On branching random walks on periodic lattices

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2nd Conference of Mathematical Centers of Russia
Moscow, 7–11 November 2022

A random walk

We consider a random walk on $\mathbb{Z}^d$. If there is a particle in a point v at some time moment t, it can either go to another point $u \in \mathbb{Z}^d$ or remain at v over a short period of time δt.

a probability of the transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t)$$

a probability of the transition $v \to v$

$$p(v, v, \delta t) = 1 + a(v, v)\delta t + o(\delta t)$$

The transition intensity 1

The value $a(v,u)$ is called the transition intensity between v and u.

(i) $a(v,u) \geqslant 0, \quad v \neq u;$

(ii) $a(v,v) < 0;$

(iii) $\sum\limits_{u \in \mathbb{Z}^d} a(v,u) = 0;$

Let $g_1, \ldots, g_d$ be a family of linearly independent (not necessarily orthogonal) vectors with integer coordinates. By a lattice we mean a set

$$\Gamma = \Big\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^{d} n_j g_j, \; n_j \in \mathbb{Z}, \, j = 1, \ldots, d \Big\}.$$

(iv) $a(v,u) = a(u,v) = a(v+g, u+g), \quad \forall g \in \Gamma;$

(v) the graph $G = (\mathbb{Z}^d, \mathcal{E})$ with the vertex set $\mathbb{Z}^d$ and edge set

$$\mathcal{E} = \{(v,u) : \; a(v,u) > 0, \; v, u \in \mathbb{Z}^d\}$$

is connected.

The transition intensity 2

(vi) $\sum_{u \in \mathbb{Z}^d} \|u\|^2 |a(v,u)| < \infty, \; v \in \mathbb{Z}^d$;

Let $x = \{x_1, x_2, \ldots, x_d\} \in \mathbb{R}^d$. We denote by $\|x\|_\infty$ the following norm

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots |x_d|\}.$$

Let P be a transition matrix from the standard basis $\{e_1, \ldots, e_d\}$ to the basis $\{g_1, \ldots, g_d\}$, i.e. $g_j = Pe_j$.

(vi') There is $\alpha \in (0,2)$ such that

$$a(v_j + g, v_k) \|P^{-1} g\|_\infty^{d+\alpha} \to h_{jk} \text{ for } \|g\| \to +\infty,$$

where $h_{jk} \in [0, +\infty)$, for all $j, k = 1, \ldots, p$ and at least for one pair j, k corresponding h_{jk} is strictly positive.

(vii') $a(v, u - g) = a(v, u + g)$ for any $v, u \in \Omega$ and $g \in \Gamma$.

A branching source

Suppose there is a particle at a point with a branching source and it can't move anywhere from there. We assume that a particle can generate several descendants over a short period of time δt.

a probability of generating $k \neq 1$ descendants

$$p_k = b_k \delta t + o(\delta t).$$

a probability of generating $k = 1$ descendant

$$p_1 = 1 + b_1 \delta t + o(\delta t).$$

The branching intensity

(1) $b_k(v) \geqslant 0, \quad k \neq 1;$

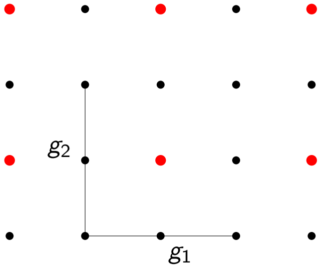
(2) $b_1(v) \leqslant 0;$

(3) $\sum\limits_{k=0}^{+\infty} b_k(v) = 0;$

(4) $\beta(v) = \sum\limits_{k=1}^{+\infty} k b_k(v) < \infty;$

(5) $\beta(v+g) = \beta(v), \quad g \in \Gamma.$

The branching sources in red vertices are "same"



A random walk with a periodic set of branching sources

Each particle located at a point $v \in \mathbb{Z}^d$ at time t can either move to a point $u \neq v$ or remain at the source and produce $k \neq 1$ descendants located at the point v (for $k = 0$ we assume that the number of descendants is 0; that is, the particle dies) or remain unchanged (that is, no changes occur) over a short period of time $[t; t + \delta t)$.

a probability of transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of generating $k \neq 1$ offsprings in v

$$p_k(v, \delta t) = b_k(v)\delta t + o(\delta t).$$

a probability of remaining unchanged

$$p(v, \delta t) = 1 + a(v, v)\delta t + b_1(v)\delta t + o(\delta t).$$

Finite number of branching sources

Yarovaya E., et. all [1998-2007]

One source. Asymptotic behaviour of all moments. Limit theorems.

Vatutin V., Topchii V. [2005]

Limit theorem for catalytic BRW on $\mathbb{Z}$ with one source of branching in super critical case.

Khristolyubov I., Yarovaya E. [2019]

N sources. Asymptotic behaviour of all moments in subcritical and supercritical cases. Limit theorems.

Smorodina N., Yarovaya E. [2022]

Compact perturbation. Martingale method for investigation of branching random walks.

Mean value of particles

By $M(v,u,t)$ we denote the mean number of particles at a point u at time t, provided that at the initial time $t=0$ there was one particle at a point v. The function $M(v,u,t)$ satisfies the following Cauchy problem:

The Cauchy problem

$$\begin{cases} M_t'(v,u,t) = \mathcal{A}M(v,u,t), \\ M(v,u,0) = \delta_u(v). \end{cases}$$

The operator $\mathcal{A}$

$$\mathcal{A} = \mathcal{A}_0 + Q,$$

$$(\mathcal{A}_0 f)(v) = \sum_{w\in\mathbb{Z}^d} a(v,w)f(w),$$

$$(Qf)(v) = \beta(v)f(v).$$

Properties of $\mathcal{A}$

The operator $\mathcal{A} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ satisfies the following properties:

- $\mathcal{A}$ is bounded;
- $\mathcal{A}$ is self-adjoint;
- $\mathcal{A}_0$ is non-positive;
- $\mathcal{A}$ is periodic with respect to the lattice Γ.

Connection with a discrete Laplacian

If condition (vi) is replaced with stronger condition that for any $v \in \mathbb{Z}^d$ there are only finite number of transition probabilities $a(v, u)$ are not zero, then operator $-\mathcal{A}_0$ is a discrete combinatorial Laplacian on the graph G defined in (v). In this case operator $-\mathcal{A}$ is a discrete Schödinger with periodic potential.

Asymptotic behaviour of $M(v, u, t)$

Theorem 1

Let a BRW satisfy conditions (i – vi) and (1 – 5). The function $M(v, u, t)$ has the following asymptotic behaviour as $t \to \infty$

$$M(v, u, t) = m(v, u) e^{t \sup \sigma(\mathcal{A})} t^{-\frac{d}{2}} (1 + o(1)).$$

Theorem 2

Let a BRW satisfy conditions (i – v), (vi' – vii') and (1 – 5). The function $M(v, u, t)$ has the following asymptotic behaviour as $t \to \infty$

$$M(v, u, t) = m(v, u) e^{t \sup \sigma(\mathcal{A})} t^{-\frac{d}{\alpha}} (1 + o(1)),$$

where $\alpha \in (0, 2)$ is defined in condition (vi') and $m(v, u)$ can be computed explicitly.

Notations

Fundamental vertex set

We can choose a set of vertices $\Omega = \{v_1, \ldots, v_p\}$ such that for any $u \in \mathbb{Z}^d$ there is an unique representation

$$u = \omega_u + \gamma_u, \quad \omega_u \in \Omega, \gamma_u \in \Gamma.$$

Dual basis

$$\langle \widetilde{g}_i, g_j \rangle = 2\pi\delta_{ij}.$$

Dual cell

$$\widetilde{\mathcal{C}} = \{\theta \in \mathbb{R}^d : \theta = \sum_{j=1}^{d} \theta_j \widetilde{g}_j, \ -1/2 \leqslant \theta_j < 1/2, \ j = 1, \ldots, d\}.$$

Direct integral decomposition

We define an operator $U : \ell^2(\mathbb{Z}^d) \to L^2(\widetilde{\mathcal{C}}, \mathbb{C}^p)$ by

$$(Uf)(v,\theta) = |\widetilde{\mathcal{C}}|^{-1/2} \sum_{g \in \Gamma} e^{-i\langle g,\theta\rangle} f(v+g), \quad v \in \Omega.$$

The operator $\mathcal{A}$ is unitary equivalent to the direct integral of matrices $A(\theta)$

$$U\mathcal{A}U^{-1} = \int_{\widetilde{\mathcal{C}}} \oplus A(\theta)\, d\theta.$$

It means that for every $f \in \ell^2(\mathbb{Z}^d)$

$$U\mathcal{A}f(v,\theta) = A(\theta)Uf(v,\theta).$$

$$A(\theta) = \begin{pmatrix} \widetilde{a}_{11}(\theta) + \beta_1 & \widetilde{a}_{12}(\theta) & \cdots & \widetilde{a}_{1p}(\theta) \\ \widetilde{a}_{21}(\theta) & \widetilde{a}_{22}(\theta) + \beta_2 & \cdots & \widetilde{a}_{2p}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{p1}(\theta) & \widetilde{a}_{p2}(\theta) & \cdots & \widetilde{a}_{pp}(\theta) + \beta_p \end{pmatrix},$$

where the functions $\widetilde{a}_{jk}(\theta)$ and constants β_j is defined by

$$\widetilde{a}_{jk}(\theta) = \sum_{g\in\Gamma} e^{-i\langle g,\theta\rangle} a(v_j + g, v_k), \qquad \beta_j = \beta(v_j).$$

Connection between $\mathcal{A}$ and $A(\theta)$

Let the eigenvalues of the matrix family $A(\theta)$ be ordered in non-increasing order for every parameter θ: $\lambda_1(\theta) \geqslant \ldots \geqslant \lambda_p(\theta)$.

$$\sigma(\mathcal{A}) = \bigcup_{j=1}^{p} \bigcup_{\theta\in\widetilde{\mathcal{C}}} \lambda_j(\theta).$$

Since $M(v,u,t)$ is a solution of the Cauchy problem

$$M(v,u,t) = e^{\mathcal{A}t}\delta_u(v) = \langle e^{\mathcal{A}t}\delta_u(\cdot), \delta_v(\cdot)\rangle_{\ell^2(\mathbb{Z}^d)}.$$

Since U is unitary

$$M(v,u,t) = \frac{1}{|\widetilde{\mathcal{C}}|}\int\limits_{\widetilde{\mathcal{C}}} \sum_{j=1}^{p} e^{\lambda_j(\theta)t}\, e^{i\langle\gamma_v-\gamma_u,\theta\rangle}\, \overline{\psi_j}(\omega_u,\theta)\, \psi_j(\omega_v,\theta)\, d\theta.$$

Properties of $\lambda_1(\theta)$

Theorem 3

For $\lambda_1(\theta)$ the following statements hold:

a) For all $\theta \in \widetilde{\mathcal{C}}$
$$\lambda_1(0) - \lambda_1(\theta) \geqslant 0.$$
The equality is achieved only for $\theta = 0$.

b) The distance between the right edge of the spectrum of $\mathcal{A}$ and the right edge of the second spectral band is positive, i.e.
$$\lambda_1(0) - \sup_{\theta \in \widetilde{\mathcal{C}}} \lambda_2(\theta) > 0.$$

c) $\lambda_1(0)$ is not an eigenvalue of $\mathcal{A}$.

d) If the conditions of the theorem 1 are valid, then the determinant of the Hessian matrix of $\lambda_1(\theta)$ does not vanish at $\theta = 0$, i.e.
$$\det \left\{ \frac{\partial^2 \lambda_1(\theta)}{\partial \theta^2} \bigg|_{\theta=0} \right\} \neq 0.$$

Key arguments to asymptotic 1

Theorem (Rytova A. and Yarovaya E. 2016)

Let

$$L(t) = \int\limits_{[-\pi,\pi]^d} f(x)e^{-tS(x)}\,dx,$$

where $f(\cdot)$, $S(\cdot)$ are continuous functions such that $f(0) \neq 0$, $S(x) > 0$ for all $x \neq 0$. Let the following functions be equivalent for $\|x\| \to 0$:

$$S(x) \sim \eta\left(\frac{x}{\|x\|}\right)\|x\|^{\alpha},$$

for some $\alpha > 0$ and some for positive and continuous function on the unit sphere $\eta(\cdot)$. Then there is such $C > 0$ that the following functions are equivalent for $t \to \infty$

$$L(t) \sim Cf(0)t^{-d/\alpha}.$$

Key arguments to asymptotic 2

Theorem (Kozyakin V. 2016)

Let $\alpha \in (0,2)$ and

$$F(\theta) = \sum_{z \in \mathbb{Z}^d \setminus \{0\}} a_z(1 - \cos\langle z, \theta\rangle), \quad \theta \in [-\pi, \pi]^d,$$

with a_z satisfying

$$a_z \|z\|_\infty^{d+\alpha} \to 1$$

for $\|z\| \to +\infty$. Then the following functions are equivalent for $\|\theta\| \to 0$

$$F(\theta) \sim \frac{2}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\|\theta\|^\alpha f\left(\frac{\theta}{\|\theta\|}\right),$$

where $f(\cdot)$ is positive continuous function on an unit sphere.

The moment of order n (supercritical case)

Let $M_n(v,u,t)$ be moment of order n for our BRW. Assume that additionally to (4) and (5) the following conditions are satisfied:

$(4')$ $\beta^{(n)}(v) = \sum\limits_{k=n}^{+\infty} k(k-1)\dots(k-n+1)b_k(v) < \infty;$

$(5')$ $\beta^{(k)}(v+g) = \beta^{(k)}(v), \quad g \in \Gamma, \quad l = k, \dots, n.$

The Cauchy problem for M_n

$$\begin{cases} \partial_t M_n(v,u,t) & = (\mathcal{A}M_n)(v,u,t) + R_n(v,u,t), \\ M_n(v,u,0) & = \delta_u(v), \end{cases}$$

where

$$\mathcal{A}f(v) = \sum_{u \in \mathbb{Z}^d} a(v,u)f(u) + \beta(v)f(v),$$

$$R_n(v,u,t) = \begin{cases} \sum\limits_{r=2}^{n} \frac{\beta^{(r)}(v)}{r!} \sum\limits_{\substack{i_1,\dots,i_r>0 \\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\dots i_r!} M_{i_1}(v,u,t)\dots M_{i_r}(v,u,t), & n \geq 2, \\ 0, & n = 1. \end{cases}$$

Asymptotic behaviour of $M_n(v, u, t)$

Theorem

Suppose that $\lambda_1(0) > 0$. Then the function $M_n(v, u, t)$ has the following asymptotic behaviour as $t \to \infty$

$$\ln M_n(v, u, t) = n\lambda_1(0)t - \frac{dn}{2} \ln t + O(1).$$

Thank you for your attention!