Non-classical waves admissibility study using vanishing viscosity method and their dependence on dissipation coefficients in the Riemann problem for the chemical flood model.

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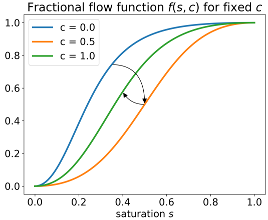
Problem statement

Chemical flooding conservation laws system ($x \in \mathbb{R}, t > 0$):

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs + a(c))_t + (cf(s,c))_x &= 0. \end{aligned} \tag{1}$$

- $s = s(x,t)$ — water saturation;
- $c = c(x,t)$ — concentration of the chemical agent in water;
- $f(s,c)$ — fractional flow function (usually S-shaped in s);
- $a(c)$ — chemical agent adsorption (increasing, concave).



Initial data: $$(s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim: Solve problem (1)–(2) when f is non-monotone in c.

Reduced problem: find an admissible shock wave

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc + a(c))_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases}$$

Proposition (Johansen-Winther, 1988 (JW))

There exist $u^- = (s^-, 1)$ and $u^+ = (s^+, 0)$ such that the solution to a Riemann problem has the following structure:

$$(1,1) \xrightarrow{c=1} u^- \xrightarrow{c \text{ jumps from } 1 \text{ to } 0} u^+ \xrightarrow{c=0} (0,0). \tag{3}$$

- JW considered $f(s,c)$ monotone in c. Found a unique vanishing viscosity solution.
- When $f(s,c)$ is non-monotone in c, multiple vanishing viscosity solutions are possible. Examples can be found in Shen (2017); see also Entov-Kerimov (1986) for a non-rigorous consideration of the non-monotone case.

Non-strictly hyperbolic system and Lax condition

$$\begin{pmatrix} s_t \\ c_t \end{pmatrix} + \begin{pmatrix} f_s(s,c) & f_c(s,c) \\ 0 & \frac{f(s,c)}{s+a'(c)} \end{pmatrix} \begin{pmatrix} s_x \\ c_x \end{pmatrix} = 0. \quad \lambda_s = f_s(s,c),\ \lambda_c = \frac{f(s,c)}{s+a'(c)}.$$

Monotone (in c) case is well-studied (Johansen-Winther, 1988).
Vanishing viscosity gives unique solution satisfying Lax condition.
Lax condition in non-monotone case is sometimes physically meaningless.

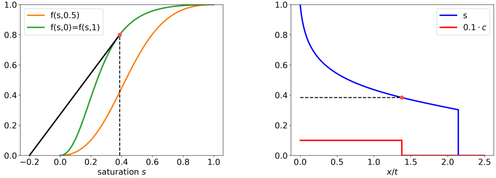


Figure 1: "boomerang" model: Lax admissible solution

Thus, we study c-shocks that have structure (vanishing viscocity travelling wave solutions).

Dissipative system

To define an admissible shock between u^- and u^+ we consider dissipative system:

$$s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x,$$
$$(cs+\alpha)_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx},$$

- ε_c — dimensionless capillary pressure coefficient
- ε_d — dimentionless diffusion coefficient
- $A(s,c)$ — capillary curve

$$\begin{pmatrix} s \\ sc + a(c) \end{pmatrix}_t + \begin{pmatrix} f(s,c) \\ cf(s,c) \end{pmatrix}_x = \varepsilon_c \left(B_\kappa(s,c) \begin{pmatrix} s \\ c \end{pmatrix}_x \right)_x,$$

$$B_\kappa(s,c) = \begin{pmatrix} A(s,c) & 0 \\ cA(s,c) & \kappa \end{pmatrix}, \qquad \kappa = \frac{\varepsilon_d}{\varepsilon_c}.$$

Restrictions on f and a

(F1) $f \in C^2([0,1]^2)$; $f(0,.) = 0$; $f(1,.) = 1$;

(F2) $f_s(s,c) > 0$ for $s \in (0,1)$, $c \in [0,1]$;
$f_s(0,c) = f_s(1,c) = 0$;

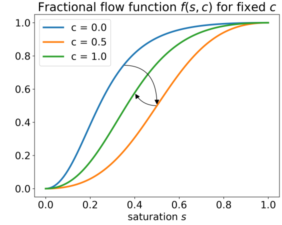
(F3) f is S-shaped in s;

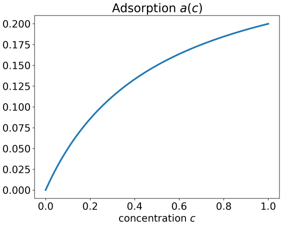
(F4) f is non-monotone in c:
$\forall s \in (0,1)\, \exists c^*(s) \in (0,1)$:

- $f_c(s,c) < 0$ for $0 < c < c^*(s)$;
- $f_c(s,c) > 0$ for $c^*(s) < c < 1$;

(A) A is bounded from zero and infinity;

(a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.





Travelling wave dynamical system

$$s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x,$$
$$(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}.$$

Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions $s(\pm\infty) = s^{\pm}$, $c(-\infty) = c^- = 1$, $c(+\infty) = c^+ = 0$, we arrive at

$$\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned} \tag{4}$$

- Here $\kappa = \dfrac{\varepsilon_d}{\varepsilon_c}$, $d_1 = \dfrac{a(c^-) - a(c^+)}{c^- - c^+} = a(1)$, $d_2 = \dfrac{c^+ a(c^-) - c^- a(c^+)}{c^- - c^+} = 0$;
- Note that $u^{\pm} = (s^{\pm}, c^{\pm})$ are fixed points of dyn.sys. (4);
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds in

$$(1,1) \to u^- \xrightarrow{\text{c-shock}} u^+ \to (0,0).$$

Main result

Consider dyn.sys. (4) under assumptions (F1)–(F4), (A), (a):

$$A(s,c)s_\xi = f(s,c) - v(s+d_1),$$
$$\kappa c_\xi = v(d_1 c - d_2 - a(c)).$$

Theorem (Bakharev, Enin, Petrova, R., 2021)

There exist $0 < v_{\min} < v_{\max} < \infty$*, such that for every* $\kappa = \varepsilon_d/\varepsilon_c \in (0,+\infty)$*, there exist unique*

- *points* $s^-(\kappa) \in [0,1]$ *and* $s^+(\kappa) \in [0,1]$*;*
- *velocity* $v(\kappa) \in [v_{\min}, v_{\max}]$*,*

such that there exists a travelling wave, connecting two saddle points $u^-(\kappa) = (s^-(\kappa), 1)$ *and* $u^+(\kappa) = (s^+(\kappa), 0)$ *with velocity* $v(\kappa)$*. Moreover,* $v(\kappa)$ *is monotone and continuous;* $v(\kappa) \to v_{\min}$ *as* $\kappa \to \infty$*;* $v(\kappa) \to v_{\max}$ *as* $\kappa \to 0$*.*

Remark

It is more convenient to prove the existence and corresponding properties of the inverse function $\kappa(v)$ instead.

Nullcline configurations

To study the solutions of

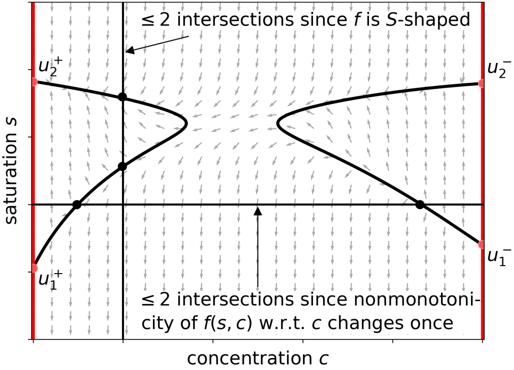
$$A(s,c)s_\xi = f(s,c) - v(s+d_1),$$
$$\kappa c_\xi = v(d_1 c - d_2 - a(c)),$$

we draw nullcline configurations in (s,c) plane:
red lines are

$$d_1 c - d_2 - a(c) = 0,$$

black lines are

$$f(s,c) - v(s+d_1) = 0.$$



Here u_1^+ and u_2^- are saddle points; u_2^+ and u_1^- are nodes.
Aim: Find pairs (κ, v) for which there is a trajectory $u_2^- \to u_1^+$.

Nullcline configurations. Classification

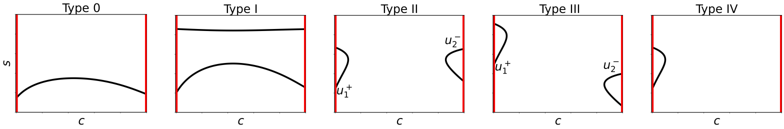


Figure 2: Five wide classes of nullcline configurations

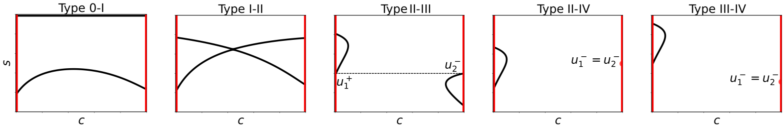


Figure 3: Intermediate types of nullcline configurations

- Only Type II nullcline configuration has saddle-to-saddle connections.
- Type I-II corresponds to $v_{\min}$.
- Type II-III or Type II-IV correspond to $v_{\max}$.

Type II: for every v there exist κ

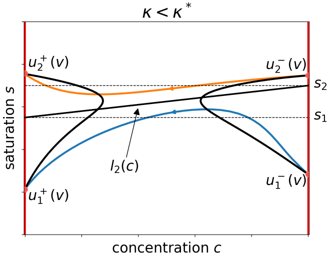


Figure 4: $\kappa << 1$.

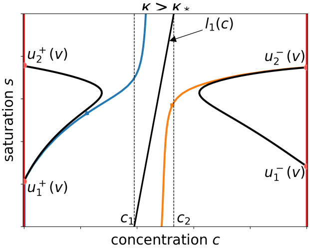
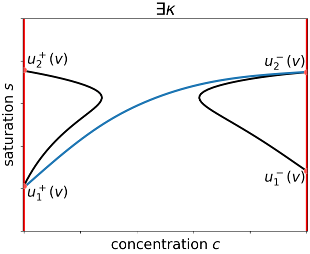


Figure 5: $\kappa >> 1$.

$$
\begin{aligned}
A(s,c)s_\xi &= f(s,c) - v(s + d_1),\\
\kappa c_\xi &= v(d_1 c - d_2 - a(c)),
\end{aligned}
$$

Used property: continuous and monotonous dependence of trajectories on κ.



Solution construction algorithm

1. From κ we calculate $v(\kappa)$ (binary search).
2. From v we determine $s^-(v)$ and $s^+(v)$ via Rankine-Hugoniot condition.
3. Construct waves $(1,1) \to (s^-(v),1)$ and $(s^+(v),0) \to (0,0)$.

Example: "boomerang" model:

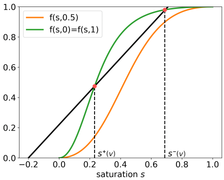


Figure 7: Flux functions

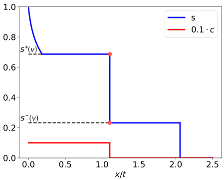


Figure 8: Solution s, c

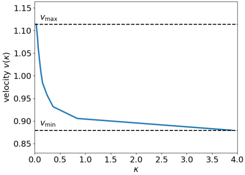


Figure 9: Function $v(\kappa)$

Possible directions for future research

- Additional dissipative forces;
- General classes of f and a;
- Construct solutions to any Riemann problem (in progress);
- Asymptotic stability as $t \to \infty$:
  Is it true that a solution of a Cauchy problem tends to a solution of a Riemann problem?

Thank you for your attention.