ is compact almost spectral if and only if $\mathbb{X}$ is homeomorphic to the spectrum of a distributive 1-lattice.*
3. *$\mathbb{X}$ is sober almost spectral if and only if $\mathbb{X}$ is homeomorphic to the spectrum of a distributive 0-lattice.*
4. *$\mathbb{X}$ is almost spectral if and only if $\mathbb{X}$ is homeomorphic to the spectrum of a distributive lattice.*

These two theorems extend to a full duality.

The functor T

$$\mathsf{T}\colon \mathbf{DP} \longrightarrow \mathbf{AS};$$
$$\mathsf{T}\colon \mathcal{P} \longmapsto \mathbb{S}\mathrm{pec}\,\mathcal{P};$$
$$\text{if } f\colon \mathcal{P}_0 \longrightarrow \mathcal{P}_1 \text{ then } \mathsf{T}(f)\colon I \longmapsto f^{-1}(I).$$

The functor P

$$\mathrm{P}\colon \mathbf{AS} \to \mathbf{DP};$$
$$\mathrm{P}\colon \mathbb{X} \mapsto \langle \mathcal{B}(\mathbb{X}); \subseteq, \varphi_{\mathcal{B}} \rangle;$$
$$\text{if } f\colon \mathbb{X}_0 \to \mathbb{X}_1 \text{ then } \mathrm{T}(f)\colon U \mapsto f^{-1}(U).$$

Theorem

The categories $\mathbf{DP}$ *and* $\mathbf{AS}$ *are dually equivalent via* P *and* T*. The categories* $\mathbf{DP}_{fin}$ *and* $\mathbf{AS}_{fin}$ *are therefore also dually equivalent.*

An instance:

Corollary

P *and* T *establish the dual equivalence of categories* $\mathbf{DSL}_0^{\wedge}$ *and* $\mathbf{ASp}_s$.

Dualities for $(0,1)$-lattices: quasivarieties generated by finite $(0,1)$-lattices

The quasivariety $\mathbf{SP}(N_5)$

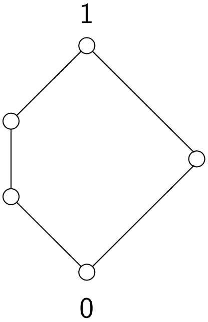

Figure: Lattice N_5

Definition (W. A. Dziobiak, MS)

A structure $\mathbb{S} = \langle X, Y, \leq, f \rangle$ is an *N_5-space*, if

(s1) $X \cap Y = \varnothing$ and $X \cup Y \neq \varnothing$;

(s2) $\leq$ is a partial order on $X \cup Y$;

(s3) $f\colon Y \to X^2$ is a function and for all $y \in Y$ with $f(y) = (a, b)$, the following conditions hold:

- $a \leq y$ and $\{a, b\}$, $\{y, b\}$ are antichains;
- if $a, b \leq z$ for some $z \in X \cup Y$ then $y \leq z$;
- if $z \leq y$ for some $z \in X \cup Y$ then either $z \leq a$ or $z \leq b$, or $z \in Y$ and $\{u, v\} \ll \{a, b\}$ where $f(z) = (u, v)$.

Definition (W. A. Dziobiak, MS)

Let $\mathbb{S} = \langle X, Y, \leq, f \rangle$ and $\mathbb{S}' = \langle X', Y', \leq, f \rangle$ be N_5-spaces. Then $\varphi \colon \mathbb{S} \to \mathbb{S}'$ is a *N_5-morphism*, if the following conditions hold:

(m1) φ maps $X \cup Y$ into $X' \cup Y' \cup \{\{a, b\} \mid a, b \in X'\}$;

(m2) if $u, v \in X \cup Y$ are such that $\varphi(u), \varphi(v) \in X' \cup Y'$ and $u \leq v$ then $\varphi(u) \leq \varphi(v)$;

(m3) for all $x \in X$, $\varphi(x) \in X'$;

(m4) for all $y \in Y$ with $f(y) = (a, b)$, the following holds:

- if $\varphi(y) \in X'$ then either $\varphi(y) = \varphi(a)$ or $\varphi(y) \leq \varphi(b)$;
- if $\varphi(y) \in Y'$ then $f(\varphi(y)) = (\varphi(a), \varphi(b))$;
- if $\varphi(y) \notin X' \cup Y'$ then $\varphi(y) = \{\varphi(a), \varphi(b)\}$ is an antichain and $\{\varphi(a), \varphi(b)\} \ll \varphi(z)$ for all $z \in X \cup Y$ with $y \leq z$.

Objects in N_5 are bi-algebraic $(0,1)$-lattices belonging to $\mathbf{SP}(N_5)$. Morphisms in N_5 are complete $(0,1)$-lattice homomorphisms.

Objects in B_5 are N_5-spaces. Morphisms in B_5 are N_5-morphisms.

Theorem (W. A. Dziobiak, MS)

The categories N_5 *and* B_5 *are dually equivalent.*

Definition (W. A. Dziobiak, MS)

A structure $\mathbb{S} = \langle X, Y, \leq, f, \mathcal{T}\rangle$ is a *topological N_5-space*, if the following conditions hold:

- $\langle X, Y, \leq, f\rangle$ is an N_5-space;
- $\mathcal{T}$ is a T_0-topology on $S = X \cup Y$ having a multiplicative basis of open ideals;
- the specialization order is $\leq^\partial$;
- the set $\mathcal{T}_I(\mathbb{S})$ of open ideals of $\langle \mathbb{S}, \mathcal{T}\rangle$ forms a complete arithmetic lattice with respect to $\subseteq$ and is a $(0, 1)$-sublattice in the ideal lattice of $\mathbb{S}$; $X \cup Y$ is a lattice-compact ideal of $\mathbb{S}$;

The $(0, 1)$-lattice of compact elements of $\mathcal{T}_I(\mathbb{S})$ (including the empty set $\varnothing$), we denote by $\mathcal{T}_c(\mathbb{S})$.

Definition (W. A. Dziobiak, MS)

Let $\mathbb{S}_0$ and $\mathbb{S}_1$ be topological N_5-spaces and let $\varphi\colon \mathbb{S}_0 \to \mathbb{S}_1$ be an N_5-morphism. Then φ is a *spectral N_5-morphism* if

$$\varphi^{-1}(I) = \{z \in S_0 \mid \varphi(z) \in I\} \cup \{z \in Y_0 \mid \varphi(z) \subseteq I\} \in \mathcal{T}_c(\mathbb{S}_0)$$

for each $I \in \mathcal{T}_c(\mathbb{S}_1)$.

Definition (W. A. Dziobiak, MS)

If $\mathbb{S} \leq \mathbb{T}$ is an extension of N_5-spaces then $\mathbb{S}$ is an N_5-*subspace* of $\mathbb{T}$ if the inclusion map is an N_5-morphism. An extension $\mathbb{S} \leq \mathbb{T}$ of topological N_5-spaces is a uN_5-*extension* if:

1. $\mathbb{S}$ is a topological subspace of $\mathbb{T}$;
2. the inclusion map is a spectral N_5-morphism;
3. $I = (I \cap S)^*$ for all $I \in \mathcal{T}_c(\mathbb{T})$.

Definition (W. A. Dziobiak, MS)

A topological N_5-space $\mathbb{S}$ is a N_5-*spectral space*, if $\mathbb{S}$ has no proper uN_5-extensions.

Objects in L_5 are $(0,1)$-lattices belonging to $\mathbf{SP}(N_5)$. Morphisms in L_5 are $(0,1)$-lattice homomorphisms.

Objects in T_5 are spectral N_5-spaces. Morphisms in T_5 are spectral N_5-morphisms.

Theorem (W. A. Dziobiak, MS)

The categories L_5 *and* T_5 *are dually equivalent.*

Similar duality results are also established for:

- the quasivariety $\mathbf{SP}(L_6)$ (O. A. Kadyrova, MS);
- the quasivariety $\mathbf{SP}(M_3)$ (W. A. Dziobiak, MS).

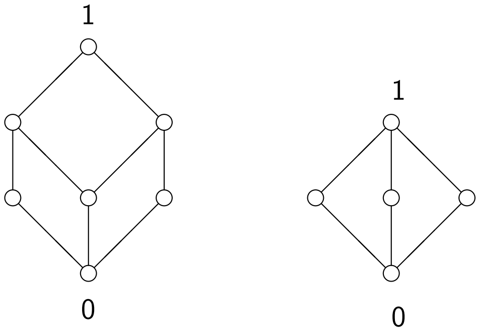

Figure: Lattices L_6 and M_3

Restrictions of all dualities to distributive lattices yield the Stone duality.