Суперсингулярные и большие решения полулинейных эллиптических и параболических уравнений

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1. What is large solution?

Let $\Omega \in \mathbb{R}^N$ be bounded domain, $f : \overline{\Omega} \times \mathbb{R}^1 \to \mathbb{R}^1_+$ be a continuous function: $f(x, 0) = 0$ and $r \to f(x, r)$ is nondecreasing $\forall\, x \in \overline{\Omega}$, $f(x, r) > 0$ in some $\Omega' \subset \Omega : \partial\Omega \subset \partial\Omega'$ and $r > 0$. Consider:

$$-\Delta u + f(x, u) = 0 \quad \text{in } \Omega, \tag{1}$$

$$\lim_{d(x)\to 0} u(x) = \infty, \quad d(x) = dist(x, \partial\Omega). \tag{2}$$

Such solution is called large or explosive solution. After ground-breaking papers by A.E. Perkins (1991), E.B. Dynkin (1991), J.-F. Le Gall (1993) large solutions for $f(x, u) = u^p$, $p > 1$, attract a lot of attention from probabilistic (spatial branching processes), conformal differential geometry. If $N = 1$ and $f(x, u) = f(u)$,then (1), (2) admit a unique solution even if $f(u)$ is not monotonic (J.Lopez-Gomez,L.Mair(2018,2019)). If $N > 1$ then uniqueness is proved only in some partial cases even in autonomous case.

2. Existence of large solutions

If $f = f(u)$, then in the case of C^1 domain Ω, $f \in C^1(0,\infty)$, $f' \geqslant 0$, $f(0) = 0$, the following Keller–Osserman condition ([1],[2]) is necessary and sufficient condition for existence of large solution:

$$\varphi(a) =: \int_a^\infty \frac{ds}{(F(s) - F(a))^{\frac{1}{2}}} < \infty \text{ for some } a > 0, \;\; F(s) := \int_0^s f(t)dt. \tag{3}$$

Sharpened KO-condition (S.Dumont,L.Dupaigne,O.Goubet,V.Radulesku(2007)): $\varphi(a) \to 0$ as $a \to \infty$, guarantees existence of large solution in the ball of arbitrary radius without monotonicity condition on $f(\cdot)$. Existence and uniqueness of large solution for general equation:

$$-\sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u + f(u) = h(x)$$

was studied by L.Veron(1992), M.Marcus, L.Veron(2006), A.Mohammed, G.Porru(2019) and other.

——————————————————————————— [1] Keller, J.B., *Commun. Pure Appl. Math.*, **X** (1957), 503–510. [2] Osserman, R., *Pac. J. Math.*, **7** (1957), 1641–1647.

Generalization of KO-condition for higher order semilinear inequalities and equations [3]:

$$Lu := \sum_{|\alpha|=m} D^{\alpha} a_{\alpha}(x,u) \geqslant f(|u|) \quad \text{in } \mathbb{R}^N,\ N \geqslant 1,\ m \geqslant 1,$$

$$|a_{\alpha}(x,s)| \leqslant c_1 |s|, \quad \forall\, s \in \mathbb{R}^1,\ \forall\, \alpha : |\alpha| = m,\ \forall\, x \in \mathbb{R}^N;\ f(s) > 0\ \forall\, s > 0.$$

Theorem 1.[3] *Let the following generalized KO-condition holds:*

$$\Phi_m(a) := \int_a^{\infty} f(s)^{-\frac{1}{m}} s^{\frac{1}{m}-1} ds < \infty \text{ for some } a > 0, \text{ and}$$

$$\liminf_{s \to 0+} \Phi_m(s)^{N-m} s < \infty.$$

Then problem under consideration does not have any nontrivial global weak solution $u \in L_{1,loc}(\mathbb{R}^N)$.

Remark 2. If $m = 2$ and $f(s) > 0\ \forall\, s > 0$ then $\Phi_2(a) = \int_a^{\infty} (f(s)s)^{-\frac{1}{2}} ds \approx \varphi(a)\ \forall\, a > 0$,where $\varphi(a)$ is from (3)

[3] Kon'kov A., Shishkov A. Generalization of the Keller-Osserman theorem for higher order differential inequalities. *Nonlinearity*, **32** (2019), 3012–3022.

Corollary 3. (Theorem of Keller, Osserman) Let $\Phi_2(a) < \infty$ for some $a > 0$. Then any non-negative global weak solution of semilinear elliptic inequality:

$$\Delta u \geqslant f(u) \quad \text{in } \mathbb{R}^N$$

is trivial: $u \equiv 0$.

Remark 4 Generalized KO-condition is sharp. For the inequality

$$\Delta^{\frac{m}{2}} u \geqslant c_0 |u| \left(\ln(2 + |u|)\right)^{\nu} \;\text{ in } \mathbb{R}^N,\; c_0 = const > 0, m = 2l, l \in \mathbb{N}$$

1) if $\nu > m$,then,by virtue of the Theorem 1,there is no nontrivial global solution;
2) if $\nu \leqslant m$,then there is constructed nontrivial global solution of the form:

$$u(x) = \exp\left(\exp\left(k(1 + |x|^2)^{\frac{1}{2}}\right)\right).$$

Local version of Keller-Osserman (KO-loc) condition was introduced by J.Lopez-Gomez (2000), L.Veron (2006) for equation (1) with $f = f(x,r)$: $-\Delta u + f(x,u) = 0$ in Ω:
For arbitrary compact $K \subset \Omega$ there exists continuous nondecreasing function $h_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$f(x,r) \geqslant h_k(r) \geqslant 0 \quad \forall\, x \in K,\ \forall\, r \geqslant 0 \text{ and}$$
$$\exists a > 0 : \int_a^\infty \frac{ds}{\sqrt{H_k(s) - H_k(a)}} < \infty,\ H_k(s) := \int_0^s h_k(t)dt.$$

(KO-loc) condition guarantees the existence of maximal solution u^{max} to (1):

$$u^{max}(x) := \lim_{n\to\infty} u_n(x),$$

where $\{u_n(x)\}$ is decreasing sequence of large solutions

$$u_n(x) \text{ in } \Omega_n : \overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega,\ \cup_{n\geqslant 1}\Omega_n = \Omega,\ \{\Omega_n\} \text{ are smooth.}$$

Question: Whether $u^{max}(x)$ is large solution in Ω?

Answer: not always! It depends essentially on the regularity of the domain Ω. If $f(x,u) = u^p$, $p > 1$, then necessary and sufficient condition for the property: u^{max} is large solution (!) was obtained in [4]:

$$\int_0^1 \frac{C_{p'}(\Omega^c \cap B_x(r))}{r^{n-2}} \frac{dr}{r} = \infty \text{ for all } x \in \Omega^c := \mathbb{R}^n \setminus \Omega,$$

where $p' = \frac{p}{p-1}$, $C_{p'}(K)$ is p'-capacity of compact K.

Moreover, if $f(x,r)|_{\partial\Omega} > 0 \; \forall \, r > 0$ and (KO-loc) condition is satisfied in some domain $V : \partial\Omega \subset V$, then under above Wiener regularity condition on $\partial\Omega$ there exists (M.Marcus,L.Veron(2009)) minimal large solution $u^{min}(x) := \lim_{n\to\infty} u'_n(x)$, where $\{u'_n(x)\}$, $n = 1, 2, \ldots$, is increasing sequence of large solutions in smooth domains Ω_n:

$$\overline{\Omega}'_{n+1} \subset \Omega'_n \quad \forall \, n \in \mathbb{N}, \quad \cap_{n\geqslant 1} \Omega'_n = \Omega.$$

Main property: any solution $u(\cdot)$ of (1), (2), should it exists, satisfies:

$$u^{min}(x) \leqslant u(x) \leqslant u^{max}(x) \quad \forall \, x \in \Omega.$$

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[4] Labutin, D.: Wiener regularity for large solutions of nonlinear equations. *Ark. Mat.* **41** (2003), 307–339.

3. Conditions of uniqueness of large solution

The problem of uniqueness reduces to property: $u^{max} = u^{min}$. For smooth domains and $f(x,u) = u^p$ with value $p = \frac{n+2}{n-2}$, which arises in conformal differential geometry, uniqueness of large solution was proved by C. Loevner and L. Nirenberg (1974). For arbitrary $p > 1$ uniqueness was firstly proved by C.Bandle and M.Marcus[5], using asymptotic expansion of any large solution near to $\partial\Omega$.
Developing methods from [5] for regular domains and $f(x,u) = a(x)u^p$, $a(x) \geqslant 0$, uniqueness was investigated by F.Cirstea and V.Radulescu(2002,2003),J.Lopez-Gomez(2006),O.Costin,L.Dupaigne(2010) and other. These methods, generated by [5] and based on asymptotic expansion of solution, require many assumptions on $f(\cdot)$ and regularity of $\partial\Omega$.

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[5] Bandle, C., Marcus, M., *J. Anal. Math.*, **58** (1992), 9–24.

There are other methods, which admit application to equations with general nonlinearity $f(x,u)$ and less smooth (particularly, lipshitz) domains: Y.Du,Q.Huang(1999),J.Garcia-Melian,R.Letelier,J.C.Sabina de Lis(2001),T.Ouyang, Z.Xie (2006),Z.Zhang, Y.Ma, L.Mi, X.Li (2010), M.Marcus,L.Veron(1997),J.Lopez-Gomez(2007). In [6] there was proposed method based on the strong barrier property of corresponding equation.

Definition 5. Let $z \in \partial\Omega$. Equation (1) possesses a strong barrier at z if there exists a number $r_0 > 0$ such that, for every $r \in (0, r_0)$ there exists a positive supersolution $v = v_{r,z}$ of (1) in $B_r(z) \cap \Omega$ such that $v \in C(B_r(z) \cap \overline{\Omega}) : \lim_{y \to x,\ y \in \Omega} v(y) = \infty\ \forall\, x \in \partial B_r(z) \cap \Omega$

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[6] Marcus, M., Veron, L., *Commun. Pure Appl. Math.*, **LVI** (2003),

It was proved by Marcus–Veron ([6],2003) that strong barrier property yields the uniqueness of large solution for equation under consideration. Additionally they proved that if

$$\partial\Omega \text{ is } C^2 \text{ and } f(x,r) \geqslant d(x)^\alpha r^p,\ p>1,\ \alpha>0,\ d(x)=dist(x,\partial\Omega),$$

then strong barrier property holds and, as consequence, problem (1), (2) has unique solution. Moreover, it was proved in [7] that, if

$$f(x,r) \leqslant \exp\left(-\frac{k}{d(x)}\right) r^p, \quad p>1,$$

then strong barrier property does not hold. Additionally, in [7] they hypothesized that condition:
$f(x,r) \geqslant \exp\left(-\frac{C}{d(x)^\alpha}\right) r^p$, $0<\alpha<1$, $p>1$, $C = const > 0$, *yields strong barrier property and, consequently, uniqueness of large solution.*

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[7] Lopez-Gomez, J., Mair, L., Veron, L., *ZAMP*, **71:109** (2020)

We proved this hypothesis and even more strong statement
Theorem 6 [8]: *Let nonlinearity $f(x,r)$ in equation* (1) *satisfies:*

$$f(x,r) \geqslant \exp\left(-\frac{\omega(d(x))}{d(x)}\right) r^p, \quad p > 1,\ d(x) = dist(x, \partial\Omega),$$

where $\omega(\cdot)$ is arbitrary nondecreasing continuous function, satisfying technical condition:

$$s^\gamma \leqslant \omega(s) < \omega_0 = const < \infty \quad \forall\, s \in (0, s_0),\ s_0 = const,\ 0 < \gamma < 1,$$

and the Dini-like condition:

$$\int_0^c \frac{\omega(s)}{s} ds < \infty. \tag{4}$$

Then equation (1) *possesses a strong barrier property for arbitrary point $z \in \partial\Omega$ and, consequently, problem* (1), (2) *has unique large solution.*

The proof does not use any comparison technique and is based on some new local integral estimates of solutions near to the $\partial\Omega$.

Problem: whether condition (4) is necessary for the uniqueness?

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[8] Shishkov,A. Very singular and large solutions of semilinear elliptic equations with degenerate absorption. *Calc.Var.PDE*, **61:102** (2022), 27p.

4. Very singular solutions for semilinear elliptic equations

Let Ω be bounded C^2 domain in $\mathbb{R}^N$, $N \geqslant 2$.

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\begin{aligned}
&-\Delta u + f(x,u) = 0 \quad \text{in } \Omega,\ q>1,\ g(x,u) \geqslant 0\ \forall\, u \geqslant 0,\ x \in \overline{\Omega} \\
&\quad u = k\delta_a \quad \text{on } \partial\Omega,\ \delta_a \text{ — Dirac measure},\ k>0,\ a \in \partial\Omega.
\end{aligned} \tag{5}
$$

If $$\int_\Omega f(x, kK(x,a))d(x)dx < \infty,\ K = K_a \text{ is Poisson kernel for } \Omega, \tag{6}$$

$d(x) = dist(x, \partial\Omega)$, then there exists a unique weak solution $u = u_{k,a}$ for any $k < \infty$ (Gmira-Veron(1991)).

Problem: what is $u_{\infty,a}(x) := \lim_{k\to\infty} u_{k,a}(x)$? If $f(x,u) = H(x)u^q$, $q > 1$ and $H(x) > 0\ \forall\, x \in \Omega$, then there exists maximal solution $U(\cdot)$ of equation (5). Additionally, if $H(x) \leqslant H_0(\rho(x))$, where $H_0(\cdot)$ is nonincreasing and:

$$\int_0^1 H_0(s)^{\frac{1}{2}} ds < \infty,$$

then U is large solution: $\lim_{d(x)\to 0} U(x) = \infty$(Ratto-Rigoli-Veron(1994)). Thus, the maximal solution U^{max} may not turn out to be a large solution both due to the irregularity of $\partial\Omega$ (see Labutin's result) and due to the strong growth of the absorption potential near the boundary.

Remark that $K(x,y) = c_N \frac{x_N}{\left(|x'-y'|^2+x_N^2\right)^{\frac{N}{2}}}$ if $\Omega = R_+^N$. Therefore, simple computation shows that condition (6) is satisfied for $f(x,u) = H(x)u^q$, for example, if:

1) $H(x) = const > 0$ and $1 < q < 1 + 2(N-1)^{-1}$;
2) $H(x) = d(x)^\alpha$, $\alpha > 0$ and $1 < q < 1 + 2(\alpha+1)(N-1)^{-1}$;
3) $0 < H(x) < C \exp\left(-\omega_0 d(x)^{-1}\right)$, $\omega_0 = const > 0$ and $1 < q < \infty$.

Thus, $u_{k,a}(x) \leqslant U^{max}(x)\ \forall\, x \in \Omega$, $\forall\, k \in \mathbb{N}$, if U^{max} is large solution.
QUESTION: what is $\lim_{k\to\infty} u_{k,a}(x)$ if condition (6) holds?
Theorem 7.[9]*If $f(x,u) = H(x)u^q$, $q > 1$, and potential $H(x)$ satisfies:*

$$H(x) \geqslant h(d(x)) \quad \forall\, x \in \Omega, \quad h(s) := \exp\left(-s^{-1}\omega(s)\right), d(x) = dist(x, \partial\Omega),$$

where nondecreasing function ω satisfies Dini condition:

$$\int_0^c \frac{\omega(s)}{s} ds < \infty,\ 0 < c < \infty. \tag{7}$$

Then $u_{\infty,a}(x) := \lim_{k\to\infty} u_{k,a}(x)$ is a very singular solution of equation (5) with more strong then Poisson kernel boundary singularity at a and $\lim_{x\to y} u_{\infty,a}(x) = 0\ \forall\, y \in \partial\Omega \setminus \{a\}$.

———————————————————— [9] A. Shishkov, L. Veron *JMAA*. **352** (2009), 206–217.

About sharpness of Dini condition for the existence of v.s. solution.

Theorem 8.(Sh.[8],2022) *Let* $1 < q < 1 + \frac{2}{N-1}$ *and potential* $H(\cdot)$ *from* ***Th.7*** *satisfies estimate:*

$$0 \leq H(x) \leq ch(d(x)) \quad in\ \Omega, \quad c = const < \infty,$$

where $h(s) = \exp\left(-s^{-1}\omega(s)\right)$, *and nondecreasing function* $\omega(\cdot) > 0$*:* $\omega(s) \to 0$ *as* $s \to 0$, *satisfies technical condition:*

$$\limsup_{j\to\infty} \mu(2^{-j+1})\mu(2^{-j})^{-1} < 1, \quad \mu(s) := \frac{\omega(s)}{s}.$$

Then under condition: $\int_0^1 \frac{\omega(s)}{s} ds = \infty$, *function* $u_{\infty,a}(x) := \lim_{k\to\infty} u_{k,a}(x)$ *is large solution, i.e.*

$$\lim_{x\to y} u_{\infty,a}(x) = \infty \quad \forall\, y \in \partial\Omega.$$

Thus for $q \in (1, 1 + \frac{2}{N-1})$ *Dini condition is criterion (necessary and sufficient condition) for existence of v.s. solution.*

Remark 9. Remember that Dini condition is also sufficient condition for uniqueness of large solution, and our conjecture is that Dini condition is also necessary condition for uniqueness of large solution

5. Large and v.s. solutions when absorption degenerates on some manifolds $\Gamma \subset \Omega : \bar{\Gamma} \cap \partial\Omega \neq \emptyset$

Firstly this problem was considered in parabolic setting [10], [11]:

$$u_t - \Delta u + h(x)u^q = 0 \text{ in } \mathbb{R}_+^{N+1},\ q > 1,$$
$$u(0,x) = k\delta(x), \quad k \in \mathbb{N}. \quad h(x) \to 0 \text{ as } |x| \to 0.$$

What is $u_\infty(t,x) := \lim_{k\to\infty} u_k(t,x)$?

Elliptic case was considered in [12], [13].

$$-\Delta u + h(x)u^q = 0 \text{ in } \mathbb{R}_+^N = \mathbb{R}^{N-1} \times \mathbb{R}_+^1,\ q > 1,\ h \in C(\overline{\mathbb{R}}_+^N). \qquad (8)$$

If $h(\cdot) > 0$ in $\mathbb{R}_+^N$ then there exists maximal solution U^{max} in $\mathbb{R}_+^N$:

$$\lim_{x_N\to 0, |x|<M} U^{max}(x) = \infty \quad \forall\, M > 0 \ \to \ U^{max}(\cdot) \text{ is large solution.}$$

Now we consider potential $h(\cdot) : h(x) = 0\ \forall\, x \in F = \{(0, x_N) \in \mathbb{R}_+^N : x_N > 0\}$!
Let $h_0(s) \in C^1[0,\infty)$ be arbitrary such that: $h_0(0) = 0,\ h_0(s) > 0\ \forall\, s > 0$.

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[10] Shishkov A., Veron L., *Calc. Var. Part. Differ. Equat.*, **33** (2008), 343–375.
[11] Marcus M., Shishkov A., *Ann. Sc. Norm. Super. Pisa, Cl. Sc., Ser. V*, **16** (2016), 1019–1047.
[12] Marcus M., Shishkov A., *Ann. I. H. Poincare-AN*, **30** (2013), 315–336.
[13] Marcus M., Shishkov A., *Ann. I. H. Poincare-AN (Erratum)*, (2013).

and $\bar{h}(s) := \exp(-s^{-1}\omega(s))$, where $\omega(s)$ satisfies following conditions

a) $\omega \in C(0,\infty)$ is positive nondecreasing function,

b) $s \to \mu(s) := s^{-1}\omega(s)$ is monotone decreasing on $\mathbb{R}^1_+$,

c) $\lim_{s\to 0}\mu(s) = \infty$.

Since $h(x) = 0 \;\forall\, x \in F \subset \mathbb{R}^N_+$ classical results do not guarantee existence of maximal solution of equation (8).

Theorem 10.(M.Marcus–A.Sh.[12],[13]) *Suppose that* $c_0 h_0(|x'|) \geqslant h(x) \geqslant c_1 \bar{h}(|x'|) \;\forall\, x \in \mathbb{R}^N_+$, $c_1 > 0$ *and*

$$\int_0^1 t^{-1}\omega(t)dt < \infty.$$

If $\{u_n\}$ *is sequence of positive solutions of* (8) *in* $\mathbb{R}^N_+$ *converging pointwise in* $\Omega = \mathbb{R}^N_+ \setminus F$ *!, then mentioned sequence* $\{u_n\}$ *converges in* $\mathbb{R}^N_+$ *and its limits is solution in* $\mathbb{R}^N_+$. *Particularly, equation* (8) *possesses a maximal solution* $U^{max}(x)$ *in* $\mathbb{R}^N_+$ *and it is large solution. Consequently, sequence of solutions* $\{u_k(x)\}$ *of mentioned equation with*

$$u_k(x',0) = k\delta(x'), \quad k = 1,2,...$$

converges to v.s. solution $u_\infty(x)$ *in* $\mathbb{R}^N_+$*, satisfying bound. condition:*

$$u_\infty(x',0) = 0 \quad \forall\, x' \neq 0, \quad \int_{R^{n-1}} u_\infty(x',x_N)dx' \to \infty \text{ as } x_N \to 0.$$

Theorem 11.(Marcus–Sh.[12],[13]) *Suppose that* $1 < q < 1 + \frac{2}{N-1}$,

$$h(x) \leqslant c\bar{h}(|x'|) \quad \forall\, x \in \mathbb{R}^N_+,\ c = const < \infty,$$

where $\bar{h}(s) = \exp(-\mu(s))$, $\mu(s) = s^{-1}\omega(s)$. *Assume that conditions a), b), c) and*

$$\limsup_{j\to\infty} \frac{\mu(a^{-j+1})}{\mu(a^{-j})} < 1 \text{ for some } a > 1$$

hold.Under these assumptions, if

$$\int_0^1 t^{-1}\omega(t)dt = \infty,$$

then $u_\infty(x) = \lim_{k\to\infty} u_k(x)$ *is solution of considered equation in* $\Omega := \mathbb{R}^N_+ \setminus F$ *only, and* $u_\infty(x) = \infty\ \forall\, x \in F = \{(0, x_N) : x_N > 0\}$ *(razor blade solution).*

Corollary 12. Suppose that $c^{-1}\bar{h}(|x'|) \leqslant h(x) \leqslant c\bar{h}(|x'|)\ \forall\, x \in \mathbb{R}^N_+$. Then Dini condition is necessary and sufficient condition for existence of both large and v.s.solutions u_∞.

Thank you for your attention!