Generalized Computable Numberings

and Fixed Points

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НОМЦ ПФО

Numberings, reducibilities

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Let β be a numbering of $R \subseteq S$. We say that β is reducible to α ($\beta \leqslant \alpha$), is there is a computable f s.t. $\beta(x) = \alpha(f(x))$ for every $x \in \mathbb{N}$.

Numberings α and β are said to be equivalent if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$.

(Generalized) computable numberings

Let $\mathcal{R} \subseteq 2^{\mathbb{N}}$ be a countable family of sets and α be its numbering. The numbering α is said to be computable ($\alpha \in \mathrm{Com}(\mathcal{R})$) if the set $G_\alpha = \{\langle x, y\rangle : y \in \alpha(x)\}$ is c.e.

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Definition (S.S. Goncharov, A. Sorbi, 1997)

A numberinng α is said to be Σ^0_n-computable ($\alpha \in \mathrm{Com}^0_\mathrm{n}(\mathcal{R})$), $n \geqslant 1$, if $G_\alpha \in \Sigma^0_n$.

Definition (S.A. Badaev, S.S. Goncharov, 2014)

A numbering α is said to be A-computable ($\alpha \in \mathrm{Com}^\mathrm{A}(\mathcal{R})$), $A \subseteq \mathbb{N}$, if G_α is A-c.e.

Principal and (pre)complete numberings

A numbering $\nu \in \mathrm{Com}^{\mathrm{A}}(\mathcal{R})$ is called principal (covering, universal), if $\alpha \leqslant \nu$ for each $\alpha \in \mathrm{Com}^{\mathrm{A}}(\mathcal{R})$.
For instance, the numbering $\nu : x \mapsto W_x$ of $\mathcal{E}$ is a principal numbering.

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Definition (A.I. Mal'tsev, 1961)

A numbering ν of a set S is said to be complete w.r.t $a \in S$ if for every p.c. function ψ there exists a computable f s.t. $\nu(f(x)) = \nu(\psi(x))$, if $\psi(x)\downarrow$, and $\nu(f(x)) = a$, if $\psi(x)\uparrow$.

For instance, the numbering $\nu : x \mapsto W_x$ of $\mathcal{E}$ is a complete numbering w.r.t. $\emptyset$.

Principal and (pre)complete numberings

Definition (Yu.L. Ershov, 1970)

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Theorem (Yu.L. Ershov, 1970)

A numbering ν of a set S is precomplete iff there is a computable h s.t. $\varphi_e(h(e))\downarrow \Rightarrow \nu(h(e)) = \nu(\varphi_e(h(e)))$ for each e.

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Theorem (Yu.L. Ershov, 1970)

If a numbering ν is precomplete, then $(\nu \equiv \alpha \oplus \beta \Rightarrow \nu \equiv \alpha \vee \nu \equiv \beta)$ for all numberings α, β.

Principal and (pre)complete numberings

Theorem (V.L. Selivanov, 1982)

$\emptyset' \leqslant_T A$ iff every A-computable principal numbering is complete w.r.t. each element.

Theorem (S.A. Badaev, S.S. Goncharov, 2014)

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Remark

For every set A, if an A-computable principal family has the least element under inclusion B, then its A-computable principal numbering is complete w.r.t. B.

Principal and (pre)complete numberings

Question (S.Yu. Podzorov, 2004)

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Theorem (S.A. Badaev, S.S. Goncharov, 2014)

For any set A, there exists a non-trivial A-computable principal family $\mathcal{F}$, $\emptyset \notin \mathcal{F}$, of pairwise disjoint sets.

Principal and (pre)complete numberings

Theorem

If A is a high set ($\emptyset'' \leqslant_T A'$), then every A-computable principal numbering is precomplete.

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- An infinite set A is called hyperimmune if there is no strongly computable sequence of pairwise disjoint sets $\{F_n\}_{n \in \mathbb{N}}$ such that $F_n \cap A \neq \emptyset$ for all n.
- If $\emptyset <_T A \leqslant_T \emptyset'$ or $\emptyset' \leqslant_T A$, then $\deg_T(A)$ is hyperimmune (contains a hyperimmune set).

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Theorem

$\deg_T(A)$ is hyperimmune iff every A-computable principal numbering of a finite family is precomplete.

Principal numberings and the Recursion Theorem

Theorem

There is a c.e. set $A \not\leq_T \emptyset$ and an A-computable principal numbering that is not precomplete.

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We say that a numbering ν satisfies the Recursion Theorem if for every computable f there is an integer n s.t. $\nu(f(n)) = \nu(n)$.

Theorem

$\deg_T(A)$ is hyperimmune iff every A-computable principal numbering satisfies the Recursion Theorem.

Principal numberings and the Recursion Theorem

Definition (T. Payne, 1973; S.A. Badaev, 1991)

A numbering ν of a set S is said to be weakly precomplete if there is a computable f s.t. $\nu(f(e)) = \nu(\varphi_e(f(e)))$ for every total function φ_e.

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We say that a numbering ν satisfies the Recursion Theorem with parameters if for every binary computable function h there is a computable f s.t. $\nu(h(n, f(n))) = \nu(f(n))$ for each n.

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Theorem (A.I. Mal'tsev, Yu.L. Ershov, T. Payne, S.A. Badaev . . .)

All implications are proper.

Principal numberings and the Recursion Theorem

Theorem (M.M. Arslanov, 1977)

Let A be a c.e. set. Then $\emptyset' \leqslant_T A$ iff there is a function $f \leqslant_T A$ s.t. $W_{f(x)} \neq W_x$ for each x.

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There are generalizations of the Completeness Criterion by V.L. Selivanov (1988), S. Terwijn and H. Barendregt (2019).

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For a c.e. set A the following statements are equivalent:

1. A is noncomputable,
2. every A-computable principal numbering is weakly precomplete,

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For a c.e. set A the following statements are equivalent:

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3. every A-computable principal numbering satisfies the Recursion Theorem with parameters,
4. every A-computable principal numbering satisfies the Recursion Theorem,
5. every A-computable principal numbering ν is nonsplittable ($\nu \equiv \alpha \oplus \beta \Rightarrow \nu \equiv \alpha \vee \nu \equiv \beta$).

Numberings and the Recursion Theorem

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Let A be a c.e. set. Then $\emptyset' \leqslant_T A$ iff every A-computable family has a precomplete A-computable numbering.

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For a c.e. set A the following statements are equivalent:

1. A is noncomputable,
2. every A-computable family has a weakly precomplete A-computable numbering,
3. every A-computable family has an A-computable numbering that satisfies the Recursion Theorem (with parameters),
4. every A-computable family has a nonsplittable A-computable numbering ν ($\nu \equiv \alpha \oplus \beta \Rightarrow \nu \equiv \alpha \vee \nu \equiv \beta$).

Minimal numberings and the Recursion Theorem

A numbering μ of a set S is called

- single-valued (a Friedberg numbering) if it is injective,
- positive if the equivalence $\eta_\mu = \{\langle x, y\rangle : \mu(x) = \mu(y)\}$ is c.e.,
- minimal if $\mu \leqslant \alpha$ for every numbering $\alpha \leqslant \mu$ of S.

μ is single-valued $\Rightarrow$ μ is positive $\Rightarrow$ μ is minimal.

Minimal numberings and the Recursion Theorem

Theorem (A.I. Mal'tsev, 1963)

There are no complete positive numberings.

Theorem (Yu.L. Ershov, 1977)

There exists a precomplete positive computable numbering of a non-trivial family.

Minimal numberings and the Recursion Theorem

Theorem (A.I. Mal'tsev, 1963)

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Theorem (Yu.L. Ershov, 1977)

There exists a precomplete positive computable numbering of a non-trivial family.

Theorem (V.V. V'yugin, 1973)

There is a computable family without minimal computable numberings.

Theorem (S.A. Badaev, S.S. Goncharov, 2001)

Every infinite Σ^0_{n+2}-computable family has an infinite number of minimal Σ^0_{n+2}-computable numberings.

Minimal numberings and the Recursion Theorem

Theorem

If A is a high (or computes a noncomputable c.e. set) set, then every infinite A-computable family has an infinite number of minimal A-computable numberings that satisfy the Recursion Theorem.

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If A is a high (or computes a noncomputable c.e. set) set, then every infinite A-computable family has an infinite number of minimal A-computable numberings that satisfy the Recursion Theorem.

Question (S.A. Badaev, S.S. Goncharov, A. Sorbi, 2003)

Are there complete minimal numberings?

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Question (S.A. Badaev, S.S. Goncharov, A. Sorbi, 2003)

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Theorem

Let $\emptyset'' \leqslant_T A$. Then for every infinite A-computable family $\mathcal{F}$ and for every $B \in \mathcal{F}$ there is a minimal A-computable numbering of $\mathcal{F}$ that is complete w.r.t. B.

The end.

Thank you for attention!