Tropical Somos sequences and billiards

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Somos–(4)

Somos–(4) sequence $\{s_n\}$ is defined by initial data

$$s_1 = s_2 = s_3 = s_4 = 1$$

and recurrence relation

$$s_{n+2}s_{n-2} = s_{n+1}s_{n-1} + s_n^2.$$

It begins with

$$\dots, 2, 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \dots$$

(Obviously $s_n = s_{5-n}$.)

Somos–(6)

Somos first introduced the sequence Somos-(6) such that

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 1$$

and

$$s_{n+3}s_{n-3} = s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

He raised the question whether all the terms are integer:

1470*. *Proposed by Michael Somos, Cleveland, Ohio.*

Consider the sequence (a_n) where $a_0 = a_1 = \cdots = a_5 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2}{a_{n-6}}$$

for $n \geq 6$. Computer calculations show that $a_6, a_7, \ldots, a_{100}$ are all integers. Consequently it is conjectured that all the a_n are integers. Prove or disprove.

Somos M. Problem 1470. Crux Mathematicorum, 15: 7 (1989), p. 208.

Integrality

The integrality of Somos–(4) and Somos–(5) was proved by Janice Malouf, Enrico Bombieri and Dean Hickerson (1990).

The integrality of Somos–(6) was proved by Dean Hickerson (April 1990).

The integrality of Somos–(7) was proved by Ben Lotto (May 1990).

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Somos–(8) is a sequence with initial data

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7 = s_8 = 1$$

satisfying recurrence relation

$$s_{n+4}s_{n-4} = s_{n+3}s_{n-3} + s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

Somos–(8) is NOT an integer sequence:

$$\ldots, 1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61, 187, 775, 5827, 14815, \frac{420514}{7}, \ldots$$

Somos–(8) is a wild object. Probably it has no properties at all.

Somos sequences

Definition

For integer $k \geq 4$ Somos–k sequence is a sequence generated by quadratic recurrence relation of the form

$$s_{n+k}s_n = \sum_{j=1}^{[k/2]} \alpha_j s_{n+k-j}s_{n+j},$$

where α_j are constants and $s_0, \ldots, s_{k-1}$ are initial data.

In particular Somos-4 is defined by initial data s_0, s_1, s_2, s_3 and fourth-order recurrence

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2,$$

Somos-6 is defined by initial data $s_0, \ldots, s_5$ and sixth-order recurrence

$$s_{n+3}s_{n-3} = \alpha s_{n+2}s_{n-2} + \beta s_{n+1}s_{n-1} + \gamma s_n^2.$$

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A sigma-function solution for Somos–4

The integrality of Somos–(4)$\cdots$(7) may be proven by elementary methods. But elementary proofs don't spread any light on the nature of Somos sequences: there is some elliptic curve hidden behind Somos–4–5 and hyperelliptic curve of genus 2 behind Somos–6–7.

The general solution of Somos–4 recurrence relation is given by C. Swart (2003) and A. Hone (2005)

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where $z, z_0 \in \mathbb{C}^*$, and

$$\sigma(z) = \sigma_\Gamma(z) = z \prod_{w \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2}$$

is Weierstrass sigma-function associated to plane lattice Γ.

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Integrality condition

Theorem (Fomin and Zelevinsky, 2002)

For a Somos–k sequences ($k = 4, 5, 6$ and 7) all of the terms in the sequences are Laurent polynomials in these initial data whose coefficients are in $\mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}]$, so that

$$s_n \in \mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}, s_1^{\pm 1}, \dots, s_k^{\pm 1}] \text{ for all } n \in \mathbb{Z}.$$

Fomin S. and Zelevinsky A. "The Laurent Phenomenon", Adv. Appl. Math. 28 (2002) 119–144.

Integrality of original Somos–(k) sequences follows from the theorem with

$$\alpha_1 = \dots = \alpha_{[k/2]} = s_1 = \dots = s_k = 1.$$

But this Theorem is not a final step.

Theorem (Bykovskii, AU, 2019)

Let Somos-4 be defined by

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2,$$

where

$$\alpha = s_0 s_1 s_2 s_3 a, \qquad \beta = s_0 s_1 s_2 s_3 b$$

and s_0, s_1, s_2, s_3, a, b *are independent formal variables. Then*

$$s_n \in \mathbb{Z}[s_0, s_1, s_2, s_3, a, b].$$

Similar results can be proven for Somos-5-6-7 sequences as well.

Polynomial Somos sequences

Another example of polynomial Somos sequences is **division polynomials** associated to the curve $y^2 = x^3 + Ax + B$:

$$\psi_0 = 0, \qquad \psi_1 = 1, \qquad \psi_2 = 2y$$
$$\psi_3 = 3x^4 + 6Ax^2 + 12Bx - A^2$$
$$\psi_4 = 4y\left(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3\right)$$
$$\dots$$
$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \text{ for } m \geq 2$$
$$\psi_{2m} = \left(\frac{\psi_m}{2y}\right) \cdot \left(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2\right) \text{ for } m \geq 3$$

They satisfy the Somos-4 equation

$$\psi_{n+2}\psi_{n-2} = \alpha\psi_{n+1}\psi_{n-1} + \beta\psi_n^2$$

with $\alpha = \psi_2^2$ and $\beta = -\psi_3$.
Similar polynomials are known for higher genus curves (Cantor, 1994).

Tropical arithmetic

Our basic object is the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. As a set this is just the real numbers $\mathbb{R}$, together with an extra element ∞ that represents infinity. The basic arithmetic operations of addition and multiplication of real numbers are redefined as follows:

$$m \oplus n := \max\{m, n\}, \qquad m \odot n := m + n.$$

The *tropicalization* of a mathematical formula is the formal procedure s.t.

$$"+" \to "\oplus" \qquad \text{and} \qquad "\times" \to "\odot".$$

Tropical subtraction is NOT defined.

Example. If $P(x)$ and $Q(x)$ are polynomials with non-negative coefficients then

$$\deg(P + Q) = \max\{\deg P, \deg Q\}, \quad \text{and} \quad \deg(P \cdot Q) = \deg P + \deg Q$$

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Tropicalization (something reasonable about polynomials) keeps information about degrees and loses all other information.

One of the greatest advantage of Tropical Geometry is that most classical problems become much simpler after their tropicalization (if such a tropicalization exists!).

This simplicity comes from the piecewise-linear nature of the tropical objects.

Tropical arithmetic

Non-trivial example taken from the lecture ”Growth and geometry in SL(2;Z) dynamics” given by A. Veselov

$$x^2 + y^2 + z^2 = 3xyz \to \max\{2x, 2y, 2z\} = x + y + z.$$

If $x \leq y \leq z$ then we have the equation $x + y = z$ corresponding to Euclidean algorithm. Markoff triples

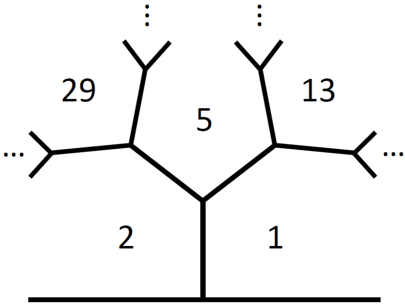
$$(1, 1, 1) \to (1, 1, 2) \to (1, 2, 5) \to \dots$$

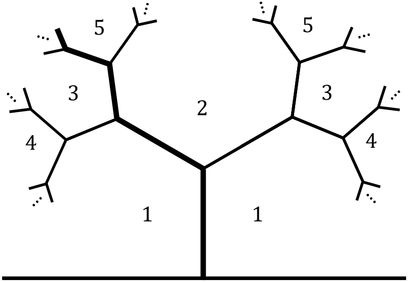
are generated by Vieta’s involution

$$\tau : (x, y, z) \to (x, y, 3xy - z)$$

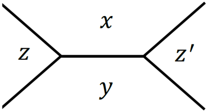
and permutations.

Markoff and Euclid trees

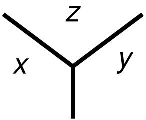




The rules are:



$z + z' = 3xy$



$z = x + y$

How can we go back from Euclid tree ($z = x + y$) to Markoff tree ($z' = 3xy - z$)? We need "detropicalization"="quantization":

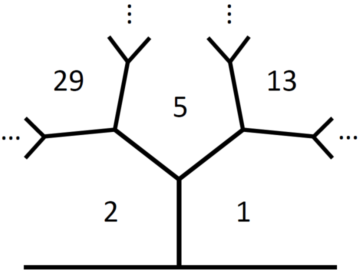
$$x + y = z \quad \to \quad AB = C$$

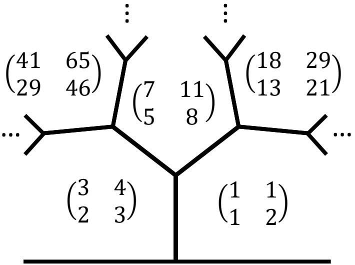
where A, B, C are already noncommutative objects, $A, B, C \in SL_2(\mathbb{Z})$. The correct choice of initial conditions is

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \langle A, B \rangle = SL_2'(\mathbb{Z}),$$

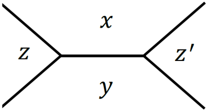
i.e. A, B are generators of commutator subgroup of $SL_2(\mathbb{Z})$.

Markoff and H. Cohn trees

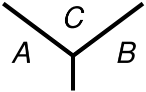




The rules are: $m(A) = \frac{1}{3}\mathrm{tr}(A)$,



$z + z' = 3xy$



$C = AB$

Tropical Somos sequences

Tropicalization of Somos-4 equation

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2$$

is

$$d_{n+2} + d_{n-2} = \min\{\tilde{\alpha} + d_{n+1} + d_{n-1}, \tilde{\beta} + 2d_n\}.$$

For

$$\alpha = s_0 s_1 s_2 s_3 a, \qquad \beta = s_0 s_1 s_2 s_3 b$$

we know that $s_n \in \mathbb{Z}[s_0, s_1, s_2, s_3, a, b]$. So we can say that d_n is a degree of $s_n(s_0, s_1, s_2, s_3)$ as a polynomial of s_0. In this case $\deg \alpha = \deg \beta = 1$ and we get the equation

$$d_{n+2} + d_{n-2} = 1 + \min\{d_{n+1} + d_{n-1}, 2d_n\}.$$

Tropicalization is well defined because Somos polynomials satisfy "positivity conjecture" (their coefficients are non-negative).

General solution of Somos-4 recurrence is

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where σ is the Weierstrass sigma function associated to the curve $y^2 = 4x^3 - g_2 x - g_3$.
So

$$s_n = e^{an^2+bn+c} \times \text{(some oscillating function)}.$$

Gauge transformations

Somos-4 sequences

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2$$

are invariant under the two-parameter abelian group of gauge transformations defined by

$$s_n \to \widetilde{s}_n = A \cdot B^n \cdot s_n.$$

Thus it is natural to introduce the gauge-invariant variables

$$f_n = \frac{s_{n-1}s_{n+1}}{s_n^2}$$

satsfying

$$f_{n+1}f_n^2 f_{n-1} = \alpha f_n + \beta.$$

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A general solution of the equation

$$f_{n+1}f_n^2 f_{n-1} = \alpha f_n + \beta$$

has the form [Hone, 2005; Swart, 2003]

$$f_n = \wp(z) - \wp(z_0 + nz),$$

where $\wp$ is the Weierstrass elliptic function associated to the curve $y^2 = 4x^3 - g_2 x - g_3$.

The tropical case

A tropical analogue of substitution $f_n = \frac{s_{n-1}s_{n+1}}{s_n^2}$ is

$$a_n = d_{n+1} - 2d_n + d_{n-1}.$$

It transforms tropical Somos-4 recurrence

$$d_{n+2} + d_{n-2} = 1 + \min\{d_{n+1} + d_{n-1}, 2d_n\}$$

into the equation

$$a_{n-1} + 2a_n + a_{n+1} = 1 + \min\{a_n, 0\}$$

which is tropical analogue of

$$f_{n+1}f_n^2 f_{n-1} = \alpha f_n + \beta.$$

Theorem

Let the sequence $\{a_n\}$ *be defined by the recurrence relation*

$$a_{n-1} + 2a_n + a_{n+1} = 1 + \min\{a_n, 0\}.$$

Then it is periodic with the period $5 - 8T$*, where*

$$T = \min\{a_0 + a_1, 1 - a_0, 1 - a_1, 1 - a_0 - a_1\}.$$

In particular it means that for original sequence $\{d_n\}$ we have

$$d_n = An^2 + Bn + ((5 - 8T)\text{-periodic function}).$$

Proof.

The proof= Tropicalization (proof of Swart – Hone theorem written without substractions). □

For the equation

$$f_{n+1}f_n^2f_{n-1} = \alpha f_n + \beta$$

we know the invariant (first integral)

$$T = f_nf_{n-1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n-1}}\right) + \frac{\beta}{f_nf_{n-1}}.$$

In particular this equation defines an elliptic curve in the (f_n, f_{n-1}) plane.

This observation admits direct tropicalization: the equation

$$a_{n-1} + 2a_n + a_{n+1} = 1 + \min\{a_n, 0\}$$

has the invariant

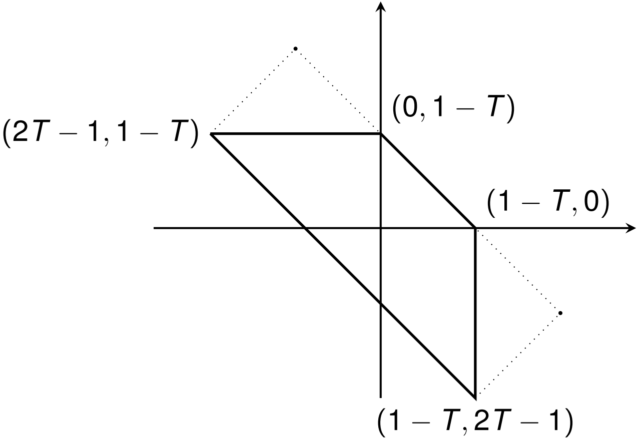
$$T = \min\{a_n + a_{n+1}, 1 - a_n, 1 - a_{n+1}, 1 - a_n - a_{n+1}\}.$$

This equation defines some line in the (a_n, a_{n+1}) plane.

The equation

$$\min\{1-x, 1-y, x+y, 1-x-y\} = T$$

defines a boarder of trapezoid in the (x, y) plane:

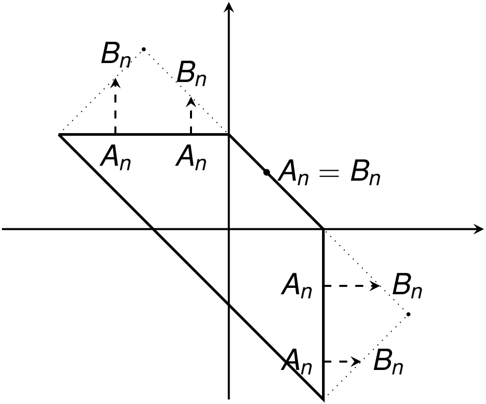


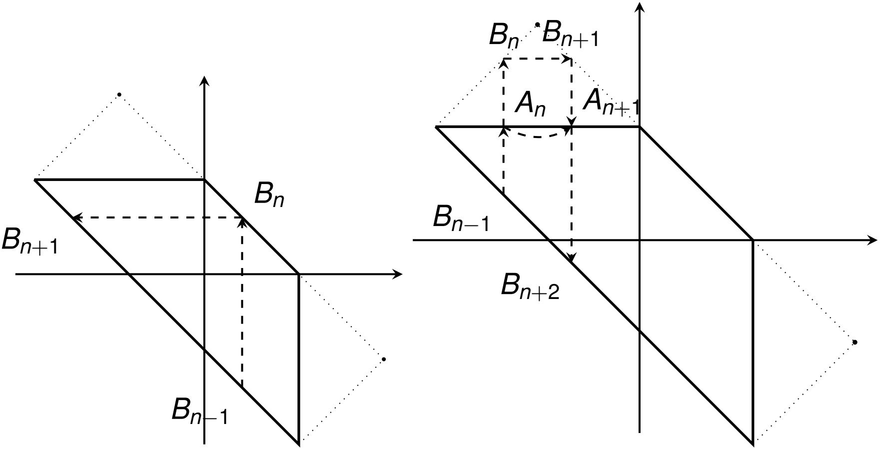
Points (a_n, a_{n+1}) are jumping along this boarder.

Let

$$A_n = \begin{cases} (a_n, a_{n+1}), & \text{if } n \text{ is even;} \\ (a_{n+1}, a_n), & \text{if } n \text{ is odd.} \end{cases}$$

And let B_n be projections of A_n from the legs of trapezoid ($B_n = A_n$ for A_n on bases).





The broken line $\dots B_{-2}B_{-1}B_0B_1B_2\dots$ is a billiard trajectory in rectangle with sides $d_1 = 4 - 6T$ and $d_2 = 1 - 2T$. The period is $d_1 + d_2 = 5 - 8T$.

Conclusions

- Somos sequences (and their properties) have natural tropical generalizations which are more simple and more visible.
- For Somos-4 and Somos-5 sequences (in genus 1 case) oscillations of points on the real locus of elliptic curves become oscillations of billiard trajectories in rectangles.
- Tropical pictures allows to predict untropical results and their proofs.

Further directions

Properties which seemed to be more or less equivalent for general Somos sequences (different sides of Integrality):

- Laurent phenomenon
- Positivity
- Reasonable tropicalization
- Periodicity $\pmod N$
- General formula in terms of theta-functions

A natural candidate for experiments is the Gale – Robinson sequence generated by

$$s_{m+n}s_m = \alpha s_{m+r}s_{m+n-r} + \beta s_{m+p}s_{m+n-p} + \gamma s_{m+q}s_{m+n-q},$$

where $r + p + q = n$.

Questions?

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Questions?

The smallest number which can be represented as a sum of three squares is 28:

$$28 = 5^2 + 1^2 + 1^2 + 1^2 = 4^2 + 2^2 + 2^2 + 2^2 = 3^2 + 3^2 + 3^2 + 1^2.$$

These three quadruples have many arithmetic properties. For example, if we take the product of the four numbers of each quadruple we get

$$5 = 5 \cdot 1 \cdot 1 \cdot 1, \qquad 32 = 4 \cdot 2 \cdot 2 \cdot 2, \qquad 27 = 3 \cdot 3 \cdot 3 \cdot 1,$$

and note that $5 = 32 - 27$.

Somos notes: strange motivation

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Now replace n by $s_n = F_{2n}$, so

$$s_0, s_1, \ldots = 0, 1, 3, 8, 21, 55, 144, 377, \ldots$$

We get

$$55 = 55 \cdot 1 \cdot 1 \cdot 1, \qquad 567 = 21 \cdot 3 \cdot 3 \cdot 3, \qquad 512 = 8 \cdot 8 \cdot 8 \cdot 1$$

and $55 = 567 - 512$.

Somos notes: strange motivation

Next example is 42:

$$42 = 6^2 + 2^2 + 1^2 + 1^2 = 5^2 + 3^2 + 2^2 + 2^2 = 4^2 + 4^2 + 3^2 + 1^2.$$

We have

$$12 = 6 \cdot 2 \cdot 1 \cdot 1, \qquad 60 = 5 \cdot 3 \cdot 2 \cdot 2, \qquad 48 = 4 \cdot 4 \cdot 3 \cdot 2,$$

and $12 = 60 - 48$. and the same for Fibonacci numbers:

$$432 = 144 \cdot 3 \cdot 1 \cdot 1, \qquad 3960 = 55 \cdot 8 \cdot 3 \cdot 3, \qquad 3528 = 21 \cdot 21 \cdot 8 \cdot 1$$

and $432 = 3960 - 3528$.

Quadruples

$$[5,1,1,1], \qquad [4,2,2,2], \qquad [3,3,3,1]$$
$$[6,2,1,1], \qquad [5,3,2,2], \qquad [4,4,3,1]$$

are the special case from more general sequence

$$[n+2,n-2,1,1], \qquad [n+1,n-1,2,2], \qquad [n,n,3,1].$$

So we can consider a general sequence satisfying recurrence

$$a_{n+2}a_{n-2}a_1^2 = a_{n+1}a_{n-1}a_2^2 + a_n^2 a_3 a_1 . \qquad (*)$$

If $a_0 = a_1 = a_2 = a_3 = 1$ then we get Somos-(4).

Somos M. In the Elliptic Realm

The recurrence (*) was know before. It was studied by Ward (in context of elliptic divisibility sequences) and by Lucas (mostly unpublished).

Somos notes: strange motivation

Quadruples

$$[5,1,1,1],\qquad [4,2,2,2],\qquad [3,3,3,1]$$
$$[6,2,1,1],\qquad [5,3,2,2],\qquad [4,4,3,1]$$

are the special case from more general sequence

$$[n+2,n-2,1,1],\qquad [n+1,n-1,2,2],\qquad [n,n,3,1].$$

So we can consider a general sequence satisfying recurrence

$$a_{n+2}a_{n-2}a_1^2 = a_{n+1}a_{n-1}a_2^2 + a_n^2 a_3 a_1. \qquad (*)$$

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