Дифференциальные преобразования характеристических функций и их свойства

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Notation

Let $V(\mathbb{R})$ be the set of left continuous functions of bounded variation on $\mathbb{R}$, $\mathfrak{F} \subset V(\mathbb{R})$ be the set of all d.f.'s on $\mathbb{R}$. For $F \in V(\mathbb{R})$ denote

$$\beta_r(F) := \int_{-\infty}^{\infty} |x|^r \, dV_{-\infty}^{x}(F), \quad r \geqslant 0,$$

$$\alpha_n(F) = \int_{-\infty}^{\infty} x^n \, dF(x), \quad n \in \mathbb{N},$$

$$f(t) = \int_{-\infty}^{\infty} e^{itx} \, dF(x), \quad t \in \mathbb{R}.$$

In particular,

$$\alpha_0(F) = \int_{-\infty}^{\infty} dF(x) = F(\infty) - F(-\infty) = f(0),$$

If $\beta_n(F) < \infty$, then $f \in \mathcal{C}^n(\mathbb{R})$ with

$$f^{(n)}(0) = i^n \alpha_n(F), \quad n \in \mathbb{N}.$$

For $F \in \mathfrak{F}$ let X be a r.v. with the d.f. F.

Definition. For $F \in V(\mathbb{R})$ with $\beta_n < \infty$, $\alpha_n \neq 0$ define **size-bias** as

$$F^{n\triangleright}(x) := \frac{1}{\alpha_n(F)} \int_{-\infty}^{x} y^n \, dF(y), \quad x \in \mathbb{R}.$$

For $n = 2m$, $F^{m\square} := F^{n\triangleright}$ is called **square-bias**.

For $F \in \mathfrak{F}$ with $F(0) = 0$ and $n = 1$ size-bias was introduced in (Lukacs, 1970) and studied in (Goldstein&Rinott, 1996), (Goldstein&Rinott, 2005), (Arratia, Goldstein, Kochman, 2013)

$n = 2$: (Lukacs, 1970), (Dreier, 1992), (Goldstein, 2007)

Example from (Arratia&Goldstein, 2021):
A fixed group of students eats in a canteen with fixed number of tables. If

X = # of students eating at a table selected at random, then
$X^{\triangleright}$ = # of students eating at the table where a student selected at random eats.

Indeed, $\mathrm{P}(X = k) = ck\mathrm{P}(X^{\triangleright} = k)$, where $c = 1/\,\mathrm{E}\,X$.

Size- and square-bias transform: properties

- **Iterated transform:** $F^{n\triangleright} = (F^{(n-1)\triangleright})^{\triangleright}$, if $\alpha_{n-1}, \alpha_n \neq 0$.
- **Ch.f.:** $f^{n\triangleright}(t) := \int_{-\infty}^{\infty} e^{itx} dF^{n\triangleright}(x) = \dfrac{f^{(n)}(t)}{f^{(n)}(0)} = \dfrac{f^{(n)}(t)}{i^n \alpha_n}$.
- **$\mathfrak{F}$-closure:** $F^{n\triangleright} \in \mathfrak{F} \;\Leftrightarrow\; \alpha_n (\operatorname{sign} x)^n F(x) \uparrow$ for all $x \in \mathbb{R} \setminus \{0\}$. In particular, if $F \in \mathfrak{F}$, then $F^{n\triangleright} \in \mathfrak{F} \;\Leftrightarrow n$ is even or n is odd and $X \overset{\text{п.н.}}{\geqslant} 0$ or $X \overset{\text{п.н.}}{\leqslant} 0$.
- **Integral equations on a set of test functions:** for every bounded g

$$\alpha_n(F) \int_{-\infty}^{\infty} g(x)\, dF^{n\triangleright}(x) = \int_{-\infty}^{\infty} x^n g(x)\, dF(x),$$

or, in terms of r.v.'s, if $X \geqslant 0$,

$$\boxed{\mathrm{E}\, X^n\, \mathrm{E}\, g(X^{n\triangleright}) = \mathrm{E}\, X^n g(X) \quad \forall g \text{ – bounded}}$$

(for $n = 1, 2$ see also (Goldstein&Rinott, 1996), (Goldstein, 2007)).

- **Relations:**
  $X^{\square} \overset{d}{=} (X^{\triangleright})^{\triangleright}$, $X^{n\square} \overset{d}{=} X^{(2n)\triangleright}$ ($n=1$: (Pekoz, Röllin, Ross, 2013)),
  $|X|^{\square} \overset{d}{=} \left|X^{\square}\right|$,
  $\left(X^2\right)^{\triangleright} \overset{d}{=} \left(X^{\square}\right)^2$,
  $(XY)^{n\triangleright} = X^{n\triangleright} Y^{n\triangleright}, \quad (XY)^{n\square} = X^{n\square} Y^{n\square}$,
- **Moments:**

$$\int_{-\infty}^{\infty} x^k \, dF^{n\triangleright}(x) = \frac{1}{\alpha_n} \int_{-\infty}^{\infty} x^{k+n} \, dF(x), \quad \text{or} \quad \mathsf{E}(X^{n\triangleright})^k = \frac{\mathsf{E}\, X^{n+k}}{\alpha_n}.$$

- **Homogeneity:** for every $c \neq 0$

$$(f(ct))^{n\triangleright} = f^{n\triangleright}(ct), \quad \text{or} \quad (cX)^{n\triangleright} \overset{d}{=} c \cdot X^{n\triangleright}.$$

- **Symmetry preserving:** if $f \in \mathbb{R}$, then $f^{n\triangleright} \in \mathbb{R}$, or, in terms of r.v.'s,

$$\text{if} \quad X \overset{d}{=} -X, \quad \text{then} \quad X^{m\square} \overset{d}{=} -X^{m\square}.$$

- **Monotonicity:** for $F \in \mathfrak{F}$ and odd $n \Rightarrow \mathrm{sign}(\alpha_n) \cdot (F - F^{n\triangleright}) \geqslant 0$.

In particular, if $X \overset{\text{п.н.}}{\geqslant} 0$ ($X \overset{\text{п.н.}}{\leqslant} 0$), then for every $n \in \mathbb{N}$

$$X^{n\triangleright} \geqslant_{\mathrm{st}} X \quad (X^{n\triangleright} \leqslant_{\mathrm{st}} X).$$

- **Fixed points:** $F^{n\triangleright} = F$ iff
  - for odd n: $F \in \mathfrak{F}$ and $\mathrm{P}(X = \sqrt[n]{\alpha_n}) = 1$.
  - for even n and $\alpha_n > 0$:

$$F(x) = pE_0(x - \sqrt[n]{\alpha_n}) + (1-p)E_0(x + \sqrt[n]{\alpha_n}), \quad p \in \mathbb{R}.$$

  Moreover, $F = F^{n\triangleright} \in \mathfrak{F} \;\Leftrightarrow\; p \in [0,1]$, i.e. $\mathrm{P}(|X| = \sqrt[n]{\alpha_n}) = 1$.
  - for even n and $\alpha_n < 0$ fixed points do not exist.
- **Single summand property:** for $F_1, \ldots, F_n \in V(\mathbb{R})$ with finite $a_k = \alpha_1(F_k) \neq 0$, $k = \overline{1,n}$ we have

$$(F_1 * \ldots * F_n)^{\triangleright} = \sum_{k=1}^{n} \frac{a_k}{A}(F_1 * \ldots * F_{k-1} * F_k^{\triangleright} * F_{k+1} * \ldots * F_n),$$

where $A = \sum_{j=1}^{n} a_j$. In particular, if $X_1, \ldots, X_n$ are i.i.d. nonnegative r.v.'s with $a_k = \mathrm{E}\, X_k \in (0,\infty)$ for all $k = \overline{1,n}$ and $S_n = X_1 + \ldots + X_n$, then

$$S_n^{\triangleright} \overset{d}{=} X_I^{\triangleright} + \sum_{k \neq I} X_k, \quad \text{where} \quad \mathrm{P}(I = k) = \frac{a_k}{A},\; k = \overline{1,n}.$$

and all the r.v.'s are independent.

Distance $L_1(X^{n\triangleright}, X)$

Recall that

$$L_1(X,Y) := \inf_{X' \overset{d}{=} X,\, Y' \overset{d}{=} Y} \mathsf{E}\,|X'-Y'| = \int_0^1 |F_X^{-1}(u) - F_Y^{-1}(u)|\mathrm{d}u$$

$$= \int_{-\infty}^{\infty} |F_X(t) - F_Y(t)|\mathrm{d}t =: \varkappa(F_X, F_Y)$$

$$= \sup_{f\in \mathrm{Lip}(1)} |\mathsf{E}\, f(X) - \mathsf{E}\, f(Y)| =: \zeta_1(X,Y)$$

- **Distance $L_1(X^{n\triangleright}, X)$:** if n is odd, or $X \overset{\text{п.н.}}{\geqslant} 0$, or $X \overset{\text{п.н.}}{\leqslant} 0$, then

$$L_1(X^{n\triangleright}, X) = \Big| \quad \Big| = \left| \frac{\alpha_{n+1}}{\alpha_n} - \alpha_1 \right|.$$

Distance $L_1(X^{\Box}, X)$

For $\rho \geqslant 1$ by X_ρ denote a two-point r.v. with

$$\mathsf{E}\,X_\rho = 0, \quad \mathsf{E}\,X_\rho^2 = 1, \quad \mathsf{E}\,|X_\rho|^3 = \rho, \quad \mathsf{E}\,X_\rho^3 \geqslant 0, \text{ i.e.}$$

$$\mathsf{P}\left(X_\rho = -\sqrt{\tfrac{p}{q}}\right) = 1 - p =: q, \quad \mathsf{P}\left(X_\rho = \sqrt{\tfrac{q}{p}}\right) = p \in (0,1).$$

Let

$$B(\rho) := \mathsf{E}\,X_\rho^3 = \sqrt{\frac{\rho}{2}\sqrt{\rho^2+8} + \frac{\rho^2}{2} - 2} < \rho, \quad \rho \geqslant 1.$$

Then $B(\rho)$ is concave and increasing with $B(1) = 0$, $\lim\limits_{\rho\to\infty} B(\rho)/\rho = 1$.

Lemma. (see (Shevtsova, 2013, DKM), (Shevtsova, 2014, JMAA)).
For every r.v. X with $\mathsf{E}\,X = 0$, $\mathsf{E}\,X^2 = 1$, and $\rho := \mathsf{E}\,|X|^3 < \infty$

$$\left|\mathsf{E}\,X^3\right| \leqslant B(\rho),$$

with equality attained at every two-point distribution of X.

Theorem. For every r.v. X with $\mathsf{E}\,X = 0$, $\mathsf{E}\,X^2 = 1$, $\rho := \mathsf{E}\,|X|^3 < \infty$

$$L_1(X, X^{\Box}) \leqslant B(\rho),$$

with equality attained at every two-point distribution of X.

(Korolev&Sh., 2010): if $\mathsf{E}\,|X| < \infty$ and $\mathsf{E}\,|Y| < \infty$, then

$$\left|\mathsf{E}\,e^{itX} - \mathsf{E}\,e^{itY}\right| \leqslant 2\sin\left(L_1(X,Y)\frac{|t|}{2} \wedge \frac{\pi}{2}\right), \quad t \in \mathbb{R}. \qquad (*)$$

Let X be a r.v. with ch.f. $f(t) := \mathsf{E}\,e^{itX}$ and

$$\mathsf{E}\,X = 0, \quad \mathsf{E}\,X^2 = 1, \quad \rho := \mathsf{E}\,|X|^3 < \infty.$$

The previous theorem yields the following result by $(*)$ with $Y = X^{\square}$.

Theorem. For all $t \in \mathbb{R}$

$$|f(t) + f''(t)| \leqslant 2\sin\frac{B(\rho)|t| \wedge \pi}{2}.$$

Applications: Bounds for ch.f.

For $t \in \mathbb{R}$ and X with $\mathsf{E}\,X = 0$, $\mathsf{E}\,X^2 = 1$, $\rho := \mathsf{E}\,|X|^3$ denote

$$r(t) := f(t) - e^{-t^2/2} = \mathsf{E}\left(e^{itX} - 1 - itX - \frac{(itX)^2}{2}\right) - \left(e^{-t^2/2} - 1 + \frac{t^2}{2}\right) =$$
$$= -i\frac{t^3\,\mathsf{E}\,X^3}{6} + \mathcal{O}(t^4), \quad t \to 0.$$

Recall that $B(\rho) = \mathsf{E}\,X_\rho^3 < \rho$.

- **Taylor's formula** $\Rightarrow |r(t)| \leqslant \dfrac{\rho|t|^3}{6} + \mathcal{O}(t^4)$.
- **(Mattner, Shevtsova, 2017)** $\Rightarrow$

$$|r(t)| \leqslant \frac{B(\rho)|t|^3}{6} + |\cos t - e^{-t^2/2}| =: R_{\mathrm{MS}}(t) = \frac{B(\rho)|t|^3}{6} + \mathcal{O}(t^4).$$

- **(Tyurin, 2009), (Korolev, Shevtsova, 2010)** $\Rightarrow$

$$|r(t)| \leqslant \int_0^{|t|} e^{(s^2-t^2)/2} \cdot 2s \cdot \sin\left(\frac{\rho s}{4} \wedge \frac{\pi}{2}\right) \mathrm{d}s =: R_{\mathrm{KS}}(t) = \frac{\rho|t|^3}{6} + \mathcal{O}(t^4).$$

Theorem. For all $t \in \mathbb{R}$

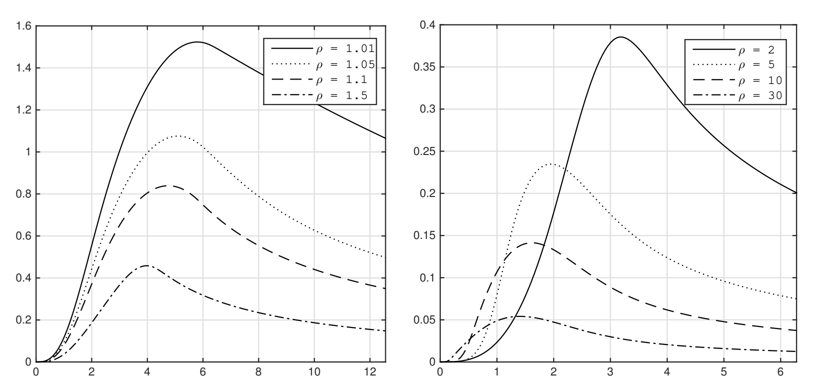
$$|r(t)| \leqslant \int_0^{|t|} e^{(u^2-t^2)/2}\,du \int_0^u 2\sin\frac{B(\rho)s \wedge \pi}{2}\mathrm{d}s + \mathcal{O}(t^4) =: R(t).$$

$$\widetilde{R}_{\mathrm{MS}}(t) := \frac{B(\rho)|t|^3}{6} = R_{\mathrm{MS}}(t) + \mathcal{O}(t^4),$$

$$R_{\mathrm{KS}}(t) := \int_0^{|t|} e^{(s^2-t^2)/2} \cdot 2s \cdot \sin\left(\frac{\rho s}{4} \wedge \frac{\pi}{2}\right)\mathrm{d}s,$$

$$\widetilde{R}(t) := \int_0^{|t|} e^{(u^2-t^2)/2}\,du \int_0^u 2\sin\frac{B(\rho)s \wedge \pi}{2}\mathrm{d}s = R(t) + \mathcal{O}(t^4) < \widetilde{R}_{\mathrm{MS}}(t).$$

Plots of the difference $R_{\mathrm{KS}}(t) - \widetilde{R}(t)$ are below:

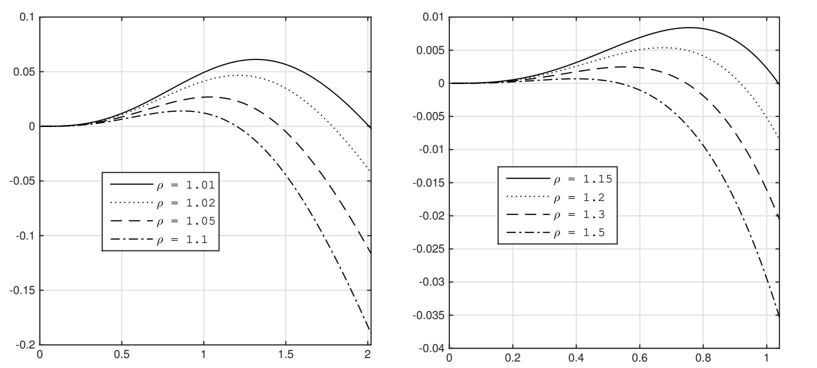


$$R_{\mathrm{MS}}(t) = \frac{B(\rho)|t|^3}{6} + |\cos t - e^{-t^2/2}|,$$

$$R_{\mathrm{KS}}(t) = \int_0^{|t|} e^{(s^2-t^2)/2} \cdot 2s \cdot \sin\left(\frac{\rho s}{4} \wedge \frac{\pi}{2}\right) \mathrm{d}s,$$

$$R(t) = \int_0^{|t|} e^{(u^2-t^2)/2}\, du \int_0^u 2\sin \frac{B(\rho)s \wedge \pi}{2} \mathrm{d}s + \mathcal{O}(t^4) < R_{\mathrm{MS}}(t).$$

Plots of the difference $R_{\mathrm{KS}}(t) - R(t)$ are below:



Equilibrium transform

Definition. (Well-known). Let $X \overset{\text{п.н.}}{\geqslant} 0$ with $\alpha_1 = \mathsf{E}\, X \in (0, \infty)$ and $F(x) = \mathsf{P}(X < x)$. Then a r.v. X^e is said to have an **equilibrium biased** (or **stationary renewal**, or **integrated tail**) distribution if it's d.f. is given by

$$F^e(x) = \frac{1}{\alpha_1} \int_0^x (1 - F(y))dy \cdot \mathbf{1}(x > 0),$$

$$p^e(x) := (F^e(x))' = \alpha_1^{-1}(1 - F(x))\mathbf{1}(x > 0).$$

The transform $F \mapsto F^e$ is also called **stationary excess operator**.

The d.f. F^e originally arised in the renewal theory as the distribution of the **initial delay** of a renewal process which makes its **renewal rate constant** (Feller, 1970, Vol. 2, Chapter 11, § 4) and, more generally, the **renewal process stationary** (Asmussen, 2003, Chapter 5, § 3). The distribution F^e appear also as the limit distribution of both **forward and backward recurrence times** in a renewal process with interarrival d.f. F (Feller, 1970) and in the celebrated **Pollaczek–Khinchin–Beekman** formula which expresses the ruin probability in the classical risk process in terms of geometric random sum of i.i.d. r.v.'s whose common d.f. is the equilibrium transform of the common distribution of claims. It is also used in **Stein' method** with limiting **exponential** distribution (Pěkoz&Röllin, 2011).

Iterated equilibrium transform: $n = 2$

For $n \in \mathbb{N}$ let $X^{ne} := (X^e)^{(n-1)e}$ with $\alpha_n := \mathsf{E}\, X^n \in (0, \infty)$ have the d.f.

$$F^{ne}(x) := (F^e)^{(n-1)e}(x) = \left(1 - \frac{n}{\alpha_n} \int_x^\infty (y - x)^n dF(y)\right) \mathbf{1}(x > 0).$$

For even n the assumption $X \overset{\text{п.н.}}{\geqslant} 0$ may be omitted.

The case $n = 2$ was called the **centered equilibrium** distribution in (Pike&Ren, 2014), where it was re-introduced in non-explicit form under the additional assumption that $\boxed{\mathsf{E}\, X = 0}$ and used in **Stein's method** with limiting **Laplace** distribution.

Under the assumption $\boxed{X \overset{\text{п.н.}}{\geqslant} 0}$ for all n:

• **(Massey&Whitt, 1993), (Lin, 1994):** For any $g \in \mathcal{C}^{n-1}(\mathbb{R})$ with Borel $g^{(n)}$ s.t. $\mathrm{E}\,|g^{(n)}(X^{ne})| < \infty$ we have

$$\mathrm{E}\,g(X) = \sum_{k=0}^{n-1} \frac{\mathrm{E}\,X^k}{k!} g^{(k)}(0) + \frac{\mathrm{E}\,X^n}{n!}\,\mathrm{E}\,g^{(n)}(X^{ne}).$$

• Starting from the works of **Shantaram and Harkness** (1969, 1972, 1975) many authors investigated limit theorems for $c_n X^{ne}$, described limit laws (mixture of exponential with log-normal) and considered arising moment problems: (**van Beek&Braat**, 1973), (**Huang&Lin**, 1995), (**Pakes**, 2007), (**López-García**, 2009), (**Jedidi, Harthi, Bouzaffour**, 2020) and distributional characterizations through scaling relations (**Pakes**, 1996), (**Pakes&Navarro**, 2007), (**López-García**, 2009).

Iterated equilibrium: hints to generalization

The ch.f. of X^e with the density $p^e(x) = \alpha_1^{-1}(1 - F(x))\mathbf{1}(x > 0)$ and ch.f. f is

$$f^e(t) = \mathsf{E}\, e^{itX} = \frac{1}{\alpha_1}\int_0^\infty e^{itx}(1 - F(x))dx = \frac{f(t) - 1}{it\alpha_1} = \frac{f(t) - 1}{tf'(0)}.$$

Iterating yields

$$f^{ne}(t) = \frac{f(t) - 1 - \ldots - f^{(n-1)}(0)t^{n-1}/(n-1)!}{f^{(n)}(0)t^n/n!}. \qquad (*)$$

In particular, for $X \overset{\text{п.н.}}{=} 1$ we have $f(t) = e^{it}$, $f^{(n)}(0) = i^n$, and

$$f^{ne}(t) = \frac{f(t) - 1 - \ldots - (it)^{n-1}/(n-1)!}{(it)^n/n!} \sim B(1, n)$$

with the density

$$1^{ne} \sim n(1 - x)^{n-1}\mathbf{1}(0 < x < 1), \quad 1^e =: U \sim U(0, 1)$$

We take $(*)$ as the definition of iterated equilibrium transform of arbitrary $F \in V(\mathbb{R})$ with the Fourier–Stieltjes transform f.

Generalized equilibrium transform, case $n = 1$

For $F \in V(\mathbb{R})$ with finite $\alpha_1 = \int x dF(x) \neq 0$ we have:

- **Ch.f.:**

$$f^e(t) := \mathrm{E}\, e^{itX^e} = \frac{f(t) - 1}{tf'(0)} = \frac{f(t) - 1}{it\alpha_1}.$$

- **Density and d.f.:**

$$p^e(x) = -\tfrac{1}{\alpha_1}(F(x) - F(-\infty))\mathbf{1}(x < 0) + \tfrac{1}{\alpha_1}(F(\infty) - F(x))\mathbf{1}(x > 0),$$

$$F^e(x) = \begin{cases} -\dfrac{1}{\alpha_1}\displaystyle\int_{-\infty}^{x}(F(y) - F(-\infty))dy, & x \leqslant 0, \\ 1 - \dfrac{1}{\alpha_1}\displaystyle\int_{x}^{\infty}(F(\infty) - F(y))dy, & x > 0, \end{cases}$$

- **Fixed points:** $f^e(t) = f(t)$ iff

$$f(t) = \frac{1}{1 - it\alpha_1} \quad \Leftrightarrow \quad \boxed{F \sim exp(1/\alpha_1)}$$

$n = 1$: interesting properties

- **Connection with $F^{\triangleright}$:**

$$F^e(x) = \int_{-\infty}^{\infty} \mathsf{P}(U \cdot y < x)dF^{\triangleright}(y),$$

in particular,

$$X^e \stackrel{d}{=} UX^{\triangleright} \stackrel{d}{=} 1^e X^{\triangleright}, \quad U \perp X^{\triangleright}.$$

- **Single factor property:** $(X_1 X_2 \cdots X_n)^e \stackrel{d}{=} X_1^e X_2^{\triangleright} \cdots X_n^{\triangleright}$,
- **Closeness of X^e and X** (Sh.&Tselischev, 2020): for $F \in \mathfrak{F}$ we have

$$\zeta_1(F, F^e) \leqslant \frac{1}{2} \cdot \frac{\alpha_2}{|\alpha_1|} - |\alpha_1|(1 - F(0)).$$

- **ζ-metrics** (Kalashnikov, 1997, p.37): for r.v.'s $X \overset{\text{п.н.}}{\geqslant} 0$, $Y \overset{\text{п.н.}}{\geqslant} 0$ with finite second moments and $\mathsf{E}\, X = \mathsf{E}\, Y = a > 0$

$$\zeta_2(X, Y) = a\zeta_1(X^e, Y^e).$$

- **Test functions** (Pekoz, Röllin, Ross, 2011): for $X \overset{a.s.}{\geqslant} 0$

$$\boxed{\mathsf{E}\, X\, \mathsf{E}\, g'(X^e) = \mathsf{E}\, g(X) - g(0) \quad \forall g \in \mathrm{Lip}}$$

Generalized centered equilibrium transform ($n = 2$)

For $F \in V(\mathbb{R})$ with finite $\alpha_2 = \int x^2 dF(x) \neq 0$ we have:

- **Ch.f.:**

$$f^L(t) := \mathsf{E}\, e^{itX^L} = \frac{f(t) - 1 - tf'(0)}{f''(0)t^2/2} = \frac{f(t) - 1 - it\alpha_1}{-\alpha_2 t^2/2} \qquad \text{(Lukacs, 1970)}$$

- **Density:**

$$p^L(x) = \frac{2}{\alpha_2}\left[\int_{-\infty}^{x} (x - y)\, dF(y)\, \mathbf{1}(x \leqslant 0) + \int_{x}^{\infty} (y - x)\, dF(y)\, \mathbf{1}(x > 0)\right].$$

- **Fixed points:** $f^L(t) = f(t) \;\Leftrightarrow\; f(t) = \dfrac{1 - it\alpha_1}{1 + \alpha_2 t^2/2} \;\Leftrightarrow$

$$F'(x) = \frac{1}{\sqrt{2\alpha_2}} e^{-|x|\sqrt{2/\alpha_2}}\left(1 - \operatorname{sign} x \cdot \sqrt{2}\frac{\alpha_1}{\sqrt{\alpha_2}}\right)$$

In particular, $F' \geqslant 0 \;\Leftrightarrow\; 2\alpha_1^2 \leqslant \alpha_2$. Moreover, if $\alpha_1 = 0$, then

$$F^L = F \quad \Leftrightarrow \quad F'(x) = \frac{1}{\sqrt{2\alpha_2}} e^{-|x|\sqrt{2/\alpha_2}} \sim L(\sqrt{2/\alpha_2}),$$

while if $\alpha_1 = \sqrt{\alpha_2/2}$, then $X \sim exp(1/\alpha_1)$.

$n = 2$: interesting properties

- **Connection with $F^{\square}$:**

$$F^L(x) = \int_{-\infty}^{\infty} \mathrm{P}(1^{2e} \cdot y < x) dF^{\square}(y),$$

in particular, if the r.v.'s below are well-defined (for $\mathrm{E}\, X = 0$ see also (Pike&Ren, 2014)),

$$X^L \stackrel{d}{=} 1^{2e} X^{\square} \stackrel{d}{=} 1^{2e} X^{2\triangleright} \stackrel{d}{=} X^{2e}, \quad X^{\square} \perp 1^{2e} \sim B(1,2).$$

- **Single factor property:** $(X_1 X_2 \cdots X_n)^L \stackrel{d}{=} {X_1}^L X_2^{\square} \cdots X_n^{\square}$,
- **Test functions**:

$$\boxed{\mathrm{E}\, X^2\, \mathrm{E}\, g''(X^L) = 2\, \mathrm{E}(g(X) - g(0) - Xg'(0)) \;\; \forall g \in \mathrm{Lip}} \qquad (*)$$

(for $\mathrm{E}\, X = 0$ see also (Pike&Ren, 2014)).

- **Test functions**:

$$\mathsf{E}\, X^2\, \mathsf{E}\, g''(X^L) = 2\, \mathsf{E}(g(X) - g(0) - Xg'(0)) \ \ \forall g \in \mathrm{Lip} \quad \Leftrightarrow \quad (*)$$

(Döbler,2015), Prop. 5 on p. 12: $\forall X:\ \mathsf{E}\, X^2 \in (0, \infty)$, $\forall a \in \mathbb{R}\ \exists!\ \hat{X}_a$:

$$\mathsf{E}(X-a)^2\, \mathsf{E}\, g''(\hat{X}_a) = 2\, \mathsf{E}(g(X) - g(a) - g'(a)(X-a))$$

for all $f \in \mathcal{C}^{(1)}(\mathbb{R})$ with $f' \in \mathrm{Lip}$; moreover, $\hat{X}_a$ is a.c. This means that

$$\hat{X}_a \stackrel{d}{=} (X-a)^L + a$$

$\forall a \in \mathbb{R}$ if X^L is defined for $\forall\, \mathsf{E}\, X$ and with $a = \mathsf{E}\, X$ only if X^L is defined for $\mathsf{E}\, X = 0$.

Zero-bias transformation

- **Ch.f.:** For a r.v. X with $\alpha_2 := \mathrm{E}\,X^2 \in (0,\infty)$, we set X-**zero-biased** r.v.

$$X^* :\overset{d}{=} UX^{\square}, \quad U \perp X^{\square}, \quad U \sim \mathcal{U}(0,1),$$

with the ch.f.

$$f^*(t) := \int_{-\infty}^{\infty} \frac{e^{itx}-1}{itx}\,dF^{\square}(x) = \frac{f'(t)-f'(0)}{tf''(0)} = \mathrm{E}\,e^{itX^*}. \qquad \text{(Lukacs, 1970)}$$

- **Density:** (Goldstein&Reinert, 1997)

$$\frac{d\mathrm{P}(X^*<t)}{dt} = \int_{-\infty}^{\infty} \frac{d}{dt}\mathrm{P}(U\cdot x<t)\,dF^{\square}(x) = \begin{cases} \mathrm{E}\,|X|\mathbf{1}(X<t)/\alpha_2, & t<0, \\ \mathrm{E}\,X\mathbf{1}(X>t)/\alpha_2, & t\geqslant 0. \end{cases}$$

- **Test functions:** (Goldstein&Reinert, 1997), (Goldstein, 2007) for $\mathrm{E}\,X=0$, (Döbler, 2013)

$$\boxed{\mathrm{E}\,X^2\,\mathrm{E}\,g'(X^*) = \mathrm{E}\,X(g(X)-g(0)) \quad \forall g\colon\ \mathrm{E}\,g'(X^*)\exists}$$

see also (Chen, Goldstein, Shao, 2011), (Ross, 2011).

- **Moments.** Take $g(x)=x^k$ and $g(x)=|x|^k$, then

$$\mathrm{E}\,X^2\,\mathrm{E}(X^*)^n = \mathrm{E}\,X^{n+2}/(n+1), \quad \mathrm{E}\,X^2\,\mathrm{E}\,|X^*|^r = \mathrm{E}\,|X|^{r+2}/(r+1).$$

Closeness of X^* and X

- **L_1 distance between X^* and X:** if $\mathsf{E}\,X = 0$, $\mathsf{E}\,X^2 = 1$, then

$$L_1(X^*, X) \leqslant 0.5 \cdot \mathsf{E}\,|X|^3 \quad \textbf{(Tyurin, 2009)}$$

- **Fixed points:** take any left-continuous $F \in V(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} x^2 dF(x) = 1, \quad \int_{-\infty}^{\infty} x dF(x) = a \in (-1, 1),$$

then $f^*(t) = f(t) \quad \Leftrightarrow \quad \dfrac{f'(t) - f'(0)}{tf''(0)} = f(t) \quad \Leftrightarrow$

$$f(t) = e^{-\alpha_2 t^2/2} + i\alpha_1 e^{-\alpha_2 t^2/2} \int_0^t e^{-\alpha_2 u^2/2} du \quad \Leftrightarrow$$

$$p(x) := \frac{d}{dx} \mathsf{P}(X < x) = \left(1 + \sqrt{\tfrac{\pi}{2}}\, a \operatorname{sign} x\right) \varphi(x),$$

moreover, $p(x) \geqslant 0$ iff $a^2 \leqslant 2\pi^{-1} = 0.6366\ldots$ In the latter case

$$X \stackrel{d}{=} Y \cdot |Z|, \qquad \text{where}$$

$$Y \perp Z, \quad Z \sim \mathcal{N}(0, 1), \quad \mathsf{P}(Y = 1) = \frac{1}{2} + \frac{a}{2}\sqrt{\frac{\pi}{2}} = 1 - \mathsf{P}(Y = -1).$$

- **Single factor property:** $(X_1 X_2 \cdots X_n)^* \overset{d}{=} X_1^* X_2^{\square} \cdots X_n^{\square}$

- $X^* :\overset{d}{=} UX^{\square} \overset{d}{=} (X^{\triangleright})^e$

Thank you!