Interpretations of Büchi arithmetics in themselves

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Büchi Arithmetics

Definition

A Büchi arithmetic BA_n*,* $n \geq 2$*, is the theory* $\mathrm{Th}(\mathbb{N}, =, +, V_n)$ *where* V_n *is an unary functional symbol such that* $V_n(x)$ *is the largest power of n that divides* x *(*$V_n(0) := 0$ *by definition).*

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Büchi arithmetics were proposed by J. Büchi in order to describe the recognizability of sets of natural numbers by finite automata via logical means. From the definition, BA_n are complete. It is known the theories are also decidable. The Cobham-Semenov theorem states that for multiplicatively independent natural numbers n, m (two numbers n, m are called multiplicatively independent if the equation $n^k = m^l$ has no integer solutions beside $k = l = 0$), any set definable in BA_n and BA_m is actually definable in Presburger arithmetic $\mathrm{PrA} = \mathrm{Th}(\mathbb{N}, =, +)$.

Bruyère Theorem

Let a finite automaton receive natural numbers in their n-ary expansion, right to left, or a tuple of natural numbers simultaneously, all the last digits, followed by the penultimate ones, etc.

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Theorem

Let $\varphi(x_1, \dots, x_m)$ be a BA_n-formula. Then there is an effectively constructed automaton $\mathcal{A}$ such that $(a_1, \dots, a_m)$ is accepted by $\mathcal{A}$ iff $\varphi \models (a_1, \dots, a_m)$. Contrariwise, let $\mathcal{A}$ be a finite automaton working on m-tuples of n-ary natural numbers. Then there is an effectively constructed BA_n-formula $\varphi(x_1, \dots, x_m)$ such that $\varphi \models (a_1, \dots, a_m)$ iff $(a_1, \dots, a_m)$ is accepted by $\mathcal{A}$.

Interpretations of models

Let $\mathcal{K}$, $\mathcal{L}$ be two first-order languages.

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*A **non-parametric** m-**dimensional interpretation** ι of $\mathcal{K}$ in an $\mathcal{L}$-structure $\mathfrak{B}$ consists of the following $\mathcal{L}$-formulas:*

1. $D_\iota(\overline{y})$ *defining the set* $\mathrm{D}_\iota \subseteq \mathfrak{B}^m$ *(domain of interpreted model);*
2. $P_\iota(\overline{x}_1, \ldots, \overline{x}_n)$*, for each predicate symbol* $P(x_1, \ldots, x_n)$ *in* $\mathcal{K}$ *including equality;*
3. $f_\iota(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$*, for each functional symbol* $f(x_1, \ldots, x_n)$ *in* $\mathcal{K}$*.*

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All vectors of variables $\overline{x}$ are of length m, f_ι's should define graphs of some functions modulo interpretation of equality.

The definition inductively generalizes into the traslation of each $\mathcal{K}$-formula.

Naturally, ι and $\mathfrak{B}$ define a $\mathcal{K}$-structure $\mathfrak{A}$ with the support $\mathrm{D}_\iota/\sim_\iota$, $\sim_\iota$ is given by $=_\iota (\overline{x}_1, \overline{x}_2)$. $\mathfrak{A}$ is called the *internal model*.

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1. m_1-dimensional interpretation ι_1 and m_2-dimensional interpretation ι_2 are **isomorphic**, if there is an isomorphism f between the corresponding internal models;
2. If f can be expressed by a $(m_1 + m_2)$-ary formula, F this isomorphism is called *definable*.

Interpretations of BA_n

We consider the interpretations of BA_n in themselves. A. Visser has proposed the following question: *given an weak arithmetical theory T without ability to encode syntax but with full induction, does it hold that each interpretation (one-dimensional or multi-dimensional) of T into itself is isomorphic to the trivial one, and, if it is, is the isomorphism always expressible by a formula of T?*

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The corresponding question has been answered positively for Presburger arithmetic PrA in the author's joint work with F. Pakhomov. Being conservative extensions of PrA with a new functional symbol, Büchi arithmetics are a natural series of theories for possible generalizations.

Non-Standard Models of BA_n

Note that as BA_n is the true theory of $(\mathbb{N}; =, +, V_n)$, it is sufficient to consider the interpretations into the standard model $(\mathbb{N}; =, +, V_n)$. We intend to establish whether for each interpretation ι of BA_n in $(\mathbb{N}; =, +, V_n)$ the internal model is isomorphic to the standard one. Hence, it is required to check whether some non-standard model of BA_n is interpretable in Büchi arithmetic.

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The order-types of the non-standard models of BA_n are described by the following standard result.

Theorem

Any nonstandard model $\mathcal{A} \models \mathrm{BA}_n$ has the order type $\mathbb{N} + \mathbb{Z} \cdot A$, where $\langle A, <_A \rangle$ is some dense linear order without endpoints. Thus, in particular, any countable non-standard model of BA_n has the order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$.

Theorem

The following theorem has been established:

Theorem

Let ι be an interpretation of PrA *in the standard model* $\mathbb{N}$ *of* BA_n*. Then the internal model* $\mathcal{A}$ *induced by* ι *is isomorphic to the standard one.*

This gives a partial positive answer to Visser's question.

Extension to the integers

The idea of the proof is following.

By forgetting about $V_n(x)$, it becomes possible to consider the interpretation of a non-standard model of BA_n in Büchi arithmetic as an interpretation of addition only (in other words, an interpretation of Presburger arithmetic PrA).

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Such an internal model induced by an interpretation of PrA is not an abelian group but merely a monoid, as none of the elements, except zero, has an additive inverse. But it is possible to define negative numbers separately and introduce addition per component. Thus each interpretation of PrA in BA_n, in fact, induces an interpretation of a corresponding abelian group:

Lemma

Let PrA *be interpreted in* BA_n *by interpretation* ι. *Then it can be extended to an interpretation of* $(\mathbb{Z}, +, <)$ *such that the internal model of* PrA *will correspond to the non-negative numbers.*

The non-standard internal model

Let ι be an interpretation of PrA such that the internal model is non-standard. As $\mathbb{N}$ is countable, the order-type of the interpretation equals $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. By applying the lemma, we may construct an interpretation ι' of an ordered abelian group $\mathcal{B}$ which order-type is isomorphic to $\mathbb{Z} \cdot \mathbb{Q}$.

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We may consider the galaxies

$$[c] := \{d \in \mathcal{B} \mid |c - d| \text{ is a standard natural number}\}.$$

In particular, the standard integers form one of the galaxies, the one containing zero (the neutral element of $\mathcal{B}$). In fact, the addition on galaxies $[c + d] := [c] + [d]$ is well-defined. Furthermore:

Lemma

$\mathcal{B}/\mathbb{Z}$ contains a subgroup Q isomorphic to $(\mathbb{Q}, +)$.

Automatic abelian groups

On the other hand, the following necessary condition for automatic groups was established by Braun and Strüngmann.

Theorem

Let $(A, +)$ be an automatic torsion-free abelian group. Then there exists a subgroup B of A isomorphic to $\mathbb{Z}^m$ for some natural m such that the orders of the elements in $C = A/B$ are only divisible by a finite number of different primes $p_1, \ldots, p_s$.

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We show this contradicts the existence of a subgroup Q isomorphic to $(\mathbb{Q},+)$ in $\mathcal{B}/\mathbb{Z}$.

Plans for further research

- Establish whether each isomorphism between the internal model of BA_n and $\mathbb{N}$ is automatic (second part of Visser's question).

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- Find an explicit axiomatization of BA_n, for each n.

Thank you!