Semiclassical asymptotics of the spectral function of the magnetic Schrödinger operator on a manifold of bounded geometry

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Magnetic Schrödinger operator

- The d-dimensional space $\mathbb{R}^d$ with Riemannian metric

$$g = \sum_{j,k=1}^{d} g_{jk}(x)dx_j dx_k.$$

- The magnetic potential $\mathbf{A} = \sum_{j=1}^{d} A_j(x)dx_j$.
- The electric potential $V \in C^\infty(\mathbb{R}^d)$.
- The magnetic Schrödinger operator

$$H_p = -\frac{1}{p}\sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}}\left(\frac{\partial}{\partial x_j} - ipA_j(x)\right)\times \\ \times \left[g^{jk}(x)\sqrt{\det g(x)}\left(\frac{\partial}{\partial x_j} - ipA_k(x)\right)\right] + V(x), \quad p > 1,$$

where $(g^{jk}(x))$ is the inverse matrix of $g(x) = (g_{jk}(x))$, $\det g(x)$ is the determinant of $g(x)$.

Magnetic Schrödinger operator

If g is the standard Euclidean metric

$$g = \sum_{j=1}^{d} dx_j^2,$$

then

$$H_p = -\frac{1}{p}\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} - ipA_j(x)\right)^2 + V(x)$$
$$= \frac{1}{\hbar}\left[\sum_{j=1}^{d}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_j(x)\right)^2 + \hbar V(x)\right], \quad p = \frac{1}{\hbar} > 1.$$

Assumptions

The magnetic Schrödinger operator

$$H_p = -\frac{1}{p}\sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}}\left(\frac{\partial}{\partial x_j} - ipA_j(x)\right)\times$$
$$\times\left[g^{jk}(x)\sqrt{\det g(x)}\left(\frac{\partial}{\partial x_j} - ipA_k(x)\right)\right] + V(x), \quad p>1,$$

For any $j,k=1,\ldots,d$, we have $g_{jk}\in C_b^\infty(\mathbb{R}^d)$, that is, for any $\alpha\in\mathbb{Z}_+^d$,

$$\sup_{x\in\mathbb{R}^d} |\partial^\alpha g_{jk}(x)| < \infty.$$

Moreover, $g(x)=(g_{jk}(x))$ is positive definite uniformly on $x\in\mathbb{R}^d$:

$$\inf_{x\in\mathbb{R}^d} g(x) \geqslant \varepsilon_0 > 0.$$

Assumptions

The magnetic Schrödinger operator

$$H_p = -\frac{1}{p}\sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}}\left(\frac{\partial}{\partial x_j} - ipA_j(x)\right) \times$$
$$\times \left[g^{jk}(x)\sqrt{\det g(x)}\left(\frac{\partial}{\partial x_j} - ipA_k(x)\right)\right] + V(x), \quad p > 1,$$

For the magnetic field $\mathbf{B}$ defined by

$$\mathbf{B} = d\mathbf{A} = \sum_{j<k} B_{jk}(x) dx_j \wedge dx_k, \quad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}.$$

we have $B_{jk} \in C_b^\infty(\mathbb{R}^d)$. Finally, $V \in C_b^\infty(\mathbb{R}^d)$.

Spectral problem

- The magnetic Schrödinger operator H_p is a uniformly elliptic second order differential operator.
- It is essentially self-adjoint in the Hilbert space $L^2(\mathbb{R}^d, dv_g)$ with initial domain $C_c^\infty(\mathbb{R}^d)$, where $dv_g = \sqrt{\det g(x)}dx$ is the Riemannian volume form.
- Such operators have been introduced and studied by J.-P. Demailly 1985 in connection with the holomorphic Morse inequalities for the Dolbeault cohomology associated with high tensor powers of a holomorphic Hermitian line bundle over a compact complex manifold.
- We will study spectral properties of H_p in the semiclassical limit $p \to \infty$. First, in the case, when $\mathbf{B}$ is of maximal rank ($d = 2n$), we give a rough asymptotic description of the spectrum of H_p as p goes to ∞.

Model operator

For any $x_0 \in \mathbb{R}^d$. the model operator $\mathcal{H}^{(x_0)}$ is the magnetic Schrödinger operator on $C^\infty(T_{x_0}\mathbb{R}^d) \cong C^\infty(\mathbb{R}^d)$

$$\mathcal{H}^{(x_0)} = -\sum_{j,k=1}^{d} g^{jk}(x_0)\left(\frac{\partial}{\partial v_j} - iA_{j,x_0}(v)\right)\times \\ \times\left(\frac{\partial}{\partial v_k} - iA_{k,x_0}(v)\right) + V(x_0), \quad v \in \mathbb{R}^d$$

with the magnetic potential

$$\mathbf{A}_{x_0} = \sum_{j=1}^{d} A_{j,x_0}(v)dv_j = \frac{1}{2}\sum_{j<k} B_{jk}(x_0)(v_j dv_k - v_k dv_j),$$

and constant magnetic field

$$d\mathbf{A}_{x_0} = \sum_{j<k} B_{jk}(x_0)dv_j \wedge dv_k = \mathbf{B}_{x_0}.$$

Spectrum of the model operator

- Let B_{x_0} be a skew-symmetric operator in $T_{x_0}\mathbb{R}^d \cong \mathbb{R}^d$ such that

$$\mathbf{B}_{x_0}(u,v) = g_{x_0}(B_{x_0}u, v), \quad u, v \in T_{x_0}\mathbb{R}^d.$$

Since B_{x_0} is skew-symmetric of rank $2n$, zero is an eigenvalue of multiplicity $d-2n$ and its non-zero eigenvalues have the form

$$\pm i a_j(x_0), j = 1, \ldots, n, \quad a_j(x_0) > 0,$$

- Denote

$$\Lambda_{\mathbf{k}}(x_0) = \sum_{j=1}^{n} (2k_j+1) a_j(x_0) + V(x_0), \quad \mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}_+^n.$$

Spectrum of the model operator

- In the maximal rank case $d = 2n$, the spectrum of $\mathcal{H}^{(x_0)}$ consists of eigenvalues of infinite multiplicity (Landau levels)

$$\sigma(\mathcal{H}^{(x_0)}) = \left\{\Lambda_{\mathbf{k}}(x_0) \,:\, \mathbf{k} \in \mathbb{Z}_+^n\right\}.$$

In particular, the lowest eigenvalue of $\mathcal{H}^{(x_0)}$ is

$$\Lambda_0(x_0) = \sum_{j=1}^{n} a_j(x_0) + V(x_0).$$

- In the case $d > 2n$, the spectrum is the half-line

$$\sigma(\mathcal{H}^{(x_0)}) = [\Lambda_0(x_0), +\infty),$$

Description of the spectrum

Theorem (Yu.K. 2020)

Assume that $\mathbf{B}$ *is of maximal rank* ($d = 2n$). *Then, for any* $K > 0$*, there exists* $c > 0$ *such that for any* $p \in \mathbb{N}$ *the spectrum of* H_p *in* $[0, K]$ *is contained in the* $cp^{-1/4}$*-neighborhood of*

$$\Sigma = \bigcup_{x_0 \in \mathbb{R}^d} \Sigma_{x_0} = \left\{ \Lambda_{\mathbf{k}}(x_0) \,:\, \mathbf{k} \in \mathbb{Z}_+^n, x_0 \in \mathbb{R}^d \right\}.$$

L. Charles (2021): in the compact case $cp^{-1/2}$ instead of $cp^{-1/4}$.

Band structure

For any $\mathbf{k} \in \mathbb{Z}_+^n$, we have the band $[\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}] = \{\Lambda_{\mathbf{k}}(x_0) : x_0 \in \mathbb{R}^d\}$, and Σ is a closed subset of $\mathbb{R}$:

$$\Sigma = \bigcup_{\mathbf{k} \in \mathbb{Z}_+^n} [\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}].$$

Functions of the operator

- For any $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi(H_p)$ is a smoothing operator in $L^2(\mathbb{R}^d, dv_g)$.
- $K_{\varphi(H_p)} \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ the Schwartz kernel of $\varphi(H_p)$ with respect to the Riemannian volume form dv_g:

$$\varphi(H_p)u(x) = \int_{\mathbb{R}^d} K_{\varphi(H_p)}(x, x')u(x')dv_g(x'), \quad u \in L^2(\mathbb{R}^d, dv_g).$$

- For any $\varepsilon > 0$ and $k \in \mathbb{Z}_+$,

$$\left|K_{\varphi(H_p)}(x, x')\right|_{C^k} = \mathcal{O}(p^{-\infty}), \quad |x - x'| > \varepsilon.$$

- For the proof, we use the finite propagation speed property of solutions of hyperbolic equations.

Full off-diagonal asymptotics

Theorem (Yu.K. 2022)

The following asymptotic expansion holds true as $p \to \infty$:

$$p^{-\frac{d}{2}} K_{\varphi(H_p)}(x_0+v, x_0+v') \cong \sum_{r=0}^{\infty} F_{r,x_0}(\sqrt{p}v, \sqrt{p}v')\kappa_{x_0}^{-\frac{1}{2}}(v)\kappa_{x_0}^{-\frac{1}{2}}(v')p^{-\frac{r}{2}}, \quad x_0 \in \mathbb{R}^d, v, v' \in \mathbb{R}^d.$$

Here κ_{x_0} is a smooth function on $T_{x_0}\mathbb{R}^d \cong \mathbb{R}^d$ defined by

$$\kappa_{x_0}(v) = \sqrt{\det g(x_0+v)}, \quad v \in \mathbb{R}^d.$$

Full off-diagonal expansions for the (generalized) Bergman kernels Dai-Liu-Ma04,Ma-Marinescu07,Yu.K.18, ..., goes back to Bismut-Lebeau localization technique in index theory.

Full off-diagonal asymptotics

Theorem (Yu.K. 2022)

For any $j, m, m' \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that, for any $N \in \mathbb{N}$, there exists $C > 0$ such that for any $p \geqslant 1$, $x_0 \in \mathbb{R}^d$ and $v, v' \in T_{x_0}\mathbb{R}^d$,

$$\begin{aligned}\sup_{|\alpha|+|\alpha'|\leqslant m}\Bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial v^{\alpha}\partial v'^{\alpha'}}\Bigg(&p^{-\frac{d}{2}}K_{\varphi(H_p)}(x_0+v, x_0+v')\\ &-\sum_{r=0}^{j}F_{r,x_0}(\sqrt{p}v,\sqrt{p}v')\kappa_{x_0}^{-\frac{1}{2}}(v)\kappa_{x_0}^{-\frac{1}{2}}(v')p^{-\frac{r}{2}}\Bigg)\Bigg|_{C^{m'}}\\ &\leqslant Cp^{-\frac{j-m+1}{2}}(1+\sqrt{p}|v|+\sqrt{p}|v'|)^{M}(1+\sqrt{p}|v-v'|)^{-N}.\end{aligned}$$

Here $C^{m'}$ is the $C^{m'}$-norm for the parameter $x_0 \in \mathbb{R}^d$.

On-diagonal asymptotics

Corollary

For any $x_0 \in \mathbb{R}^d$, there exists a sequence of distributions $f_r(x_0) \in \mathcal{S}'(\mathbb{R}), r \geqslant 0$, such that the following asymptotic expansion holds true as $p \to \infty$ uniformly on x_0:

$$K_{\varphi(H_p)}(x_0, x_0) \sim p^{\frac{d}{2}} \sum_{r=0}^{\infty} \langle f_r(x_0), \varphi \rangle p^{-\frac{r}{2}}, \quad \langle f_r(x_0), \varphi \rangle = F_{r,x_0}(0,0).$$

Corollary: semiclassical trace formula

In the compact case

$$\operatorname{tr} \varphi(H_p) \sim \sum_{r=0}^{\infty} \langle f_r, \varphi \rangle p^{\frac{d-r}{2}}, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

A version of the Gutzwiller trace formula (for a singular energy level).

Leading coefficient

- Recall that, for any $x_0 \in \mathbb{R}^d$. the model operator $\mathcal{H}^{(x_0)}$ is the magnetic Schrödinger operator on $C^\infty(T_{x_0}\mathbb{R}^d) \cong C^\infty(\mathbb{R}^d)$ with constant magnetic field $\mathbf{B}_{x_0}$.
- For any $\varphi \in \mathcal{S}(\mathbb{R})$, $K_{\varphi(\mathcal{H}^{(x_0)})}(v, v') \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ the smooth integral kernel of the operator $\varphi(\mathcal{H}^{(x_0)})$:

$$\varphi(\mathcal{H}^{(x_0)})u(v) = \int_{\mathbb{R}^d} K_{\varphi(\mathcal{H}^{(x_0)})}(v, v')u(v')dv', \quad u \in C_c^\infty(\mathbb{R}^d).$$

- Then

$$\langle f_0(x_0), \varphi\rangle = K_{\varphi(\mathcal{H}^{(x_0)})}(0,0), \quad x_0 \in \mathbb{R}^d.$$

Leading coefficient

In the maximal rank case $d = 2n$

$$\langle f_0(x_0), \varphi\rangle = \frac{1}{(2\pi)^n}\prod_{j=1}^n a_j(x_0)\sum_{\mathbf{k}\in\mathbb{Z}_+^n}\varphi(\Lambda_{\mathbf{k}}(x_0)),$$

where $\Lambda_{\mathbf{k}}(x_0) = \sum_{j=1}^n (2k_j+1)a_j(x_0) + V(x_0), \mathbf{k}\in\mathbb{Z}_+^n$.
In the case $d > 2n$,

$$\begin{aligned}\langle f_0(x_0), \varphi\rangle =& \frac{1}{(2\pi)^{d-n}}\prod_{j=1}^n a_j(x_0)\sum_{\mathbf{k}\in\mathbb{Z}_+^n}\int_{\mathbb{R}^{d-2n}}\varphi(\Lambda_{\mathbf{k}}(x_0)+|\xi|^2)d\xi \\ =& \frac{|S^{d-2n-1}|}{2(2\pi)^{d-n}}\prod_{j=1}^n a_j(x_0)\sum_{\mathbf{k}\in\mathbb{Z}_+^n}\int_0^{+\infty}\varphi(\tau)(\tau-\Lambda_{\mathbf{k}}(x_0))_+^{d/2-n-1}d\tau.\end{aligned}$$

Higher order coefficients

In the maximal rank case $d = 2n$

$$\langle f_r(x_0), \varphi\rangle = \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \sum_{j=1}^{N} f_{r,\mathbf{k},j}(x_0)\varphi^{(j-1)}(\Lambda_{\mathbf{k}}(x_0)).$$

In the general case $d > 2n$

$$\langle f_r(x_0), \varphi\rangle = \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \sum_{j=1}^{N} f_{r,\mathbf{k},j}(x_0) \int_0^{+\infty} \varphi^{(j-1)}(\tau)(\tau - \Lambda_{\mathbf{k}}(x_0))_+^{d/2-n-1} d\tau$$

Asymptotic localization of Schwartz kernels

Theorem

Assume that, for some $x_0 \in \mathbb{R}^d$, the rank of $\mathbf{B}_{x_0}$ equals d and an interval (α, β) does not contain any $\Lambda_{\mathbf{k}}(x_0)$ with $\mathbf{k} \in \mathbb{Z}_+^n$.
For any $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subset (\alpha, \beta)$,

$$\left| K_{\varphi(H_p)}(x_0, x_0) \right|_{C^k} = \mathcal{O}(p^{-\infty}), \quad k = 0, 1, \ldots, \quad p \to \infty.$$

Moreover, if an interval $[\alpha, \beta]$ does not contain any $\Lambda_{\mathbf{k}}(x_0)$ with $\mathbf{k} \in \mathbb{Z}_+^n$, then the Schwartz kernel of the spectral projection $E_{[\alpha,\beta]}$ of the operator H_p associated with $[\alpha, \beta]$ satisfies

$$\left| E_{[\alpha,\beta]}(x_0, x_0) \right| = \mathcal{O}(p^{-\infty}), \quad p \to \infty.$$

Interpretation of coefficients: the maximal rank case

$$\langle f_0(x_0), \varphi\rangle = \frac{1}{(2\pi)^n}\prod_{j=1}^{n} a_j(x_0) \sum_{\mathbf{k}\in\mathbb{Z}^n} \varphi(\Lambda_{\mathbf{k}}(x_0)).$$

Using the formula

$$\sum_{\mathbf{k}\in\mathbb{Z}^n_+} \varphi(\Lambda_{\mathbf{k}}) = C_1 \left\langle \frac{e^{itV}}{\prod_{j=1}^n \sin a_j(t-i0)}, \hat{\varphi}(t)\right\rangle$$

where $\hat{\varphi}(t)$ is the Fourier transform, we can write

$$\langle f_0(x_0), \varphi\rangle = \frac{C_1}{(2\pi)^n}\prod_{j=1}^{n} a_j(x_0) \left\langle \frac{e^{itV(x_0)}}{\prod_{j=1}^n \sin a_j(x_0)(t-i0)}, \hat{\varphi}(t)\right\rangle.$$

Interpretation of coefficients: the general case

Similarly, using the formula

$$\sum_{\mathbf{k}\in\mathbb{Z}^n_+}\int_0^\infty \varphi(\tau)(\tau-\Lambda_{\mathbf{k}})_+^{d/2-n-1}\,d\tau$$

$$= C\left\langle \frac{e^{itV}}{(t-i0)^{d/2-n}\prod_{j=1}^n \sin a_j(t-i0)}, \hat\varphi(t)\right\rangle.$$

we write

$$\langle f_0(x_0),\varphi\rangle$$

$$=\frac{C_2}{(2\pi)^{d-n}}\prod_{j=1}^n a_j(x_0)\left\langle \frac{e^{itV(x_0)}}{(t-i0)^{d/2-n}\prod_{j=1}^n \sin a_j(x_0)(t-i0)}, \hat\varphi(t)\right\rangle.$$

Magnetic geometry

- The semiclassical magnetic Schrödinger operator

$$\mathcal{H}^\hbar = \sum_{j=1}^{d} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - A_j(x) \right)^2 + \hbar V.$$

- The classical Hamiltonian

$$H(x,\xi) = |\xi - \mathbf{A}(x)|_g^2, \quad (x,\xi) \in T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d.$$

- The zero energy level (the characteristic set)

$$\Sigma = H^{-1}(0) = \{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d : \xi = \mathbf{A}(x)\}.$$

- $E_0 = 0$ is the critical energy level:

$$dH(x,\xi) = 0, \quad (x,\xi) \in \Sigma.$$

Magnetic geometry

- Σ is a d-dimensional submanifold of $T^*\mathbb{R}^d$ which can be identified with $\mathbb{R}^d$ using
$$j : x_0 \in \mathbb{R}^d \mapsto (x_0, \mathbf{A}(x_0)) \in \Sigma,$$
its inverse is the restriction of the projection $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ to Σ.
So the operator $\mathcal{H}^h$ is degenerate at each point $x_0 \in \mathbb{R}^d$.
- The restriction of the canonical symplectic form $\omega = d\xi \wedge dx$ to Σ is
$$\omega_\Sigma = \pi^* \mathbf{B}.$$
- If $\mathbf{B}$ is non-degenerate (maximal rank), then $(\Sigma, \pi^*\mathbf{B})$ is a symplectic manifold.
In microlocal analysis, these are double symplectic characteristics.
- If $\mathbf{B}$ has constant rank, then $(\Sigma, \pi^*\mathbf{B})$ is a presymplectic manifold.

The linearized flow

- The Hamiltonian flow of H is (equivalent to) the magnetic geodesic flow
$$\phi_t : T^*\mathbb{R}^d \to T^*\mathbb{R}^d.$$
- ϕ_t preserves Σ. Moreover, since $dH = 0$ on Σ, each point of Σ is a fixed point of the flow:
$$\phi_t(j(x_0)) = j(x_0), \quad j(x_0) = (x_0, \mathbf{A}(x_0)) \in \Sigma, \quad t \in \mathbb{R}.$$
- For each $x_0 \in \mathbb{R}^d$, one can consider the linearized flow at $j(x_0)$:
$$(d\phi_t)_{j(x_0)} : T_{j(x_0)}\Sigma \to T_{j(x_0)}\Sigma, \quad t \in \mathbb{R},$$
a linear Hamiltonian flow with quadratic Hamiltonian $d^2H_{j(x_0)}$. The eigenvalues of $d^2H_{j(x_0)}$ are $2a_k(x_0), k = 1, \ldots, n$ (and 0 if $d > 2n$).

Geometric interpretation of the formula

$$\langle f_0(x_0),\varphi\rangle = \frac{C_1}{(2\pi)^n}\prod_{j=1}^n a_j(x_0)\left\langle \frac{e^{itV(x_0)}}{\prod_{j=1}^n \sin a_j(x_0)(t-i0)}, \hat\varphi(t)\right\rangle .$$

$(d\phi_t)_{j(x_0)}$ are the rotations by the angles $2a_k(x_0)t, k=1,\ldots,n$:

$$\frac{1}{\prod_{j=1}^n \sin a_j(x_0)t} = \frac{1}{\det(I-(d\phi_t)_{j(x_0)})^{1/2}}$$

the standard contribution of a non-degenerate periodic trajectory in the trace formula (with the linear Poincare map replaced by $(d\phi_t)_{j(x_0)}$). The singularities

$$T_k(x_0) = \frac{\pi}{a_k(x_0)}$$

are the periods of $(d\phi_t)_{j(x_0)}$.