# On Homogenization *for* a Locally Periodic Maxwell Operator

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# PROBLEM FORMULATION

$$J = \{\psi \in L_2(\mathbb{R}^3; \mathbb{C}^3) \colon \operatorname{div} \psi = 0\}$$

# The Maxwell Operator

$$\mathcal{M}^{\varepsilon}(\mathbb{\varepsilon}, \mathbb{\mu}^0) = \begin{pmatrix} 0 & i\,\mathrm{rot}(\mathbb{\mu}^0)^{-1} \\ -i\,\mathrm{rot}(\mathbb{\varepsilon}(x, x/\varepsilon))^{-1} & 0 \end{pmatrix} \text{ in } J \oplus J$$

$$\mathrm{dom}\,\mathcal{M}^{\varepsilon}(\mathbb{\varepsilon}, \mathbb{\mu}^0) = \left\{\mathbb{\varepsilon}(x, x/\varepsilon)^{-1} w, (\mathbb{\mu}^0)^{-1} z \in H(\mathrm{rot})\right\}$$

$$\mathbb{\varepsilon} \in C^{0,1}(\bar{\mathbb{R}}^3; L_\infty(\mathbb{T}^3)), \qquad \mathbb{\mu}^0 = \mathrm{const},$$

$$0 < cI \leqslant \mathbb{\varepsilon}(x, y),\ \mathbb{\mu}^0 \leqslant CI,$$

$$(\mathcal{M}^{\varepsilon}(\mathbb{E}, \mathbb{\mu}^0) - i)\begin{pmatrix} w_\varepsilon \\ z_\varepsilon \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}$$

$$w_\varepsilon, z_\varepsilon, u_\varepsilon, v_\varepsilon \overset{\text{w-}L_2}{\rightsquigarrow} w_0, z_0, u_0, v_0$$

$$(\mathcal{M}^{\varepsilon}(\mathbb{\epsilon}, \mathbb{\mu}^{0}) - i)\begin{pmatrix} w_{\varepsilon} \\ z_{\varepsilon} \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

$$w_{\varepsilon}, z_{\varepsilon}, u_{\varepsilon}, v_{\varepsilon} \overset{\text{w-}L_2}{\rightsquigarrow} w_0, z_0, u_0, v_0$$

$$
\left\{
\begin{aligned}
\operatorname{rot}(\mathbb{\mu}^0)^{-1} z_\varepsilon - w_\varepsilon &= 0,\\
\operatorname{rot} \mathbb{\epsilon}(x, x/\varepsilon)^{-1} w_\varepsilon + z_\varepsilon &= ir,\\
\operatorname{div} w_\varepsilon &= 0,\\
\operatorname{div} z_\varepsilon &= 0;\\
\mathbb{\epsilon}(x, x/\varepsilon) u_\varepsilon &= w_\varepsilon,\\
\mathbb{\mu}^0 v_\varepsilon &= z_\varepsilon.
\end{aligned}
\right.
$$

$$
w_\varepsilon, z_\varepsilon, u_\varepsilon, v_\varepsilon \overset{\text{w-}L_2}{\rightsquigarrow} w_0, z_0, u_0, v_0
$$

# The Effective Maxwell Operator

$$\mathcal{M}^0(\varepsilon^0, \mu^0) = \begin{pmatrix} 0 & i\,\mathrm{rot}(\mu^0)^{-1} \\ -i\,\mathrm{rot}\,\varepsilon^0(x)^{-1} & 0 \end{pmatrix} \text{ in } J \oplus J$$

$$\varepsilon^0(x) = \int_{\mathbb{T}^3} \varepsilon(x,y)\left(\nabla_y M(x,y) + I\right) dy$$

$$\begin{cases} -\,\mathrm{div}\,\varepsilon(x,\cdot)\left(\nabla_y M(x,\cdot) + I\right) = 0 \text{ on } \mathbb{T}^3 \\ \displaystyle\int_{\mathbb{T}^3} M(x,y)\,dy = 0 \end{cases}$$

$$\mathcal{M}^0(\varepsilon^0, \mu^0) = \begin{pmatrix} 0 & i \operatorname{rot} (\mu^0)^{-1} \\ -i \operatorname{rot} \varepsilon^0(x)^{-1} & 0 \end{pmatrix} \text{ in } J \oplus J$$

$$(\mathcal{M}^0(\varepsilon^0, \mu^0) - i) \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

# MAIN RESULTS

**Theorem 1**

$$\|z_\varepsilon - z_0\|_{L_2} \leqslant C\varepsilon\|r\|_{L_2},$$

$$\|v_\varepsilon - v_0\|_{L_2} \leqslant C\varepsilon\|r\|_{L_2}.$$

**Theorem 2**

$$\|z_\varepsilon - z_0 - \varepsilon S^\varepsilon \Psi w_0\|_{H^1} \leqslant C\varepsilon \|r\|_{L_2},$$

$$\|v_\varepsilon - v_0 - \varepsilon (\mathbb{\mu}^0)^{-1} S^\varepsilon \Psi w_0\|_{H^1} \leqslant C\varepsilon \|r\|_{L_2}.$$

$$\Psi(x,y) = \operatorname{rot}_y p(x,y) \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3},$$

$$\begin{cases} \operatorname{rot} (\mathbb{\mu}^0)^{-1} \operatorname{rot} p(x,\cdot) \\ \qquad = \mathbb{\epsilon}(x,\cdot)(\nabla M(x,\cdot) + I)\mathbb{\epsilon}^0(x)^{-1} - I \text{ on } \mathbb{T}^3 \\ \operatorname{div} p(x,\cdot) = 0 \text{ on } \mathbb{T}^3, \qquad \displaystyle\int_{\mathbb{T}^3} p(x,y)\,dy = 0 \end{cases}$$

**Theorem 3**

$$\|u_\varepsilon - u_0 - S^\varepsilon(\nabla_y M)u_0\|_{L_2} \leqslant C\varepsilon\|r\|_{L_2},$$
$$\|w_\varepsilon - w_0 - S^\varepsilon \Upsilon w_0\|_{L_2} \leqslant C\varepsilon\|r\|_{L_2}.$$

$$\Upsilon(x,y) = \mathfrak{C}(x,y)(\nabla_y M(x,y) + I)\mathfrak{C}^0(x)^{-1} - I$$

# Previous Results

**Classic results: w-approx.**
[BeLePap], [BaPa], [ZhKO]

**U-approx., in $\mathbb{R}^3$**
[BSu03,05,06]; [Su04] , [BSu07], [Su07]

**U-approx., in domains**
[Su18], [Su19]

**U-approx., non-stationary M. equation**
[DoSu2021]

# SCHEME OF THE PROOF

# A Model Operator

$$
\begin{cases}
\operatorname{rot}(\mathbb{\mu}^0)^{-1} z_\varepsilon - w_\varepsilon = 0, \\
\operatorname{rot} \mathbb{\varepsilon}(x, x/\varepsilon)^{-1} w_\varepsilon + z_\varepsilon = ir, \\
\operatorname{div} z_\varepsilon = 0.
\end{cases}
$$

# A Model Operator

$$\begin{cases} \operatorname{rot} \epsilon(x, x/\varepsilon)^{-1} \operatorname{rot} (\mu^0)^{-1} z_\varepsilon + z_\varepsilon = ir, \\ \operatorname{div} z_\varepsilon = 0. \end{cases}$$

# A Model Operator

$$(\mathcal{L}^{\varepsilon} + I)((\mu^0)^{-1/2} z_{\varepsilon}) = i(\mu^0)^{-1/2} r$$

$$\begin{aligned} \mathcal{L}^{\varepsilon} &= (\mu^0)^{-1/2} \operatorname{rot} \epsilon(x, x/\varepsilon)^{-1} \operatorname{rot} (\mu^0)^{-1/2} \\ &\quad - (\mu^0)^{1/2} \nabla \operatorname{div} (\mu^0)^{1/2} \\ &= b(D)^{*} g(x, x/\varepsilon) b(D) \end{aligned}$$

# Uniform approximations for $\mathcal{L}^\varepsilon$

$$(\mathcal{L}^\varepsilon + I)^{-1} = (\mathcal{L}^0 + I)^{-1} + O(\varepsilon)$$

$$b(D)(\mathcal{L}^\varepsilon + I)^{-1} = b(D)(\mathcal{L}^0 + I)^{-1} + \varepsilon b(D) S^\varepsilon \Lambda b(D)(\mathcal{L}^0 + I)^{-1} + O(\varepsilon)$$

$$g^\varepsilon b(D)(\mathcal{L}^\varepsilon + I)^{-1} = S^\varepsilon g(b(D)\Lambda + I) b(D)(\mathcal{L}^0 + I)^{-1} + O(\varepsilon)$$

# The Weyl Decomposition

The Weyl decomposition

$$L_2(\mathbb{R}^3;\mathbb{C}^3) = \left\{\mathrm{div}(\mu^0)^{1/2}\varphi = 0\right\} \oplus \left\{(\mu^0)^{1/2}\nabla\psi\right\}$$

reduces both $(\mathcal{L}^\varepsilon + I)^{-1}$ and $(\mathcal{L}^0 + I)^{-1}$.