# On Spectrally Universal Classes of Structures

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# Degree spectrum of a structure

Let $L$ be a computable signature. We identify first-order $L$-formulas with their Gödel numbers.

For an $L$-structure $\mathcal{A}$, by $D(\mathcal{A})$ we denote the atomic diagram of $\mathcal{A}$.

## Definition

Let $\mathcal{S}$ be a countably infinite $L$-structure. The **degree spectrum** of the structure $\mathcal{S}$ is the set

$$\mathrm{DgSp}(\mathcal{S}) = \{\deg_T(D(\mathcal{A})) : \mathcal{A} \cong \mathcal{S}, \text{ and } \mathrm{dom}(\mathcal{A}) = \omega\}.$$

# A brief glance at degree spectra

A structure $\mathcal{S}$ is *automorphically trivial* if there exists a finite set $F \subseteq \mathrm{dom}(\mathcal{S})$ with the following property: any permutation $f$ of $\mathrm{dom}(\mathcal{S})$, which keeps the elements of $F$ fixed, is an automorphism of $\mathcal{S}$.

E.g., a complete graph is automorphically trivial.

## Theorem (Knight 1986)

If a countable structure $\mathcal{S}$ is not automorphically trivial, then its spectrum $\mathrm{DgSp}(\mathcal{S})$ is closed upwards.

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Example (Richter 1981). By $(p_i)_{i\in\omega}$ we denote the list of all prime numbers in increasing order.

Let $X$ be a subset of $\omega$. The torsion abelian group

$$\bigoplus_{i \in X \oplus \overline{X}} \mathbb{Z}_{p_i}$$

has degree spectrum $\{\mathbf{d} : \mathbf{d} \geq \deg_T(X)\}$.

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Nowadays, there is a plethora of known examples of degree spectra:

- $\{\mathbf{d} : \mathbf{d} \not\leq \mathbf{a}\}$, where $\mathbf{a}$ is a low degree or $\mathbf{a}$ is a c.e. degree [Kalimullin 2007, 2008];
- the set of all hyperimmune degrees [Csima, Kalimullin 2010];
- the set of all non-hyperarithmetic degrees [Greenberg, Montalbán, Slaman 2013];
- the set of all non-$K$-trivial degrees [Kalimullin, Faizrakhmanov 2018];
- ...

# Spectral universality

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A class of $L$-structures $K$ is **spectrally universal** if for any countable structure $\mathcal{S}$ (in any computable signature), there exists a structure $\mathcal{A}_{\mathcal{S}} \in K$ such that $\mathrm{DgSp}(\mathcal{A}_{\mathcal{S}}) = \mathrm{DgSp}(\mathcal{S})$.

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Classical example. The following classes are spectrally universal:

- the class of all graphs,
- the class of all undirected (i.e., symmetric and irreflexive) graphs.

We cite [Hirschfeldt, Khoussainov, Shore, Slinko 2002]. Similar constructions appear as early as [Lavrov 1963].

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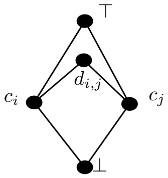
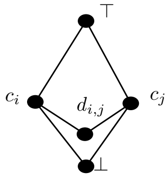
- $\mathrm{dom}(P_G) = \{\bot, \top\} \cup \{c_i : i \in \omega\} \cup \{d_{i,j} : i < j\}$.
- The nodes $c_i$, $i \in \omega$, "code" the domain of $G$.
- The edges are encoded like this:




$G \models Edge(i,j)$

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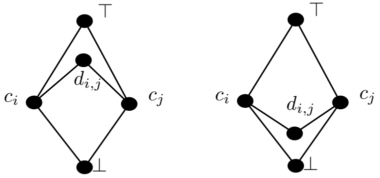


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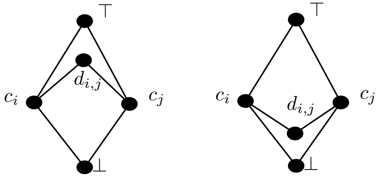


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(i) Given $G$, one can effectively construct $P_G$.

(ii) The original graph $G$ is "$\Delta^0_1$-definable" (with parameters $\bot, \top$) inside $P_G$:

(ii.a) $c_i$ are precisely those nodes that lie strictly between $\bot$ and $\top$;

(ii.b) $G \models Edge(i,j)$ iff there is an upper bound of $\{c_i, c_j\}$ which is not equal to $\top$;

(ii.c) $G \models \neg Edge(i,j)$ iff there is a lower bound of $\{c_i, c_j\}$ which is not equal to $\bot$.

# Example: Spectral universality of partial orders

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Roughly speaking, Conditions (i) and (ii) are enough to prove that

$$\mathrm{DgSp}(P_G) = \mathrm{DgSp}(G).$$

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## Problem

Find natural classes of ordered algebraic structures that are spectrally universal.

# A digression: Other notions of universality

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• In particular, HKSS-universality implies *universality with respect to effective dimensions*.

Let $\mathbf{d}$ be a Turing degree. The $\mathbf{d}$-*computable dimension* of a computable structure $\mathcal{S}$ is the number of computable copies of $\mathcal{S}$, up to $\mathbf{d}$-computable isomorphisms.

A class $K$ is *universal w.r.t. effective dimensions* if any possible $\mathbf{d}$-computable dimension can be realized by a computable structure from the class $K$.

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The class of Boolean algebras and the class of linear orders are not universal w.r.t. effective dimensions:

- Any natural number $n \geq 2$ can be realized as computable dimension of a graph [Goncharov 1977].
- A computable Boolean algebra has computable dimension either one or $\omega$. A similar fact is true for linear orders [Goncharov, Dzgoev 1980].

Spectral universality in this talk:

1. Familiar signature enrichments of Boolean algebras.
2. Mereological structures.
3. Open questions.

# Boolean algebras with operators

Let $\mathcal{B}$ be a Boolean algebra. A function $f\colon \mathrm{dom}(\mathcal{B})^n \to \mathrm{dom}(\mathcal{B})$ is an *operator* on $\mathcal{B}$ if it satisfies the following:

(a) $f(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n) = 0$ for all $k$ and $a_j \in \mathcal{B}$ [normality];

(b) $f(a_1, \ldots, a_{k-1}, b \cup c, a_{k+1}, \ldots, a_n) = f(a_1, \ldots, a_{k-1}, b, a_{k+1}, \ldots, a_n) \cup f(a_1, \ldots, a_{k-1}, c, a_{k+1}, \ldots, a_n)$ [additivity].

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For a linear order $L$, by $\mathrm{Int}(L)$ we denote the interval Boolean algebra induced by the ordering $L$.

## Theorem (Khoussainov, Kowalski 2012)

The class of Boolean algebras with operators, which have form $(\mathrm{Int}(\omega); f^2, g^1)$, is spectrally universal.

# Modal algebras

### Theorem 1 (B. 2016)

The class of polymodal algebras, which have form $(\mathrm{Int}(\omega^2); f^1, g^1, h^1, q^1)$, is spectrally universal.

The simulation technique (for modal logics) developed by Kracht and Wolter (1999) allows to obtain the following:

### Theorem 2 (B. 2021)

The class of modal algebras of the form $(\mathrm{Int}(\omega^2 \cdot 8); f^1)$ is spectrally universal.

# Contact Boolean algebras

## Definition (Dimov, Vakarelov 2006)

Let $R$ be a binary relation on a Boolean algebra $\mathcal{B}$. The structure $(\mathcal{B}; R)$ is a *contact Boolean algebra* if $R$ satisfies the following:

- $R(x,y) \to (x \neq 0)\&(y \neq 0)$;
- $R$ is symmetric;
- $R(x, y \cup z) \leftrightarrow (R(x,y) \vee R(x,z))$;
- $x \neq 0 \to R(x,x)$.

Contact algebras are considered as models of region-based theory of space.

Example. Let $(X, \tau)$ be a topological space. A set $a \subseteq X$ is regular closed if $a = \mathrm{cl}(\mathrm{int}(a))$. One puts:

- $a \vee b := a \cup b$;
- $a \wedge b := \mathrm{cl}(\mathrm{int}(a \cap b))$;
- $C(a) := \mathrm{cl}(X \setminus a)$;
- $R(a,b) \ \Leftrightarrow \ a \cap b \neq \emptyset$.

Then $(RC(X), \vee, \wedge, C, \emptyset, X; R)$ is a contact Boolean algebra.

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## Theorem 3 (B. 2019)

The class of contact Boolean algebras of the form $(\mathrm{Int}(\omega \cdot 2); R)$ is spectrally universal.

# Mereological structures

Mereological theories are based on the relation of "being a part of". The systematic investigations of mereology go back to the works of Leśniewski (1910s).

Here we give only one example of a mereological theory. In the signature $\{P^2\}$, the theory $GM2$ is given by the following axioms:

- $P$ is a partial order;
- (Fusion–2 Scheme) For a formula $\alpha(x)$, where variables $y$ and $z$ do not occur free, we have:

  $$\exists x \alpha(x) \rightarrow \exists z[\forall y(\alpha(y) \rightarrow P(y,z)) \,\&\, \forall y(P(y,z) \rightarrow \exists x(\alpha(x) \,\&\, O(y,x)))],$$

  where $O(x,y) := \exists z(P(z,x) \& P(z,y))$ is the overlap relation.

## Theorem 4 (B., Tsai 2022)

The class of models of $GM2$ is spectrally universal.

# Open problems

### Question 1

Is the class of distributive lattices spectrally universal?

We have the following result:

### Theorem 5 (B., Frolov, Kalimullin, Melnikov 2017)

There exists a distributive lattice having the Slaman–Wehner spectrum, i.e., $\{\mathbf{d} : \mathbf{d} \neq \mathbf{0}\}$.

Downey and Jockusch (1994) proved that every low Boolean algebra has a computable copy. Hence, the Slaman–Wehner spectrum cannot be realized by a Boolean algebra.

# Open problems

## Question 2

Is the class of Heyting algebras spectrally universal?

Here we have a partial result:

## Theorem 6 (B. 2022)

The class of Heyting algebras with distinguished set of atoms and set of coatoms is spectrally universal.

Note that the result of Knight and Stob (2000) implies that the Slaman–Wehner spectrum cannot be realized by a Boolean algebra with distinguished set of atoms.

# References


- D. R. Hirschfeldt, B. Khoussainov, R. A. Shore, and A. M. Slinko, *Degree spectra and computable dimensions in algebraic structures* // Annals of Pure and Applied Logic, 2002, 115:1–3, 71–113.
- I. Sh. Kalimullin, V. L. Selivanov, and A. N. Frolov, *Degree spectra of structures* // Journal of Mathematical Sciences, 2021, 256:2, 143–159.
- N. Bazhenov, *Categoricity spectra for polymodal algebras* // Studia Logica, 2016, 104:6, 1083–1097.
- N. Bazhenov, *Computable contact algebras* // Fundamenta Informaticae, 2019, 167:4, 257–269.
- N. Bazhenov and H.-C. Tsai, *On the effective universality of mereological theories* // Mathematical Logic Quarterly, 2022, 68:2, 213–226.
- N. Bazhenov, *Computable Heyting algebras with distinguished atoms and coatoms* // Journal of Logic, Language and Information, published online, doi: 10.1007/s10849-022-09371-0