Homogenization of a Non-stationary Periodic Maxwell System in the Case of Constant Magnetic Permeability

Mark Dorodnyi (joint work with Tatiana Suslina)

2nd Conference of Mathematical Centers of Russia

7–11 November, 2022

St. Petersburg State University

We study homogenization problem for the nonstationary Maxwell system with periodic rapidly oscillating coefficients. This problem was studied by traditional methods of homogenization theory. See

E. Sanchez-Palencia, *Nonhomogeneous media and vibration theory*, 1980.

V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators*, 1993.

The traditional results give weak convergence of the solutions to the solution of the homogenized system.

We study homogenization problem for the nonstationary Maxwell system with periodic rapidly oscillating coefficients. This problem was studied by traditional methods of homogenization theory. See

E. Sanchez-Palencia, *Nonhomogeneous media and vibration theory*, 1980.

V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators*, 1993.

The traditional results give weak convergence of the solutions to the solution of the homogenized system.

We assume that the magnetic permeability is constant and the dielectric permittivity is rapidly oscillating. In this case we find approximations for the solutions in the L_2-norm.

Let Γ be a lattice in $\mathbb{R}^3$, let Ω be the cell of Γ.

Statement of the problem

Let Γ be a lattice in $\mathbb{R}^3$, let Ω be the cell of Γ.

Example: $\Gamma = \mathbb{Z}^3$, $\Omega = (0, 1)^3$.

Let Γ be a lattice in $\mathbb{R}^3$, let Ω be the cell of Γ.

Example: $\Gamma = \mathbb{Z}^3$, $\Omega = (0,1)^3$.

Let $\varepsilon > 0$ be a (small) parameter. We use the notation $\phi^\varepsilon(\mathbf{x}) = \phi\left(\frac{\mathbf{x}}{\varepsilon}\right)$.

Let Γ be a lattice in $\mathbb{R}^3$, let Ω be the cell of Γ.

Example: $\Gamma = \mathbb{Z}^3$, $\Omega = (0,1)^3$.

Let $\varepsilon > 0$ be a (small) parameter. We use the notation $\phi^\varepsilon(\boldsymbol{x}) = \phi\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$.

Suppose that the dielectric permittivity is given by the rapidly oscillating matrix $\eta^\varepsilon(\boldsymbol{x})$ and the magnetic permeability is equal to μ.

- $\eta(\boldsymbol{x})$ is a Γ-periodic symmetric (3×3)-matrix-valued function with real entries,

$$c'\mathbb{1} \leqslant \eta(\boldsymbol{x}) \leqslant c''\mathbb{1}, \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad 0 < c' \leqslant c'' < \infty.$$

- μ is a positive symmetric matrix with real entries.

Statement of the problem

min

We study the Cauchy problem for the nonstationary Maxwell system:

$$
(1) \quad \begin{cases} \partial_\tau \boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau) = (\eta^\varepsilon(\boldsymbol{x}))^{-1} \operatorname{curl} \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau), & \operatorname{div} \eta^\varepsilon(\boldsymbol{x}) \boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \partial_\tau \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = -\mu^{-1} \operatorname{curl} \boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau), & \operatorname{div} \mu \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{E}_\varepsilon(\boldsymbol{x},0) = (P_\varepsilon \boldsymbol{f})(\boldsymbol{x}), \quad \boldsymbol{H}_\varepsilon(\boldsymbol{x},0) = \phi(\boldsymbol{x}). \end{cases}
$$

We study the Cauchy problem for the nonstationary Maxwell system:

$$
(1) \quad \begin{cases} \partial_\tau \boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau) = (\eta^\varepsilon(\boldsymbol{x}))^{-1} \operatorname{curl} \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau), & \operatorname{div} \eta^\varepsilon(\boldsymbol{x}) \boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \partial_\tau \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = -\mu^{-1} \operatorname{curl} \boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau), & \operatorname{div} \mu \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{E}_\varepsilon(\boldsymbol{x},0) = (P_\varepsilon \boldsymbol{f})(\boldsymbol{x}), \quad \boldsymbol{H}_\varepsilon(\boldsymbol{x},0) = \boldsymbol{\phi}(\boldsymbol{x}). \end{cases}
$$

Here $\boldsymbol{\phi} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div} \mu \boldsymbol{\phi}(\boldsymbol{x}) = 0$ (in the sense of distributions). Next, $\boldsymbol{f} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and P_ε is the orthogonal projection of the weighted space $L_2(\mathbb{R}^3;\mathbb{C}^3;\eta^\varepsilon)$ onto the subspace

$$
\{\boldsymbol{u} \in L_2(\mathbb{R}^3;\mathbb{C}^3) \colon \operatorname{div} \eta^\varepsilon(\boldsymbol{x}) \boldsymbol{u}(\boldsymbol{x}) = 0\}.
$$

We study the Cauchy problem for the nonstationary Maxwell system:

$$
(1) \quad \begin{cases} \partial_\tau \mathbf{E}_\varepsilon(\mathbf{x},\tau) = (\eta^\varepsilon(\mathbf{x}))^{-1} \operatorname{curl} \mathbf{H}_\varepsilon(\mathbf{x},\tau), & \operatorname{div} \eta^\varepsilon(\mathbf{x}) \mathbf{E}_\varepsilon(\mathbf{x},\tau) = 0, \\ \partial_\tau \mathbf{H}_\varepsilon(\mathbf{x},\tau) = -\mu^{-1} \operatorname{curl} \mathbf{E}_\varepsilon(\mathbf{x},\tau), & \operatorname{div} \mu \mathbf{H}_\varepsilon(\mathbf{x},\tau) = 0, \\ \mathbf{E}_\varepsilon(\mathbf{x},0) = (P_\varepsilon \mathbf{f})(\mathbf{x}), \quad \mathbf{H}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}). \end{cases}
$$

Here $\boldsymbol{\phi} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div} \mu \boldsymbol{\phi}(\mathbf{x}) = 0$ (in the sense of distributions). Next, $\mathbf{f} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and P_ε is the orthogonal projection of the weighted space $L_2(\mathbb{R}^3;\mathbb{C}^3;\eta^\varepsilon)$ onto the subspace

$$
\{\mathbf{u} \in L_2(\mathbb{R}^3;\mathbb{C}^3) \colon \operatorname{div} \eta^\varepsilon(\mathbf{x}) \mathbf{u}(\mathbf{x}) = 0\}.
$$

The projection P_ε acts as follows $(P_\varepsilon \mathbf{f})(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \nabla \omega_\varepsilon(\mathbf{x})$, where ω_ε is the solution of the equation $\operatorname{div} \eta^\varepsilon \nabla \omega_\varepsilon = \operatorname{div} \eta^\varepsilon \mathbf{f}$.

The Cauchy problem for the homogenized Maxwell system:

$$(2)\quad \begin{cases} \partial_\tau \boldsymbol{E}_0(\boldsymbol{x},\tau) = (\eta^0)^{-1}\operatorname{curl}\boldsymbol{H}_0(\boldsymbol{x},\tau), & \operatorname{div}\eta^0\boldsymbol{E}_0(\boldsymbol{x},\tau)=0,\\ \partial_\tau \boldsymbol{H}_0(\boldsymbol{x},\tau) = -\mu^{-1}\operatorname{curl}\boldsymbol{E}_0(\boldsymbol{x},\tau), & \operatorname{div}\mu\boldsymbol{H}_0(\boldsymbol{x},\tau)=0,\\ \boldsymbol{E}_0(\boldsymbol{x},0) = (P_0\boldsymbol{f})(\boldsymbol{x}), \quad \boldsymbol{H}_0(\boldsymbol{x},0)=\phi(\boldsymbol{x}). \end{cases}$$

Here $\phi, \boldsymbol{f}$ are the same as in (1). Recall that $\phi, \boldsymbol{f} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div}\mu\phi = 0$.

The Cauchy problem for the homogenized Maxwell system:

$$(2)\quad \begin{cases} \partial_\tau \boldsymbol{E}_0(\boldsymbol{x},\tau) = (\eta^0)^{-1}\operatorname{curl} \boldsymbol{H}_0(\boldsymbol{x},\tau), & \operatorname{div} \eta^0 \boldsymbol{E}_0(\boldsymbol{x},\tau) = 0, \\ \partial_\tau \boldsymbol{H}_0(\boldsymbol{x},\tau) = -\mu^{-1}\operatorname{curl} \boldsymbol{E}_0(\boldsymbol{x},\tau), & \operatorname{div} \mu \boldsymbol{H}_0(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{E}_0(\boldsymbol{x},0) = (P_0\boldsymbol{f})(\boldsymbol{x}), \quad \boldsymbol{H}_0(\boldsymbol{x},0) = \phi(\boldsymbol{x}). \end{cases}$$

Here $\phi, \boldsymbol{f}$ are the same as in (1). Recall that $\phi, \boldsymbol{f} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div} \mu\phi = 0$. Next, P_0 is the orthogonal projection of $L_2(\mathbb{R}^3;\mathbb{C}^3;\eta^0)$ onto the subspace

$$J_0 = \{\boldsymbol{u} \in L_2(\mathbb{R}^3;\mathbb{C}^3) \colon \operatorname{div} \eta^0 \boldsymbol{u} = 0\}.$$

Homogenized Maxwell system

The Cauchy problem for the homogenized Maxwell system:

$$(2)\quad \begin{cases} \partial_\tau \boldsymbol{E}_0(\boldsymbol{x},\tau) = (\eta^0)^{-1} \operatorname{curl} \boldsymbol{H}_0(\boldsymbol{x},\tau), & \operatorname{div} \eta^0 \boldsymbol{E}_0(\boldsymbol{x},\tau) = 0, \\ \partial_\tau \boldsymbol{H}_0(\boldsymbol{x},\tau) = -\mu^{-1} \operatorname{curl} \boldsymbol{E}_0(\boldsymbol{x},\tau), & \operatorname{div} \mu \boldsymbol{H}_0(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{E}_0(\boldsymbol{x},0) = (P_0 \boldsymbol{f})(\boldsymbol{x}), \quad \boldsymbol{H}_0(\boldsymbol{x},0) = \phi(\boldsymbol{x}). & \end{cases}$$

Here $\phi, \boldsymbol{f}$ are the same as in (1). Recall that $\phi, \boldsymbol{f} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div} \mu\phi = 0$. Next, P_0 is the orthogonal projection of $L_2(\mathbb{R}^3;\mathbb{C}^3;\eta^0)$ onto the subspace

$$J_0 = \{\boldsymbol{u} \in L_2(\mathbb{R}^3;\mathbb{C}^3) \colon \operatorname{div} \eta^0 \boldsymbol{u} = 0\}.$$

The projection P_0 acts as follows: $(P_0 \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \nabla\omega_0(\boldsymbol{x})$, where ω_0 is the weak solution of the equation $\operatorname{div} \eta^0 \nabla \omega_0 = \operatorname{div} \eta^0 \boldsymbol{f}$.

The auxiliary cell problem

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the standard orthonormal basis in $\mathbb{R}^3$. Let $\Phi_j(\boldsymbol{x})$ be a Γ-periodic solution of the problem

$$\operatorname{div} \eta(\boldsymbol{x})(\nabla\Phi_j(\boldsymbol{x}) + \boldsymbol{e}_j) = 0, \quad \int_\Omega \Phi_j(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

The auxiliary cell problem

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the standard orthonormal basis in $\mathbb{R}^3$. Let $\Phi_j(\boldsymbol{x})$ be a Γ-periodic solution of the problem

$$\operatorname{div} \eta(\boldsymbol{x})(\nabla\Phi_j(\boldsymbol{x}) + \boldsymbol{e}_j) = 0, \quad \int_\Omega \Phi_j(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Effective matrix

We introduce the matrix $\Sigma(\boldsymbol{x})$ with the columns $\nabla\Phi_j(\boldsymbol{x})$, $j = 1, 2, 3$, and the matrix $\widetilde{\eta}(\boldsymbol{x}) := \eta(\boldsymbol{x})(\Sigma(\boldsymbol{x}) + \mathbb{1})$.

The auxiliary cell problem

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the standard orthonormal basis in $\mathbb{R}^3$. Let $\Phi_j(\boldsymbol{x})$ be a Γ-periodic solution of the problem

$$\operatorname{div} \eta(\boldsymbol{x})(\nabla\Phi_j(\boldsymbol{x}) + \boldsymbol{e}_j) = 0, \quad \int_\Omega \Phi_j(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Effective matrix

We introduce the matrix $\Sigma(\boldsymbol{x})$ with the columns $\nabla\Phi_j(\boldsymbol{x})$, $j = 1, 2, 3$, and the matrix $\widetilde{\eta}(\boldsymbol{x}) := \eta(\boldsymbol{x})(\Sigma(\boldsymbol{x}) + \mathbb{1})$. The constant matrix

$$\eta^0 = |\Omega|^{-1} \int_\Omega \widetilde{\eta}(\boldsymbol{x})\, d\boldsymbol{x}$$

is called the effective matrix.

min

The auxiliary cell problem

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the standard orthonormal basis in $\mathbb{R}^3$. Let $\Phi_j(\boldsymbol{x})$ be a Γ-periodic solution of the problem

$$\operatorname{div} \eta(\boldsymbol{x})(\nabla \Phi_j(\boldsymbol{x}) + \boldsymbol{e}_j) = 0, \quad \int_\Omega \Phi_j(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Effective matrix

We introduce the matrix $\Sigma(\boldsymbol{x})$ with the columns $\nabla \Phi_j(\boldsymbol{x})$, $j = 1, 2, 3$, and the matrix $\widetilde{\eta}(\boldsymbol{x}) := \eta(\boldsymbol{x})(\Sigma(\boldsymbol{x}) + \mathbb{1})$. The constant matrix

$$\eta^0 = |\Omega|^{-1} \int_\Omega \widetilde{\eta}(\boldsymbol{x})\, d\boldsymbol{x}$$

is called the effective matrix.

It turns out that the matrix η^0 is positive definite.

Theorem 1 [M. Dorodnyi and T. Suslina, 2021]

1°. Let $\phi, \boldsymbol{f} \in H^2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div} \mu\phi = 0$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ the fields $\boldsymbol{H}_\varepsilon$ and $\boldsymbol{B}_\varepsilon = \mu \boldsymbol{H}_\varepsilon$ satisfy the following sharp order estimates

$$\|\boldsymbol{H}_\varepsilon(\cdot,\tau) - \boldsymbol{H}_0(\cdot,\tau)\|_{L_2} \leqslant C(1+|\tau|)\varepsilon \left(\|\phi\|_{H^2} + \|\boldsymbol{f}\|_{H^2}\right),$$
$$\|\boldsymbol{B}_\varepsilon(\cdot,\tau) - \boldsymbol{B}_0(\cdot,\tau)\|_{L_2} \leqslant C(1+|\tau|)\varepsilon \left(\|\phi\|_{H^2} + \|\boldsymbol{f}\|_{H^2}\right).$$

2°. Let $\phi, \boldsymbol{f} \in H^s(\mathbb{R}^3;\mathbb{C}^3)$ with $0 \leqslant s \leqslant 2$ and $\operatorname{div} \mu\phi = 0$. Then

$$\|\boldsymbol{H}_\varepsilon(\cdot,\tau) - \boldsymbol{H}_0(\cdot,\tau)\|_{L_2} \leqslant C(s)(1+|\tau|)^{s/2}\varepsilon^{s/2} \left(\|\phi\|_{H^s} + \|\boldsymbol{f}\|_{H^s}\right),$$
$$\|\boldsymbol{B}_\varepsilon(\cdot,\tau) - \boldsymbol{B}_0(\cdot,\tau)\|_{L_2} \leqslant C(s)(1+|\tau|)^{s/2}\varepsilon^{s/2} \left(\|\phi\|_{H^s} + \|\boldsymbol{f}\|_{H^s}\right).$$

3°. Let $\phi, \boldsymbol{f} \in L_2(\mathbb{R}^3;\mathbb{C}^3)$ and $\operatorname{div} \mu\phi = 0$. Then

$$\|\boldsymbol{H}_\varepsilon(\cdot,\tau) - \boldsymbol{H}_0(\cdot,\tau)\|_{L_2} \to 0, \qquad \|\boldsymbol{B}_\varepsilon(\cdot,\tau) - \boldsymbol{B}_0(\cdot,\tau)\|_{L_2} \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

It is possible to approximate all four fields under the additional assumption that $\phi = \mathbf{0}$.

It is possible to approximate all four fields under the additional assumption that $\phi = \mathbf{0}$.

Theorem 2 [M. Dorodnyi and T. Suslina, 2021]

1°. Let $\phi = \mathbf{0}$ and $\boldsymbol{f} \in H^3(\mathbb{R}^3;\mathbb{C}^3)$. Then for fixed $\tau \in \mathbb{R}$ and small ε the fields $\boldsymbol{E}_\varepsilon$ and $\boldsymbol{D}_\varepsilon = \eta^\varepsilon \boldsymbol{E}_\varepsilon$ satisfy

$$\|(\boldsymbol{E}_\varepsilon(\cdot,\tau) - \boldsymbol{E}_\varepsilon(\cdot,0)) - (\mathbb{1} + \Sigma^\varepsilon)(\boldsymbol{E}_0(\cdot,\tau) - \boldsymbol{E}_0(\cdot,0))\|_{L_2} \leqslant C|\tau|(1+|\tau|)\varepsilon\|\boldsymbol{f}\|_{H^3},$$

$$\|(\boldsymbol{D}_\varepsilon(\cdot,\tau) - \boldsymbol{D}_\varepsilon(\cdot,0)) - (\mathbb{1} + (\widetilde{\eta}^\varepsilon(\eta^0)^{-1} - \mathbb{1}))(\boldsymbol{D}_0(\cdot,\tau) - \boldsymbol{D}_0(\cdot,0))\|_{L_2} \leqslant C|\tau|(1+|\tau|)\varepsilon\|\boldsymbol{f}\|_{H^3}.$$

Theorem 2 [M. Dorodnyi and T. Suslina, 2021]

2°. Let $\phi = \mathbf{0}$ and $\boldsymbol{f} \in H^{1+s}(\mathbb{R}^3;\mathbb{C}^3)$ with $0 \leqslant s \leqslant 2$. Then

$$\|(\boldsymbol{E}_\varepsilon(\cdot,\tau) - \boldsymbol{E}_\varepsilon(\cdot,0)) - (\mathbb{1} + \Sigma^\varepsilon \Pi_\varepsilon)(\boldsymbol{E}_0(\cdot,\tau) - \boldsymbol{E}_0(\cdot,0))\|_{L_2} \leqslant C(s)|\tau|(1+|\tau|)^{s/2}\varepsilon^{s/2}\|\boldsymbol{f}\|_{H^{1+s}},$$

$$\|(\boldsymbol{D}_\varepsilon(\cdot,\tau) - \boldsymbol{D}_\varepsilon(\cdot,0)) - (\mathbb{1} + (\widetilde{\eta}^\varepsilon(\eta^0)^{-1} - \mathbb{1})\Pi_\varepsilon)(\boldsymbol{D}_0(\cdot,\tau) - \boldsymbol{D}_0(\cdot,0))\|_{L_2} \leqslant C(s)|\tau|(1+|\tau|)^{s/2}\varepsilon^{s/2}\|\boldsymbol{f}\|_{H^{1+s}}.$$

Here Π_ε is an auxiliary smoothing operator given by

$$(\Pi_\varepsilon \varphi)(\boldsymbol{x}) = (2\pi)^{-3/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} \widehat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

where $\widehat{\varphi}(\boldsymbol{\xi})$ is the Fourier-image of a function $\varphi(\boldsymbol{x})$, $\widetilde{\Omega}$ is the central Brillouin zone of the dual lattice.

Theorem 2 [M. Dorodnyi and T. Suslina, 2021]

3°. Let $\phi = \mathbf{0}$ and $\boldsymbol{f} \in H^1(\mathbb{R}^3;\mathbb{C}^3)$. Then

$$\|(\boldsymbol{E}_\varepsilon(\cdot,\tau)-\boldsymbol{E}_\varepsilon(\cdot,0))-(\mathbb{1}+\Sigma^\varepsilon\Pi_\varepsilon)(\boldsymbol{E}_0(\cdot,\tau)-\boldsymbol{E}_0(\cdot,0))\|_{L_2} \to 0,$$

$$\|(\boldsymbol{D}_\varepsilon(\cdot,\tau)-\boldsymbol{D}_\varepsilon(\cdot,0))-(\mathbb{1}+(\widetilde{\eta}^\varepsilon(\eta^0)^{-1}-\mathbb{1})\Pi_\varepsilon)(\boldsymbol{D}_0(\cdot,\tau)-\boldsymbol{D}_0(\cdot,0))\|_{L_2}\to 0,$$

as $\varepsilon \to 0$.

4°. Let $\phi = \mathbf{0}$ and $\boldsymbol{f} \in H^3(\mathbb{R}^3;\mathbb{C}^3)$. Then

$$\left\|\boldsymbol{H}_\varepsilon(\cdot,\tau) - \boldsymbol{H}_0(\cdot,\tau) - \varepsilon\mu^{-1}\Psi^\varepsilon \operatorname{curl} \boldsymbol{H}_0(\cdot,\tau)\right\|_{H^1} \leqslant C(1+|\tau|)\varepsilon\|\boldsymbol{f}\|_{H^3},$$

$$\left\|\boldsymbol{B}_\varepsilon(\cdot,\tau) - \boldsymbol{B}_0(\cdot,\tau) - \varepsilon\Psi^\varepsilon \operatorname{curl} \boldsymbol{H}_0(\cdot,\tau)\right\|_{H^1} \leqslant C(1+|\tau|)\varepsilon\|\boldsymbol{f}\|_{H^3},$$

where $\Psi(\boldsymbol{x})$ is the (3×3)-matrix with the columns $\operatorname{curl} \boldsymbol{p}_j(\boldsymbol{x})$, $j=1,2,3$;
$\boldsymbol{p}_j$ is the Γ-periodic solution of the problem

$$\operatorname{curl}(\mu^{-1}\operatorname{curl} \boldsymbol{p}_j(\boldsymbol{x})) = \eta(\boldsymbol{x})([\widetilde{\Sigma}(\boldsymbol{x})]_j + \boldsymbol{c}_j) - \boldsymbol{e}_j, \ \operatorname{div} \boldsymbol{p}_j(\boldsymbol{x}) = 0, \ \int_\Omega \boldsymbol{p}_j(\boldsymbol{x})\,d\boldsymbol{x} = 0;$$

$[\widetilde{\Sigma}(\boldsymbol{x})]_j$ is the j-th column of the matrix $\widetilde{\Sigma}(\boldsymbol{x}) := \Sigma(\boldsymbol{x})(\eta^0)^{-1}$; $\boldsymbol{c}_j := (\eta^0)^{-1}\boldsymbol{e}_j$.

Reduction to a second order equation

The method of investigation is based on the reduction to a second order equation. The problem for $\boldsymbol{H}_\varepsilon$:

$$\begin{cases} \mu\partial_\tau^2 \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = -\operatorname{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\operatorname{curl}\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau), \\ \operatorname{div}\mu\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{H}_\varepsilon(\boldsymbol{x},0) = \boldsymbol{\phi}(\boldsymbol{x}), \quad \mu\,\partial_\tau\boldsymbol{H}_\varepsilon(\boldsymbol{x},0) = \boldsymbol{\psi}(\boldsymbol{x}), \end{cases}$$

where $\boldsymbol{\psi} := \operatorname{curl}\boldsymbol{f}$.

The method of investigation is based on the reduction to a second order equation. The problem for $\boldsymbol{H}_\varepsilon$:

$$\begin{cases} \mu\partial_\tau^2 \boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = -\operatorname{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\operatorname{curl}\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau), \\ \operatorname{div}\mu\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{H}_\varepsilon(\boldsymbol{x},0) = \boldsymbol{\phi}(\boldsymbol{x}), \quad \mu\,\partial_\tau\boldsymbol{H}_\varepsilon(\boldsymbol{x},0) = \boldsymbol{\psi}(\boldsymbol{x}), \end{cases}$$

where $\boldsymbol{\psi} := \operatorname{curl}\boldsymbol{f}$.

The other fields are expressed in terms of $\boldsymbol{H}_\varepsilon$ as follows:

$$\boldsymbol{B}_\varepsilon(\boldsymbol{x},\tau) = \mu\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tau),$$

$$\boldsymbol{E}_\varepsilon(\boldsymbol{x},\tau) - \boldsymbol{E}_\varepsilon(\boldsymbol{x},0) = \int_0^\tau (\eta^\varepsilon(\boldsymbol{x}))^{-1}\operatorname{curl}\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tilde{\tau})\,d\tilde{\tau},$$

$$\boldsymbol{D}_\varepsilon(\boldsymbol{x},\tau) - \boldsymbol{D}_\varepsilon(\boldsymbol{x},0) = \int_0^\tau \operatorname{curl}\boldsymbol{H}_\varepsilon(\boldsymbol{x},\tilde{\tau})\,d\tilde{\tau}.$$

Reduction to a second order equation

It is convenient to symmetrize the problem. We substitute $\mu^{1/2}\boldsymbol{H}_\varepsilon = \boldsymbol{\varphi}_\varepsilon$. Then $\boldsymbol{\varphi}_\varepsilon$ is the solution of the problem

$$
(3) \qquad \begin{cases} \partial_\tau^2 \boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau) = -\mu^{-1/2}\,\mathrm{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\,\mathrm{curl}\,\mu^{-1/2}\boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau), \\ \mathrm{div}\ \mu^{1/2}\boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},0) = \mu^{1/2}\boldsymbol{\phi}(\boldsymbol{x}), \quad \partial_\tau\boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},0) = \mu^{-1/2}\boldsymbol{\psi}(\boldsymbol{x}). \end{cases}
$$

It is convenient to symmetrize the problem. We substitute $\mu^{1/2}\boldsymbol{H}_\varepsilon = \boldsymbol{\varphi}_\varepsilon$. Then $\boldsymbol{\varphi}_\varepsilon$ is the solution of the problem

$$
(3)\qquad \begin{cases} \partial_\tau^2 \boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau) = -\mu^{-1/2}\,\mathrm{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\,\mathrm{curl}\,\mu^{-1/2}\boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau), \\ \mathrm{div}\;\mu^{1/2}\boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau) = 0, \\ \boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},0) = \mu^{1/2}\boldsymbol{\phi}(\boldsymbol{x}), \quad \partial_\tau \boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},0) = \mu^{-1/2}\boldsymbol{\psi}(\boldsymbol{x}). \end{cases}
$$

Next, we extend system (3) in order to “remove” the divergence-free condition. This leads us to consider the operator

$$
\mathcal{L}_\varepsilon := \mu^{-1/2}\,\mathrm{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\,\mathrm{curl}\,\mu^{-1/2} - \mu^{1/2}\nabla\,\mathrm{div}\,\mu^{1/2},
$$

acting in $L_2(\mathbb{R}^3;\mathbb{C}^3)$. Then $\boldsymbol{\varphi}_\varepsilon$ is the solution of the hyperbolic equation

$$
\partial_\tau^2 \boldsymbol{\varphi}_\varepsilon(\boldsymbol{x},\tau) = -(\mathcal{L}_\varepsilon \boldsymbol{\varphi}_\varepsilon)(\boldsymbol{x},\tau).
$$

Reduction to a second order equation

The operator

$$\mathcal{L}_\varepsilon = \mu^{-1/2}\,\mathrm{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\,\mathrm{curl}\,\mu^{-1/2} - \mu^{1/2}\nabla\,\mathrm{div}\,\mu^{1/2}$$

is reduced by the orthogonal decomposition of $L_2(\mathbb{R}^3;\mathbb{C}^3)$ into the divergence-free and the gradient subspaces (the Weyl decomposition):

$$L_2(\mathbb{R}^3;\mathbb{C}^3) = G \oplus J,$$

where

$$G = \{\boldsymbol{u} = \mu^{1/2}\nabla\omega \colon \omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3),\ \nabla\omega \in L_2(\mathbb{R}^3,\mathbb{C}^3)\},$$
$$J = \{\boldsymbol{u} \in L_2(\mathbb{R}^3,\mathbb{C}^3)\colon\ \mathrm{div}\,\mu^{1/2}\boldsymbol{u} = 0\}.$$

Reduction to a second order equation

The operator

$$\mathcal{L}_\varepsilon = \mu^{-1/2}\,\mathrm{curl}(\eta^\varepsilon(\boldsymbol{x}))^{-1}\,\mathrm{curl}\,\mu^{-1/2} - \mu^{1/2}\nabla\,\mathrm{div}\,\mu^{1/2}$$

is reduced by the orthogonal decomposition of $L_2(\mathbb{R}^3;\mathbb{C}^3)$ into the divergence-free and the gradient subspaces (the Weyl decomposition):

$$L_2(\mathbb{R}^3;\mathbb{C}^3) = G \oplus J,$$

where

$$G = \{\boldsymbol{u} = \mu^{1/2}\nabla\omega\colon \omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3),\ \nabla\omega \in L_2(\mathbb{R}^3,\mathbb{C}^3)\},$$
$$J = \{\boldsymbol{u} \in L_2(\mathbb{R}^3,\mathbb{C}^3)\colon\ \mathrm{div}\,\mu^{1/2}\boldsymbol{u} = 0\}.$$

We are mainly interested in the divergence-free part $\mathcal{L}_{J,\varepsilon}$ of the operator $\mathcal{L}_\varepsilon$.

The solution of problem (3) is represented as:

$$\varphi_\varepsilon = \cos(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{1/2}\phi + \mathcal{L}_{J,\varepsilon}^{-1/2}\sin(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{-1/2}\psi.$$

Reduction to a second order equation

⏱ min

The solution of problem (3) is represented as:

$$\boldsymbol{\varphi}_\varepsilon = \cos(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{1/2}\boldsymbol{\phi} + \mathcal{L}_{J,\varepsilon}^{-1/2}\sin(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{-1/2}\boldsymbol{\psi}.$$

Hence, for $\boldsymbol{H}_\varepsilon = \mu^{-1/2}\boldsymbol{\varphi}_\varepsilon$ we have

$$\boldsymbol{H}_\varepsilon(\cdot,\tau) = \mu^{-1/2}\cos(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{1/2}\boldsymbol{\phi} + \mu^{-1/2}\mathcal{L}_{J,\varepsilon}^{-1/2}\sin(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{-1/2}\boldsymbol{\psi}.$$

Reduction to a second order equation

The solution of problem (3) is represented as:

$$\boldsymbol{\varphi}_\varepsilon = \cos(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{1/2}\boldsymbol{\phi} + \mathcal{L}_{J,\varepsilon}^{-1/2}\sin(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{-1/2}\boldsymbol{\psi}.$$

Hence, for $\boldsymbol{H}_\varepsilon = \mu^{-1/2}\boldsymbol{\varphi}_\varepsilon$ we have

$$\boldsymbol{H}_\varepsilon(\cdot,\tau) = \mu^{-1/2}\cos(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{1/2}\boldsymbol{\phi} + \mu^{-1/2}\mathcal{L}_{J,\varepsilon}^{-1/2}\sin(\tau \mathcal{L}_{J,\varepsilon}^{1/2})\mu^{-1/2}\boldsymbol{\psi}.$$

Thus, we have reduced our problem to studying the behavior of the operator functions

$$\cos(\tau \mathcal{L}_{J,\varepsilon}^{1/2}), \quad \mathcal{L}_{J,\varepsilon}^{-1/2}\sin(\tau \mathcal{L}_{J,\varepsilon}^{1/2}).$$

The operator $\mathcal{L}_\varepsilon$ belongs to a general class of matrix elliptic differential operators to which the operator-theoretic approach developed by M. Sh. Birman and T. A. Suslina applies.

Homogenization of hyperbolic equations with these operators was studied in the papers:

- **M. Sh. Birman, T. A. Suslina**, 2008.
- **Yu. M. Meshkova**, 2017.
- **M. A. Dorodnyi, T. A. Suslina**, 2018, 2020.

Approximation for the operator-functions

The operator $\mathcal{L}_\varepsilon$ belongs to a general class of matrix elliptic differential operators to which the operator-theoretic approach developed by M. Sh. Birman and T. A. Suslina applies.

Homogenization of hyperbolic equations with these operators was studied in the papers:

- **M. Sh. Birman, T. A. Suslina**, 2008.
- **Yu. M. Meshkova**, 2017.
- **M. A. Dorodnyi, T. A. Suslina**, 2018, 2020.

We apply results from these papers to the operator $\mathcal{L}_\varepsilon$, and then obtain results for $\mathcal{L}_{J,\varepsilon}$.

The "model" second-order operator

In $L_2(\mathbb{R}^3;\mathbb{C}^3)$ we consider the second order operator

$$\mathcal{L}_\varepsilon\big|_{\varepsilon=1} =: \mathcal{L} = \mu^{-1/2}\operatorname{curl}\eta(\boldsymbol{x})^{-1}\operatorname{curl}\mu^{-1/2} - \mu^{1/2}\nabla\operatorname{div}\mu^{1/2}.$$

The Floquet-Bloch theory $\Longrightarrow$

$$\mathcal{L} \sim \int_{\widetilde{\Omega}} \oplus \mathcal{L}(\boldsymbol{k})\,d\boldsymbol{k}.$$

The "model" second-order operator

In $L_2(\mathbb{R}^3; \mathbb{C}^3)$ we consider the second order operator

$$\mathcal{L}_\varepsilon|_{\varepsilon=1} =: \mathcal{L} = \mu^{-1/2} \operatorname{curl} \eta(\boldsymbol{x})^{-1} \operatorname{curl} \mu^{-1/2} - \mu^{1/2} \nabla \operatorname{div} \mu^{1/2}.$$

The Floquet-Bloch theory $\Longrightarrow$

$$\mathcal{L} \sim \int_{\widetilde{\Omega}} \oplus \mathcal{L}(\boldsymbol{k})\, d\boldsymbol{k}.$$

The parameter $\boldsymbol{k} \in \widetilde{\Omega}$ is called the *quasimomentum*. The operator $\mathcal{L}(\boldsymbol{k})$ acts in $L_2(\Omega; \mathbb{C}^3)$ and is defined by the expression

$$\mathcal{L}(\boldsymbol{k}) = \mu^{-1/2} \operatorname{curl}_{\boldsymbol{k}} \eta(\boldsymbol{x})^{-1} \operatorname{curl}_{\boldsymbol{k}} \mu^{-1/2} - \mu^{1/2} \nabla_{\boldsymbol{k}} \operatorname{div}_{\boldsymbol{k}} \mu^{1/2}$$

with periodic boundary conditions. Here

$$\nabla_{\boldsymbol{k}} \varphi = \nabla \varphi + i\boldsymbol{k}\varphi, \quad \operatorname{div}_{\boldsymbol{k}} \mathrm{u} = \operatorname{div} \mathrm{u} + i\boldsymbol{k} \cdot \mathrm{u}, \quad \operatorname{curl}_{\boldsymbol{k}} \mathrm{u} = \operatorname{curl} \mathrm{u} + i\boldsymbol{k} \times \mathrm{u}.$$

The operator

$$\mathcal{L}(\boldsymbol{k}) = \mu^{-1/2} \operatorname{curl}_{\boldsymbol{k}} \eta(\boldsymbol{x})^{-1} \operatorname{curl}_{\boldsymbol{k}} \mu^{-1/2} - \mu^{1/2} \nabla_{\boldsymbol{k}} \operatorname{div}_{\boldsymbol{k}} \mu^{1/2}$$

is reduced by the Weyl decomposition

$$L_2(\Omega; \mathbb{C}^3) = G(\boldsymbol{k}) \oplus J(\boldsymbol{k}),$$
$$G(\boldsymbol{k}) := \{\boldsymbol{u} = \mu^{1/2} \nabla_{\boldsymbol{k}} \varphi,\ \varphi \in \widetilde{H}^1(\Omega)\},$$
$$J(\boldsymbol{k}) := \{\boldsymbol{u} \in L_2(\Omega; \mathbb{C}^3) \colon \operatorname{div}_{\boldsymbol{k}} \mu^{1/2} \widetilde{\boldsymbol{u}}(\boldsymbol{x}) = 0\}.$$

Here $\widetilde{H}^1(\Omega)$ is the periodic subspace in $H^1(\Omega)$ and $\widetilde{\boldsymbol{u}}(\boldsymbol{x})$ stands for the Γ-periodic extension of a function $\boldsymbol{u}$ to the whole of $\mathbb{R}^3$.

The “model” second-order operator

The operator

$$\mathcal{L}(\boldsymbol{k}) = \mu^{-1/2} \operatorname{curl}_{\boldsymbol{k}} \eta(\boldsymbol{x})^{-1} \operatorname{curl}_{\boldsymbol{k}} \mu^{-1/2} - \mu^{1/2} \nabla_{\boldsymbol{k}} \operatorname{div}_{\boldsymbol{k}} \mu^{1/2}$$

is reduced by the Weyl decomposition

$$\begin{aligned} L_2(\Omega; \mathbb{C}^3) &= G(\boldsymbol{k}) \oplus J(\boldsymbol{k}), \\ G(\boldsymbol{k}) &:= \{\boldsymbol{u} = \mu^{1/2} \nabla_{\boldsymbol{k}} \varphi, \ \varphi \in \widetilde{H}^1(\Omega)\}, \\ J(\boldsymbol{k}) &:= \{\boldsymbol{u} \in L_2(\Omega; \mathbb{C}^3) \colon \operatorname{div}_{\boldsymbol{k}} \mu^{1/2} \widetilde{\boldsymbol{u}}(\boldsymbol{x}) = 0\}. \end{aligned}$$

Here $\widetilde{H}^1(\Omega)$ is the periodic subspace in $H^1(\Omega)$ and $\widetilde{\boldsymbol{u}}(\boldsymbol{x})$ stands for the Γ-periodic extension of a function $\boldsymbol{u}$ to the whole of $\mathbb{R}^3$.

It is important that $\mathcal{L}_J \sim \int_{\widetilde{\Omega}} \oplus \mathcal{L}_J(\boldsymbol{k})\, d\boldsymbol{k}$, where $\mathcal{L}_J(\boldsymbol{k})$ is the part of the operator $\mathcal{L}(\boldsymbol{k})$ in the subspace $J(\boldsymbol{k})$.

We put

$$\boldsymbol{k} = t\boldsymbol{\theta}, \quad t = |\boldsymbol{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^2.$$

and study the operator family $\mathcal{L}(\boldsymbol{k}) =: L(t; \boldsymbol{\theta})$ by means of the *analytic perturbation theory* with respect to t.

We put

$$\boldsymbol{k} = t\boldsymbol{\theta}, \quad t = |\boldsymbol{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^2.$$

and study the operator family $\mathcal{L}(\boldsymbol{k}) =: L(t;\boldsymbol{\theta})$ by means of the *analytic perturbation theory* with respect to t. The kernel of $\mathcal{L}(\mathbf{0})$ is given by

$$\operatorname{Ker} \mathcal{L}(\mathbf{0}) = \mathfrak{N} = \{\boldsymbol{u} \in L_2(\Omega;\mathbb{C}^3) \colon \boldsymbol{u} = \boldsymbol{c} \in \mathbb{C}^3\}.$$

The “model” operator: analytic perturbation theory

We put

$$\boldsymbol{k} = t\boldsymbol{\theta}, \quad t = |\boldsymbol{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^2.$$

and study the operator family $\mathcal{L}(\boldsymbol{k}) =: L(t;\boldsymbol{\theta})$ by means of the *analytic perturbation theory* with respect to t. The kernel of $\mathcal{L}(\mathbf{0})$ is given by

$$\operatorname{Ker} \mathcal{L}(\mathbf{0}) = \mathfrak{N} = \{\boldsymbol{u} \in L_2(\Omega;\mathbb{C}^3) \colon \boldsymbol{u} = \boldsymbol{c} \in \mathbb{C}^3\}.$$

So, the point $\lambda_0 = 0$ is an isolated eigenvalue of multiplicity 3 for $\mathcal{L}(\mathbf{0})$.

The "model" operator: analytic perturbation theory

⏱ min

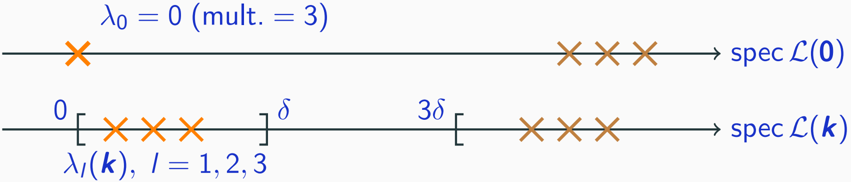
We put

$$\boldsymbol{k} = t\boldsymbol{\theta}, \quad t = |\boldsymbol{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^2.$$

and study the operator family $\mathcal{L}(\boldsymbol{k}) =: L(t;\boldsymbol{\theta})$ by means of the *analytic perturbation theory* with respect to t. The kernel of $\mathcal{L}(\mathbf{0})$ is given by

$$\operatorname{Ker}\mathcal{L}(\mathbf{0}) = \mathfrak{N} = \{\boldsymbol{u} \in L_2(\Omega;\mathbb{C}^3) \colon \boldsymbol{u} = \boldsymbol{c} \in \mathbb{C}^3\}.$$

So, the point $\lambda_0 = 0$ is an isolated eigenvalue of multiplicity 3 for $\mathcal{L}(\mathbf{0})$. Then for $|\boldsymbol{k}| \leqslant t^0$:



There exist real-analytic branches of eigenvalues and eigenvectors:

$$L(t;\boldsymbol{\theta})\varphi_l(t;\boldsymbol{\theta}) = \lambda_l(t;\boldsymbol{\theta})\varphi_l(t;\boldsymbol{\theta}), \qquad l = 1, 2, 3,$$

and the set $\{\varphi_l(t,\boldsymbol{\theta})\}_{l=1,2,3}$ forms an orthonormal basis in the eigenspace of $L(t,\boldsymbol{\theta})$ corresponding to $[0,\delta]$.

There exist real-analytic branches of eigenvalues and eigenvectors:

$$L(t;\boldsymbol{\theta})\varphi_l(t;\boldsymbol{\theta}) = \lambda_l(t;\boldsymbol{\theta})\varphi_l(t;\boldsymbol{\theta}), \qquad l = 1,2,3,$$

and the set $\{\varphi_l(t,\boldsymbol{\theta})\}_{l=1,2,3}$ forms an orthonormal basis in the eigenspace of $L(t,\boldsymbol{\theta})$ corresponding to $[0,\delta]$. Then we have the following power series expansions

$$\begin{aligned} \lambda_l(t;\boldsymbol{\theta}) &= \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \ldots, \qquad & \gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0, \\ \varphi_l(t;\boldsymbol{\theta}) &= \boldsymbol{\omega}_l(\boldsymbol{\theta}) + t\psi_l(\boldsymbol{\theta}) + \ldots, \qquad & l = 1,2,3. \end{aligned}$$

The "model" operator: analytic perturbation theory

⏱ min

There exist real-analytic branches of eigenvalues and eigenvectors:

$$L(t;\boldsymbol{\theta})\boldsymbol{\varphi}_l(t;\boldsymbol{\theta}) = \lambda_l(t;\boldsymbol{\theta})\boldsymbol{\varphi}_l(t;\boldsymbol{\theta}), \qquad l = 1,2,3,$$

and the set $\{\boldsymbol{\varphi}_l(t,\boldsymbol{\theta})\}_{l=1,2,3}$ forms an orthonormal basis in the eigenspace of $L(t,\boldsymbol{\theta})$ corresponding to $[0,\delta]$. Then we have the following power series expansions

$$\begin{aligned} \lambda_l(t;\boldsymbol{\theta}) &= \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \ldots, \qquad & \gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0, \\ \boldsymbol{\varphi}_l(t;\boldsymbol{\theta}) &= \boldsymbol{\omega}_l(\boldsymbol{\theta}) + t\boldsymbol{\psi}_l(\boldsymbol{\theta}) + \ldots, \qquad & l = 1,2,3. \end{aligned}$$

The set $\{\boldsymbol{\omega}_l(\boldsymbol{\theta})\}_{l=1}^{n}$ forms an orthonormal basis in $\mathfrak{N}$.

Spectral germ

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \qquad \gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0,$$
$$\varphi_l(t;\boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\psi_l(\boldsymbol{\theta}) + \dots, \qquad l = 1,2,3.$$

The spectral germ $S(\boldsymbol{\theta})\colon \mathfrak{N} \to \mathfrak{N}$:

$$S(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}), \quad l = 1,2,3.$$

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \qquad \gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0,$$
$$\varphi_l(t;\boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\psi_l(\boldsymbol{\theta}) + \dots, \qquad l = 1,2,3.$$

The spectral germ $S(\boldsymbol{\theta})\colon \mathfrak{N} \to \mathfrak{N}$:

$$S(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}), \quad l = 1,2,3.$$

We have

$$S(\boldsymbol{\theta}) = \mu^{-1/2} r(\boldsymbol{\theta})^t (\eta^0)^{-1} r(\boldsymbol{\theta}) \mu^{-1/2} + \mu^{1/2} \boldsymbol{\theta}\boldsymbol{\theta}^t \mu^{1/2},$$

where

$$r(\boldsymbol{\theta}) = \begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}.$$

Spectral germ

The spectral germ

$$S(\boldsymbol{\theta}) = \mu^{-1/2} r(\boldsymbol{\theta})^t (\eta^0)^{-1} r(\boldsymbol{\theta}) \mu^{-1/2} + \mu^{1/2} \boldsymbol{\theta}\boldsymbol{\theta}^t \mu^{1/2}$$

is reduced by the "Weyl decomposition"

$$\mathfrak{N} = G^0_{\boldsymbol{\theta}} \oplus J^0_{\boldsymbol{\theta}},$$

$$G^0_{\boldsymbol{\theta}} = \{\alpha \mu^{1/2} \boldsymbol{\theta} \colon \alpha \in \mathbb{C}\}, \quad \dim G^0_{\boldsymbol{\theta}} = 1,$$

$$J^0_{\boldsymbol{\theta}} = \{\mu^{1/2} \mathbf{c} \colon \mu \mathbf{c} \perp \boldsymbol{\theta}\}, \quad \dim J^0_{\boldsymbol{\theta}} = 2.$$

The spectral germ

$$S(\boldsymbol{\theta}) = \mu^{-1/2} r(\boldsymbol{\theta})^t (\eta^0)^{-1} r(\boldsymbol{\theta}) \mu^{-1/2} + \mu^{1/2} \boldsymbol{\theta}\boldsymbol{\theta}^t \mu^{1/2}$$

is reduced by the “Weyl decomposition”

$$\mathfrak{N} = G^0_{\boldsymbol{\theta}} \oplus J^0_{\boldsymbol{\theta}},$$
$$G^0_{\boldsymbol{\theta}} = \{\alpha \mu^{1/2} \boldsymbol{\theta} \colon \alpha \in \mathbb{C}\}, \quad \dim G^0_{\boldsymbol{\theta}} = 1,$$
$$J^0_{\boldsymbol{\theta}} = \{\mu^{1/2} \boldsymbol{c} \colon \mu \boldsymbol{c} \perp \boldsymbol{\theta}\}, \quad \dim J^0_{\boldsymbol{\theta}} = 2.$$

The germ has unique eigenvalue in the subspace $G^0_{\boldsymbol{\theta}}$:

$$\gamma_3(\boldsymbol{\theta}) = \langle \mu \boldsymbol{\theta}, \boldsymbol{\theta} \rangle.$$

The spectral germ

$$S(\boldsymbol{\theta}) = \mu^{-1/2} r(\boldsymbol{\theta})^t (\eta^0)^{-1} r(\boldsymbol{\theta}) \mu^{-1/2} + \mu^{1/2} \boldsymbol{\theta}\boldsymbol{\theta}^t \mu^{1/2}$$

is reduced by the “Weyl decomposition”

$$\mathfrak{N} = G^0_{\boldsymbol{\theta}} \oplus J^0_{\boldsymbol{\theta}},$$
$$G^0_{\boldsymbol{\theta}} = \{\alpha\mu^{1/2}\boldsymbol{\theta} \colon \alpha \in \mathbb{C}\}, \quad \dim G^0_{\boldsymbol{\theta}} = 1,$$
$$J^0_{\boldsymbol{\theta}} = \{\mu^{1/2}\boldsymbol{c} \colon \mu\boldsymbol{c} \perp \boldsymbol{\theta}\}, \quad \dim J^0_{\boldsymbol{\theta}} = 2.$$

The germ has unique eigenvalue in the subspace $G^0_{\boldsymbol{\theta}}$:

$$\gamma_3(\boldsymbol{\theta}) = \langle \mu\boldsymbol{\theta}, \boldsymbol{\theta} \rangle.$$

In the subspace $J^0_{\boldsymbol{\theta}}$ the germ has two eigenvalues $\gamma_1(\boldsymbol{\theta})$ and $\gamma_2(\boldsymbol{\theta})$ corresponding to the algebraic problem

$$r(\boldsymbol{\theta})^t (\eta^0)^{-1} r(\boldsymbol{\theta}) \boldsymbol{c} = \gamma\mu\boldsymbol{c}, \quad \mu\boldsymbol{c} \perp \boldsymbol{\theta}. \tag{4}$$

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \qquad l = 1,2,3.$$
$$\boldsymbol{\varphi}_l(t;\boldsymbol{\theta}) = \boldsymbol{\omega}_l(\boldsymbol{\theta}) + t\boldsymbol{\psi}_l(\boldsymbol{\theta}) + \dots,$$

- We can always choose $\{\boldsymbol{\varphi}_l(t,\boldsymbol{\theta})\}_{l=1}^3$ so that

$$\boldsymbol{\varphi}_3(t;\boldsymbol{\theta}) \in G(t\boldsymbol{\theta}); \quad \boldsymbol{\varphi}_1(t;\boldsymbol{\theta}), \boldsymbol{\varphi}_2(t;\boldsymbol{\theta}) \in J(t\boldsymbol{\theta});$$

and $\boldsymbol{\omega}_3(\boldsymbol{\theta}) \in G^0_{\boldsymbol{\theta}}$; $\boldsymbol{\omega}_1(\boldsymbol{\theta}), \boldsymbol{\omega}_2(\boldsymbol{\theta}) \in J^0_{\boldsymbol{\theta}}$.

Analytic perturbation theory

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \qquad l = 1, 2, 3.$$
$$\varphi_l(t;\boldsymbol{\theta}) = \boldsymbol{\omega}_l(\boldsymbol{\theta}) + t\boldsymbol{\psi}_l(\boldsymbol{\theta}) + \dots,$$

- We can always choose $\{\varphi_l(t,\boldsymbol{\theta})\}_{l=1}^3$ so that

$$\varphi_3(t;\boldsymbol{\theta}) \in G(t\boldsymbol{\theta}); \quad \varphi_1(t;\boldsymbol{\theta}), \varphi_2(t;\boldsymbol{\theta}) \in J(t\boldsymbol{\theta});$$

  and $\boldsymbol{\omega}_3(\boldsymbol{\theta}) \in G^0_{\boldsymbol{\theta}}$; $\boldsymbol{\omega}_1(\boldsymbol{\theta}), \boldsymbol{\omega}_2(\boldsymbol{\theta}) \in J^0_{\boldsymbol{\theta}}$.
- We always have $\varkappa_3(\boldsymbol{\theta}) = 0$.

$$\begin{aligned} \lambda_l(t;\boldsymbol{\theta}) &= \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \\ \varphi_l(t;\boldsymbol{\theta}) &= \boldsymbol{\omega}_l(\boldsymbol{\theta}) + t\boldsymbol{\psi}_l(\boldsymbol{\theta}) + \dots, \end{aligned} \qquad l = 1, 2, 3.$$

- We can always choose $\{\varphi_l(t,\boldsymbol{\theta})\}_{l=1}^3$ so that

$$\varphi_3(t;\boldsymbol{\theta}) \in G(t\boldsymbol{\theta}); \quad \varphi_1(t;\boldsymbol{\theta}), \varphi_2(t;\boldsymbol{\theta}) \in J(t\boldsymbol{\theta});$$

  and $\boldsymbol{\omega}_3(\boldsymbol{\theta}) \in G^0_{\boldsymbol{\theta}}$; $\boldsymbol{\omega}_1(\boldsymbol{\theta}), \boldsymbol{\omega}_2(\boldsymbol{\theta}) \in J^0_{\boldsymbol{\theta}}$.
- We always have $\varkappa_3(\boldsymbol{\theta}) = 0$.
- If $\gamma_1(\boldsymbol{\theta}) \neq \gamma_2(\boldsymbol{\theta})$, then $\varkappa_1(\boldsymbol{\theta}) = \varkappa_2(\boldsymbol{\theta}) = 0$.

Analytic perturbation theory

$$\begin{aligned} \lambda_l(t;\boldsymbol{\theta}) &= \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \\ \varphi_l(t;\boldsymbol{\theta}) &= \boldsymbol{\omega}_l(\boldsymbol{\theta}) + t\psi_l(\boldsymbol{\theta}) + \dots, \end{aligned} \qquad l = 1, 2, 3.$$

- We can always choose $\{\varphi_l(t,\boldsymbol{\theta})\}_{l=1}^3$ so that

$$\varphi_3(t;\boldsymbol{\theta}) \in G(t\boldsymbol{\theta}); \quad \varphi_1(t;\boldsymbol{\theta}), \varphi_2(t;\boldsymbol{\theta}) \in J(t\boldsymbol{\theta});$$

  and $\boldsymbol{\omega}_3(\boldsymbol{\theta}) \in G^0_{\boldsymbol{\theta}}$; $\boldsymbol{\omega}_1(\boldsymbol{\theta}), \boldsymbol{\omega}_2(\boldsymbol{\theta}) \in J^0_{\boldsymbol{\theta}}$.
- We always have $\varkappa_3(\boldsymbol{\theta}) = 0$.
- If $\gamma_1(\boldsymbol{\theta}) \neq \gamma_2(\boldsymbol{\theta})$, then $\varkappa_1(\boldsymbol{\theta}) = \varkappa_2(\boldsymbol{\theta}) = 0$.
- If $\gamma_1(\boldsymbol{\theta}_0) = \gamma_2(\boldsymbol{\theta}_0)$ for some $\boldsymbol{\theta}_0$, then

$$\varkappa_{1,2}(\boldsymbol{\theta}_0) = \pm F(\boldsymbol{\theta}_0)\frac{\langle \mu\boldsymbol{\theta}_0, \boldsymbol{\theta}_0\rangle^{1/2}}{(\det \mu)^{1/2}}.$$

  Here the function $F(\boldsymbol{\theta}_0)$ is homogeneous of degree 1; it can be calculated in terms of solutions of auxiliary problems.

min

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \varkappa_l(\boldsymbol{\theta})t^3 + \dots, \qquad l = 1, 2, 3.$$

Theorem 3 [M. Dorodnyi and T. Suslina, 2021]

Suppose that $\gamma_1(\boldsymbol{\theta}) \neq \gamma_2(\boldsymbol{\theta})$ for any $\boldsymbol{\theta} \in \mathbb{S}^2$ or $\gamma_1(\boldsymbol{\theta}) \equiv \gamma_2(\boldsymbol{\theta})$ and $\varkappa_{1,2}(\boldsymbol{\theta}) \equiv 0$.

If $\boldsymbol{\phi}, \boldsymbol{f} \in H^{3/2}(\mathbb{R}^3; \mathbb{C}^3)$ and $\operatorname{div} \mu\boldsymbol{\phi} = 0$, then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have

$$\left\|\boldsymbol{H}_\varepsilon(\cdot,\tau) - \boldsymbol{H}_0(\cdot,\tau)\right\|_{L_2} \leqslant C(1+|\tau|)^{1/2}\varepsilon\left(\|\boldsymbol{\phi}\|_{H^{3/2}} + \|\boldsymbol{f}\|_{H^{3/2}}\right),$$

$$\left\|\boldsymbol{B}_\varepsilon(\cdot,\tau) - \boldsymbol{B}_0(\cdot,\tau)\right\|_{L_2} \leqslant C(1+|\tau|)^{1/2}\varepsilon\left(\|\boldsymbol{\phi}\|_{H^{3/2}} + \|\boldsymbol{f}\|_{H^{3/2}}\right).$$

Theorem 4 [M. Dorodnyi and T. Suslina, 2021]

Suppose that $\gamma_1(\boldsymbol{\theta}) \neq \gamma_2(\boldsymbol{\theta})$ for any $\boldsymbol{\theta} \in \mathbb{S}^2$ or $\gamma_1(\boldsymbol{\theta}) \equiv \gamma_2(\boldsymbol{\theta})$ and $\varkappa_{1,2}(\boldsymbol{\theta}) \equiv 0$.

If $\boldsymbol{\phi} = \mathbf{0}$ and $\boldsymbol{f} \in H^{5/2}(\mathbb{R}^3;\mathbb{C}^3)$, then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have

$$\left\| \boldsymbol{H}_\varepsilon(\cdot,\tau) - \boldsymbol{H}_0(\cdot,\tau) - \varepsilon\mu^{-1}\Psi^\varepsilon \operatorname{curl} \boldsymbol{H}_0(\cdot,\tau) \right\|_{H^1} \leqslant C(1+|\tau|)^{1/2}\varepsilon \|\boldsymbol{f}\|_{H^{5/2}},$$

$$\left\| \boldsymbol{B}_\varepsilon(\cdot,\tau) - \boldsymbol{B}_0(\cdot,\tau) - \varepsilon\Psi^\varepsilon \operatorname{curl} \boldsymbol{H}_0(\cdot,\tau) \right\|_{H^1} \leqslant C(1+|\tau|)^{1/2}\varepsilon \|\boldsymbol{f}\|_{H^{5/2}},$$

$$\left\| (\boldsymbol{E}_\varepsilon(\cdot,\tau) - \boldsymbol{E}_\varepsilon(\cdot,0)) - (\mathbb{1} + \Sigma^\varepsilon)(\boldsymbol{E}_0(\cdot,\tau) - \boldsymbol{E}_0(\cdot,0)) \right\|_{L_2} \leqslant C|\tau|(1+|\tau|)^{1/2}\varepsilon \|\boldsymbol{f}\|_{H^{5/2}},$$

$$\left\| (\boldsymbol{D}_\varepsilon(\cdot,\tau) - \boldsymbol{D}_\varepsilon(\cdot,0)) - (\mathbb{1} + (\widetilde{\eta}^\varepsilon(\eta^0)^{-1} - \mathbb{1}))(\boldsymbol{D}_0(\cdot,\tau) - \boldsymbol{D}_0(\cdot,0)) \right\|_{L_2} \leqslant C|\tau|(1+|\tau|)^{1/2}\varepsilon \|\boldsymbol{f}\|_{H^{5/2}}.$$

References

- M. Sh. Birman, T. A. Suslina, *Operator error estimates in the homogenization problem for nonstationary periodic equations*, St. Petersburg Math. J., **20**:6 (2009), 873–928.
- Yu. M. Meshkova, *On operator error estimates for homogenization of hyperbolic systems with periodic coefficients*, J. Spectr. Theory, **11**:2 (2021), 587–660.
- M. A. Dorodnyi, T. A. Suslina, *Spectral approach to homogenization of hyperbolic equations with periodic coefficients*, J. Differential Equations, **264**:12 (2018), 7463–7522.
- M. A. Dorodnyi, T. A. Suslina, *Homogenization of the hyperbolic equations with periodic coefficients in $\mathbb{R}^d$: sharpness of the results*, St. Petersburg Math. J., **32**:4 (2021), 605–703.

References

- M. A. Dorodnyi, T. A. Suslina, *Homogenization of a nonstationary model equation of electrodynamics*, Math. Notes, **102**:5 (2017), 645–663.
- M. A. Dorodnyi, T. A. Suslina, *Homogenization of nonstationary periodic Maxwell system in the case of constant permeability*, J. Differential Equations, **307** (2022), 348–388.

Thank you for your attention!