Multiscale Analysis of Stationary Thermoelastic Vibrations of a Composite Material

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Thermoelastic wave model

By $u = (u_1, u_2)$ and $\sigma = \{\sigma_{ij}\}$ denote displacement vectors and strain tensors respectively. Then the equation of motion has the form

$$-\operatorname{div}\sigma(u,\theta) + \rho\,\partial_{tt}u = \rho F, \tag{1}$$

where $F = (F_1, F_2)$ is a given mass force, ρ is a media density. Linear constitutive equation (Duhamel–Neumann's Law) has the form

$$\sigma(u,\theta) = Ae(u) - B\theta, \tag{2}$$

where $e(u) = \{e_{ij}(u)\}$ is the strain tensor, $e_{ij}(u) = 1/2(\partial_{x_i}u_j + \partial_{x_j}u_i)$, θ is the temperature variation.
The heat conduction is described by the equations

$$-\operatorname{div}q(\theta) + c\,\partial_t\theta + TB : \partial_t e(u) = Q, \tag{3}$$

$$q(\theta) = K\nabla\theta, \tag{4}$$

where Q is the heat source.

We consider oscillations of the medium in time. We suppose that

$$
\begin{aligned}
F = F(x,t) &= F_c(x)\cos\omega t + F_s(x)\sin\omega t,\\
Q = Q(x,t) &= Q_c(x)\cos\omega t + Q_s(x)\sin\omega t,\\
u = u(x,t) &= u_c(x)\cos\omega t + u_s(x)\sin\omega t,\\
\theta = \theta(x,t) &= \theta_c(x)\cos\omega t + \theta_s(x)\sin\omega t,
\end{aligned}
\tag{5}
$$

where $\omega \in \mathbb{R}$ is the oscillation frequency. Then equations (1)–(3) reduce to

$$
\begin{aligned}
-\operatorname{div}\sigma(u_c,\theta_c) - \rho\omega^2 u_c = \rho F_c, &\quad -\operatorname{div} q(\theta_c) + c\omega\theta_s + T\omega B : e(u_s) = Q_c,\\
-\operatorname{div}\sigma(u_s,\theta_s) - \rho\omega^2 u_s = \rho F_s, &\quad -\operatorname{div} q(\theta_s) - c\omega\theta_c - T\omega B : e(u_c) = Q_s,
\end{aligned}
\tag{6}
$$

These are the basic relations of stationary vibrations of thermoelasticity.

Let the body occupy domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\Gamma = \partial\Omega$. Let Γ be separated into two parts Γ_D, Γ_N with the nonzero one-dimensional Hausdorff measure, on which the body edge is fixed and free. Also suppose that the edge Γ_D is isothermal, and there are no external heat sources on Γ_N. In such a case, the following boundary conditions are fulfilled

$$\begin{aligned} u_c = 0, \quad u_s = 0, \quad \theta_c = 0, \quad \theta_s = 0 \quad \text{on} \quad \Gamma_D, \\ \sigma(u_c, \theta_c)n = 0, \quad \sigma(u_s, \theta_s)n = 0, \quad q(\theta_c)n = 0, \quad q(\theta_s)n = 0 \quad \text{on} \quad \Gamma_N, \end{aligned} \tag{7}$$

where n is the unit outward normal vector to Γ. If F, Q, A, B, ρ, ω, K, c and T are given, equations (6) and (7) together provide the boundary value problem of finding u_c, u_s, θ_c, θ_s.
This problem is well-posed for some restrictions on initial data.

Model with a thin inclusion

In the two-dimensional space $\mathbb{R}^2$ referred to the coordinate system Oy_1y_2, let Ω be a bounded domain with a Lipschitz boundary $\partial\Omega$. Let $\gamma = \big(\Omega \cap \{y_2 = 0\}\big)$ be the intersection of Ω with axis Oy_2, which is supposed to have a positive one-dimensional measure:

$$\gamma = \big\{(y_1, y_2) \in \mathbb{R}^2 \colon\ 0 < y_1 < l^*,\ y_2 = 0\big\}, \quad l^* = \text{const} > 0.$$

Line segment γ divides Ω into subdomains $\Omega_\pm$ with Lipschitz boundaries $\partial\Omega_\pm$.

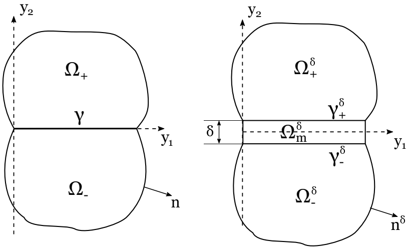

Figure: Left: γ divides Ω into two parts. Right: the assembly of thermoelastic bodies; $\Omega^\delta_\pm$ are adherents, Ω^δ_m is adhesive, parameter δ characterizes thickness of the adhesive

Let us introduce a small real parameter $\delta > 0$. We consider the assembly consisting of the three thermoelastic isotropic bodies. To describe this assembly, we introduce the following geometric objects:

$$\Omega_m^\delta = \{0 < y_1 < l^*\} \times \{-\delta/2 < y_2 < \delta/2\},$$
$$\Omega_\pm^\delta = \big\{(y_1, y_2) \in \mathbb{R}^2\colon\ (y_1, y_2 \mp \delta/2) \in \Omega_\pm\big\},$$
$$\gamma_\pm^\delta = \gamma \pm (0, \delta/2),$$

which depend on the fixed parameter δ. Also, to the entire region occupied by the material, we relate the domain $\Omega^\delta = \Omega_+^\delta \cup \Omega_-^\delta \cup \Omega_m^\delta \cup \gamma_+^\delta \cup \gamma_-^\delta$ and introduce the sets $\Gamma_\pm^\delta = \partial\Omega^\delta \cap \partial\Omega_\pm^\delta$, which are the parts of the outer boundary $\partial\Omega^\delta$.

Problem A-diff

Consider the formulation of the problem that describes stationary thermoelastic waves in the bonded adhesive and adherents. The state of the composite body is described by the displacements functions u_c^δ, u_s^δ and the temperature variation functions θ_c^δ, θ_s^δ, defined in Ω^δ. The equations of the boundary value problem describing thermoelastic waves are

$$\begin{array}{l} -\operatorname{div}\sigma(u_c^\delta,\theta_c^\delta)-\rho\omega^2 u_c^\delta=\rho F_c,\\ -\operatorname{div} q(\theta_c^\delta)+c\omega\theta_s^\delta+T\omega B:e(u_s^\delta)=Q_c,\\ -\operatorname{div}\sigma(u_s^\delta,\theta_s^\delta)-\rho\omega^2 u_s^\delta=\rho F_s,\\ -\operatorname{div} q(\theta_s^\delta)-c\omega\theta_c^\delta-T\omega B:e(u_c^\delta)=Q_s \end{array} \quad \text{in} \quad \Omega^\delta, \tag{8}$$

$$u_c^\delta=0,\quad u_s^\delta=0,\quad \theta_c^\delta=0,\quad \theta_s^\delta=0 \quad \text{on} \quad \Gamma_D^\delta=\Gamma_+^\delta\cup\Gamma_-^\delta, \tag{9}$$

$$\sigma(u_c^\delta,\theta_c^\delta)n^\delta=0,\ \sigma(u_s^\delta,\theta_s^\delta)n^\delta=0,\ q(\theta_c^\delta)n^\delta=0,\ q(\theta_s^\delta)n^\delta=0 \text{ on } \partial\Omega^\delta\backslash\overline{\Gamma_D^\delta}, \tag{10}$$

where n^δ is the unit outward normal to $\partial\Omega^\delta$.

Problem B-diff

Suppose that the adhesive layer is hard with high conductivity:

$$\lambda, \mu, k, \rho, c \sim \frac{1}{\delta} \quad \text{in} \quad \Omega_m^\delta.$$

This is related with the desire to obtain a hard thin inclusion as a limit instead of the adhesive.
Find functions u_c, u_s, θ_c, θ_s, defined in Ω, such that

$$\begin{array}{ll} -\operatorname{div}\sigma(u_c,\theta_c) - \rho\omega^2 u_c = \rho\bar{F}_c, & -\operatorname{div} q(\theta_c) + c\omega\theta_s + T\omega B : e(u_s) = \bar{Q}_c, \\ -\operatorname{div}\sigma(u_s,\theta_s) - \rho\omega^2 u_s = \rho\bar{F}_s, & -\operatorname{div} q(\theta_s) - c\omega\theta_c - T\omega B : e(u_c) = \bar{Q}_s \end{array}, \tag{11}$$

$$u_c = 0, \quad u_s = 0, \quad \theta_c = 0, \quad \theta_s = 0 \;\text{ on }\; \partial\Omega, \tag{12}$$

$$\left.\begin{array}{r} [\sigma(u_c,\theta_c)\nu] = -\partial_{x_1}(a^{in}\partial_{x_1}u_c - b^{in}\theta_c) - \rho\omega^2 u_c, \\ [\sigma(u_s,\theta_s)\nu] = -\partial_{x_1}(a^{in}\partial_{x_1}u_s - b^{in}\theta_s) - \rho\omega^2 u_s, \\ [q(\theta_c)\nu] = -\partial_{x_1}(k^{in}\partial_{x_1}\theta_c) + d^{in}\omega\theta_s + T\omega b^{in}\partial_{x_1}u_s, \\ [q(\theta_s)\nu] = -\partial_{x_1}(k^{in}\partial_{x_1}\theta_s) - d^{in}\omega\theta_c - T\omega b^{in}\partial_{x_1}u_c \end{array}\right. \quad \text{on } \gamma, \tag{13}$$

where $\nu = (0, 1)$, bracket $[\,\cdot\,]$ designates a jump on γ, tensor σ and vector q, as before, are calculated according to the constitutive relations, and scalars k^{in} and d^{in} and the elements of matrix a^{in} and vector b^{in} are the new coefficients, describing the material properties of the thin inclusion γ. We calculate:

$$a^{in} = \begin{pmatrix} 4\mu(\lambda+\mu)/(\lambda+2\mu) & 0 \\ 0 & 0 \end{pmatrix}, \quad b^{in} = \begin{pmatrix} 2\beta\mu/(\lambda+2\mu) \\ 0 \end{pmatrix}, \qquad (14)$$

$$k^{in} = k, \quad d^{in} = c + T\beta^2/(\lambda+2\mu).$$

Generalization to any finite number of thin inclusions

Using the similar arguments (with natural modifications), as in the previous sections, we construct the well-posed model of stationary vibrations of thermoelastic body incorporating a family of thin inclusions $\gamma^{\varepsilon} = \Omega \cap \{x_2 = j\varepsilon, \, j \in \mathbb{Z}\}$, which are parallel to each other and spaced apart from each other at a distance of $\varepsilon > 0$, as on Figure 2. In this case, the essential geometric requirement is only that each of the subdomains, into which the domain Ω is divided by the set γ^{ε}, has a Lipschitz boundary.

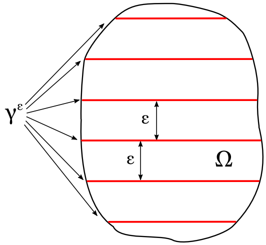

Figure: Several thin inclusions

Homogenization by the number of thin inclusions

Let thin inclusions occupy the set of physical positions

$$\gamma^\varepsilon = \Omega \cap \{x_2 = j\varepsilon,\, j \in \mathbb{Z}\}, \tag{15}$$

where $\varepsilon > 0$ is a dimensionless parameter characterizing the distance between two neighboring inclusions.
Introduce into considerations the space

$$V^\varepsilon = \Big\{(u,\theta) \in H_0^1(\Omega) \times H_0^1(\Omega) \colon\ u\big|_{\gamma^\varepsilon} \in H_0^1(\gamma^\varepsilon),\ \theta\big|_{\gamma^\varepsilon} \in H_0^1(\gamma^\varepsilon)\Big\}$$

with the standard norm and scalar product.
We consider that the coefficients in the model also depend on ε:

$$\begin{array}{c} \mu^\varepsilon := \mu(x, \frac{x_2}{\varepsilon}), \quad \lambda^\varepsilon := \lambda(x, \frac{x_2}{\varepsilon}), \quad \beta^\varepsilon := \beta(x, \frac{x_2}{\varepsilon}), \\ \rho^\varepsilon := \rho(x, \frac{x_2}{\varepsilon}), \quad k^\varepsilon := k(x, \frac{x_2}{\varepsilon}), \quad c^\varepsilon := c(x, \frac{x_2}{\varepsilon}) \end{array} \quad \text{in} \quad \Omega,$$

$$\begin{array}{r} a_\varepsilon^{in} := \varepsilon^p a^{in}(x_1), \quad b_\varepsilon^{in} := \varepsilon^p b^{in}(x_1), \quad \rho^\varepsilon := \varepsilon^p \rho^{in}(x_1), \\ k_\varepsilon^{in} := \varepsilon^p k^{in}(x_1), \quad d_\varepsilon^{in} := \varepsilon^q d^{in}(x_1) \end{array} \quad \text{on} \quad \gamma^\varepsilon,$$

where $\mu = \mu(x,\xi_2)$, $\lambda = \lambda(x,\xi_2)$, $\beta = \beta(x,\xi_2)$, $\rho = \rho(x,\xi_2)$, $k = k(x,\xi_2)$, and $c = c(x,\xi_2)$ are 1-periodic in ξ_2; parameters $p, q \in \mathbb{Z}$ are given.

Problem B_ε

For any fixed $\varepsilon > 0$ and $p, q \in \mathbb{Z}$, find a quadruple of functions $(u_c^\varepsilon, \theta_c^\varepsilon, u_s^\varepsilon, \theta_s^\varepsilon) \in V^\varepsilon \times V^\varepsilon$ satisfying the integral equalities

$$\int\limits_\Omega \Big(2\mu(x, \frac{x_2}{\varepsilon})\mathbb{D}_x(u_r^\varepsilon) : \mathbb{D}_x(v_r) + \big(\lambda(x, \frac{x_2}{\varepsilon})\mathrm{div}_x u_r^\varepsilon - \beta(x, \frac{x_2}{\varepsilon})\theta_r^\varepsilon\big)\mathrm{div}_x v_r - \omega^2 \rho(x$$

$$+\varepsilon^p \int\limits_{\gamma^\varepsilon} \Big(a_{11}^{in}(x_1)\partial_{x_1} u_{r1}^\varepsilon \partial_{x_1} v_{r1} - b_1^{in}(x_1)\theta_r^\varepsilon \partial_{x_1} v_{r1} - \omega^2 \rho^{in}(x_1) u_r^\varepsilon \cdot v_r\Big)\, d\nu^\varepsilon(x) =$$

$$= \int\limits_\Omega \rho(x, \frac{x_2}{\varepsilon}) F_r(x) \cdot v_r(x)\, dx, \quad \text{`}r\text{' stands for `}c\text{' and `}s\text{'}, \qquad (16)$$

$$\int\limits_\Omega \Big(k(x, \frac{x_2}{\varepsilon})\nabla_x \theta_c^\varepsilon \cdot \nabla_x \vartheta_c + \omega c(x, \frac{x_2}{\varepsilon})\theta_s^\varepsilon \vartheta_c + T\omega\beta(x, \frac{x_2}{\varepsilon})(\mathrm{div}_x u_s^\varepsilon)\vartheta_c\Big)\, dx$$

$$+ \varepsilon^p \int\limits_{\gamma^\varepsilon} \Big(k^{in}(x_1)\partial_{x_1}\theta_c^\varepsilon \partial_{x_1}\vartheta_c + T\omega b_1^{in}(x_1)(\partial_{x_1} u_{s1}^\varepsilon)\vartheta_c\Big)\, d\nu^\varepsilon(x)$$

$$+ \varepsilon^q \int \omega d^{in}(x_1)\theta_s^\varepsilon \vartheta_c \, d\nu^\varepsilon(x) = \int Q_c \vartheta_c(x)\, dx \qquad (17)$$

$$
\int_{\Omega} \Big(k(x, \frac{x_2}{\varepsilon}) \nabla_x \theta_s^\varepsilon \cdot \nabla_x \vartheta_s - \omega c(x, \frac{x_2}{\varepsilon}) \theta_c^\varepsilon \vartheta_s - T\omega\beta(x, \frac{x_2}{\varepsilon}) (\mathrm{div}_x u_c^\varepsilon) \vartheta_s \Big)\, dx
$$

$$
+ \varepsilon^p \int_{\gamma^\varepsilon} \Big(k^{in}(x_1) \partial_{x_1} \theta_s^\varepsilon \partial_{x_1} \vartheta_s - T\omega b_1^{in}(x_1) (\partial_{x_1} u_{c1}^\varepsilon) \vartheta_s \Big)\, d\nu^\varepsilon(x)
$$

$$
- \varepsilon^q \int_{\gamma^\varepsilon} \omega d^{in}(x_1) \theta_c^\varepsilon \vartheta_s \, d\nu^\varepsilon(x) = \int_{\Omega} Q_s \vartheta_s(x)\, dx \quad (18)
$$

for all quadruples of test functions $(v_c, \vartheta_c, v_s, \vartheta_s) \in V^\varepsilon \times V^\varepsilon$. In (16) and further, $\mathbb{D}_x$ is the symmetric part of the gradient (i.e., $\mathbb{D}_x(\phi) = (1/2)(\nabla_x \phi + (\nabla_x \phi)^T))$ for any admissible vector-function ϕ), a_{11}^{in} and b_1^{in} are the only nonzero components of matrix a^{in} and vector b^{in}, respectively.

Some assumptions

We restrict ourselves to considering the case $p = q = 1$, so we set $p = q = 1$ everywhere further.

Conditions W. **1.** Functions $\mu, \lambda, \beta, \rho, k$, and c are differentiable in (x, ξ_2) and 1-periodic in ξ_2. Functions $a_{11}^{in}, b_1^{in}, \rho^{in}, k^{in}$, and d^{in} are differentiable in x_1.

2. There exist positive constants $\mu_*, \mu^*, \lambda_*, \lambda^*, \beta_*, \beta^*, \rho_*, \rho^*, k_*, k^*, c_*$, and c^* such that the bounds

$$\mu_* \le \mu(x, \xi_2) \le \mu^*, \quad \lambda_* \le \lambda(x, \xi_2) \le \lambda^*, \quad \beta_* \le \beta(x, \xi_2) \le \beta^*,$$
$$\rho_* \le \rho(x, \xi_2) \le \rho^*, \quad k_* \le k(x, \xi_2) \le k^*, \quad c_* \le c(x, \xi_2) \le c^*$$

hold for all $x \in \Omega, \xi_2 \in \mathbb{R}$ and the bounds

$$\mu_* \le a_{11}^{in}(x_1) \le \mu^*, \ \beta_* \le b_1^{in}(x_1) \le \beta^*, \ \rho_* \le \rho^{in}(x_1) \le \rho^*, \ k_* \le k^{in}(x_1) \le k^*$$

hold for all $x_1 \in [-l_1, l_1]$, where $l_1 = \max\limits_{(x_1', x_2') \in \overline{\Omega}} |x_1'|$.

3. The elastic stiffness and thermal conductivity properties dominate over frequency of vibrations and the linear thermal extension property in the following sense.
There exist positive constants c_1 and c_2 such that

$$\mu_* > \max\left\{\frac{1}{2}C_{Korn}^2\omega^2\rho^*, \frac{1}{2}C_{Korn}^2\omega^2\rho^*C_T, \frac{\beta^*}{2}\left(\frac{1}{c_1}+\frac{T\omega}{c_2}\right)+2C_{PF}^2\omega^2\rho^*\right\},$$

$$\lambda_* > \left(\frac{1}{c_1}+\frac{T\omega}{c_2}\right)\frac{\beta^*}{2}, \quad k_* > (c_1+T\omega c_2)C_{PF}^2\beta^*,$$

where C_{Korn} is the constant from Korn's inequality on Ω, C_{PF} is the constant from Poincare–Friedrichs inequality on Ω, and C_T is the constant from the estimate of trace on γ^ε. We note that C_{Korn}, $C_{PF} = \sqrt{2}l_1$ and $C_T = 4l_2^2$ do not depend on ε. Here, $l_2 = \max\limits_{(x_1', x_2')\in\overline{\Omega}} |x_2'|$.

Theorem 1. *Let Conditions W hold. Assume that* $F_c, F_s \in L^2(\Omega)^2, Q_c, Q_s \in L^2(\Omega)$. *Then for any fixed* $\varepsilon \in (0, \varepsilon_0]$ *there exists a unique solution* $(u_c^\varepsilon, \theta_c^\varepsilon, u_s^\varepsilon, \theta_s^\varepsilon) \in V^\varepsilon \times V^\varepsilon$ *to Problem* B_ε*. Moreover, there is a constant* $c_0 > 0$ *independent of* ε *such that the set of the uniform (in* ε*) estimates holds true:*

$$\|u_r^\varepsilon\|_{H_0^1(\Omega)^2} \le c_0, \quad \|\theta_r^\varepsilon\|_{H_0^1(\Omega)^2} \le c_0, \quad \varepsilon^{\frac{1}{2}} \|\partial_{x_1} u_{r1}^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \le c_0,$$
$$\varepsilon^{\frac{1}{2}} \|u_r^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \le c_0, \quad \varepsilon^{\frac{1}{2}} \|\partial_{x_1} \theta_r^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \le c_0, \quad \varepsilon^{\frac{1}{2}} \|\theta_r^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \le c_0, \tag{19}$$

where 'r*' stands for '*c*' and '*s*'.*

Theorem 2. *Assume that* $F_c, F_s \in L^2(\Omega)^2, Q_c, Q_s \in L^2(\Omega)$.
Then the family $\{(u_c^\varepsilon, \theta_c^\varepsilon, u_s^\varepsilon, \theta_s^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ *of solutions to Problem* B_ε *as* $\varepsilon \to 0+$ *tends to the solution* $(u_c^*, \theta_c^*, u_s^*, \theta_s^*)$ *of Problem H, stated below, in the sense of the limiting relations*

$$u_c^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} u_c^* \quad u_s^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} u_s^* \ \text{ weakly in } \ H_0^1(\Omega)^2, \ \text{strongly in } \ L^2(\Omega)^2, \tag{20}$$

$$\theta_c^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} \theta_c^* \quad \theta_s^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} \theta_s^* \ \text{ weakly in } \ H_0^1(\Omega), \ \text{strongly in } \ L^2(\Omega). \tag{21}$$

Stationary vibrations of the homogenized composite

Find amplitudes $u_c^*, u_s^* \in H_0^1(\Omega)^2$ of displacements and amplitudes $\theta_c^*, \theta_s^* \in H_0^1(\Omega)$ of temperature of periodic in time thermomechanical oscillations of a composite satisfying the system of integral equalities

$$\int_\Omega \left\{\mathcal{A}_\mu(x) : \nabla_x u_r^*(x) - \mathbb{A}_\gamma(x)\theta_r^*(x)\right\} : \nabla_x v_r(x)\, dx -$$

$$- \int_\Omega \omega^2\big(\hat{\rho}(x) + \rho^{in}(x_1)\big) u_r^*(x) \cdot v_r(x)\, dx = \int_\Omega \hat{\rho}(x) F_r(x) \cdot v_r(x)\, dx,$$

$$\text{for all} \quad v_r \in H_0^1(\Omega)^2, \quad \text{`}r\text{' stands for `}c\text{' and `}s\text{'}, \qquad (22)$$

$$\int_\Omega \Big\{\mathbb{A}_k(x)\nabla_x\theta_c^*(x) \cdot \nabla_x\vartheta_c(x) + \omega\big(T\mathbb{A}_\gamma(x) : \nabla_x u_s^*(x) +$$

$$+ m_{cv}(x)\theta_s^*(x)\big)\vartheta_c(x)\Big\}\, dx = \int_\Omega Q_c(x)\vartheta_c(x)\, dx \quad \text{for all} \quad \vartheta_c \in H_0^1(\Omega),$$

$$(23)$$

$$\int_\Omega \Big\{ \mathbb{A}_k(x)\nabla_x\theta_s^*(x)\cdot\nabla_x\vartheta_s(x) - \omega\big(T\mathbb{A}_\gamma(x) : \nabla_x u_c^*(x) +$$

$$+ m_{cv}(x)\theta_c^*(x)\big)\vartheta_s(x)\Big\}\, dx = \int_\Omega Q_s(x)\vartheta_s(x)\, dx, \quad \text{for all } \vartheta_s \in H_0^1(\Omega). \tag{24}$$

Thus, the distributions of displacement and temperature in Ω for $t \in \mathbb{R}$ have the forms

$$u^*(x,t) = u_c^* \cos\omega t + u_s^* \sin\omega t, \tag{25}$$

$$\theta^*(x,t) = \theta_c^* \cos\omega t + \theta_s^* \sin\omega t. \tag{26}$$

Variational formulation

$$\int_\Omega \Big\{ \big(\hat{\rho}(x) + \rho^{in}(x_1)\big) \frac{\partial^2 u^*(x,t)}{\partial t^2} \cdot v(x) + $$
$$+ \big(\mathcal{A}_\mu(x) : \nabla_x u^*(x,t) - \mathbb{A}_\gamma(x)\theta^*(x,t)\big) : \nabla_x v(x) \Big\} \, dx =$$
$$= \int_\Omega \hat{\rho}(x) F(x,t) \cdot v(x) \, dx \quad \text{for all} \quad v \in H_0^1(\Omega)^2, \quad t \in \mathbb{R}. \quad (27)$$

$$\int_\Omega \Big\{ m_{cv}(x) \frac{\partial \theta^*(x,t)}{\partial t} \vartheta(x) + $$
$$+ \mathbb{A}_k(x) \nabla_x \theta^*(x,t) \cdot \nabla_x \vartheta(x) + T \mathbb{A}_\gamma(x) \nabla_x \frac{\partial u^*(x,t)}{\partial t} \vartheta(x) \Big\} \, dx =$$
$$= \int_\Omega Q(x,t) \vartheta(x) \, dx \quad \text{for all} \quad \vartheta \in H_0^1(\Omega), \quad t \in \mathbb{R}. \quad (28)$$

Differential formulation

In the sense of distributions, these two integral equalities are equivalent to the partial differential equations

$$(\hat{\rho} + \rho^{in})\frac{\partial^2 u^*}{\partial t^2} - \mathsf{div}_x(\mathcal{A}_\mu : \nabla_x u^* - \mathbb{A}_\gamma \theta^*) = \hat{\rho} F, \quad x \in \Omega, \quad t \in \mathbb{R}, \tag{29}$$

$$m_{cv}\frac{\partial \theta^*}{\partial t} - \mathsf{div}_x(\mathbb{A}_k \nabla_x \theta^*) + T\mathbb{A}_\gamma \nabla_x \frac{\partial u^*}{\partial t} = Q, \quad x \in \Omega, \quad t \in \mathbb{R}, \tag{30}$$

respectively.

Here we use the following notation:
For a 1-periodic in ξ_2 function $\phi = \phi(x, \xi)$ by $\hat{\phi}$ we denote its mean in ξ_2 over the period $[0, 1)$:

$$\hat{\phi}(x, \xi_1) := \int_0^1 \phi(x, \xi)\, d\xi_2 \tag{31}$$

We denote

$$\Lambda_{\lambda^2}(x) := \int_0^1 \frac{\lambda^2(x, \xi_2)\, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}, \quad \Lambda_{\lambda}(x) := \int_0^1 \frac{\lambda(x, \xi_2)\, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)},$$
$$\Lambda_1(x) := \int_0^1 \frac{d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}, \quad \Lambda_{\lambda\gamma}(x) := \int_0^1 \frac{\lambda(x, \xi_2)\beta(x, \xi_2)\, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)},$$
$$\Lambda_{\gamma}(x) := \int_0^1 \frac{\beta(x, \xi_2)\, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}, \quad \Lambda_{\gamma^2}(x) := \int_0^1 \frac{\beta^2(x, \xi_2)\, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}.$$

With these notations, the components of tensor $\mathcal{A}_\mu = \left(a^\mu_{ijkl}(x)\right)_{i,j,k,l=1,2}$ are

$$
\begin{aligned}
a^\mu_{1111}(x) = 2\hat{\mu}(x) + \hat{\lambda}(x) + a^{in}_{11}(x_1) - \Lambda_{\lambda^2}(x) + \Lambda^2_\lambda \Lambda^{-1}_1(x), \\
a^\mu_{1122}(x) = a^\mu_{2211}(x) = \Lambda_\lambda(x)\Lambda^{-1}_1(x), \\
a^\mu_{1212}(x) = a^\mu_{2121}(x) = a^\mu_{1221}(x) = a^\mu_{2112}(x) = \widehat{(\mu^{-1})}(x), \\
a^\mu_{2222}(x) = \Lambda^{-1}_1(x), \\
a^\mu_{2111}(x) = a^\mu_{1211}(x) = a^\mu_{1112}(x) = a^\mu_{2212}(x) = a^\mu_{1121}(x) = \\
= a^\mu_{2221}(x) = a^\mu_{1222}(x) = a^\mu_{2122}(x) \equiv 0,
\end{aligned}
\tag{32}
$$

the matrix-valued functions $\mathbb{A}_\gamma$ and $\mathbb{A}_k$ have the representations

$$
\mathbb{A}_\gamma = \begin{pmatrix} \hat{\beta}(x) + b^{in}_1(x_1) + \Lambda_\lambda(x)\Lambda_\gamma(x)\Lambda^{-1}_1(x) - \Lambda_{\lambda\gamma}(x) & 0 \\ 0 & \Lambda_\gamma(x)\Lambda^{-1}_1(x) \end{pmatrix},
\tag{33}
$$

$$
\mathbb{A}_k = \begin{pmatrix} \hat{k}(x) + k^{in}(x_1) & 0 \\ 0 & \widehat{(k^{-1})}(x) \end{pmatrix},
\tag{34}
$$

and the scalar functions $\hat{\rho}$ and m_{cv} have the forms

$$\hat{\rho}(x) = \int_0^1 \rho(x, \xi_2)\, d\xi_2, \tag{35}$$

$$m_{cv}(x) = \hat{c}(x) + d^{in}(x_1) + T\Lambda_{\gamma^2}(x) - T\Lambda_\gamma^2(x)\Lambda_1^{-1}(x). \tag{36}$$

Theorem

(i) Let Conditions W hold and $\mathcal{A}_\mu, \mathbb{A}_\gamma, \mathbb{A}_k$, $\hat{\rho}$, *and* m_{cv} *be defined by formulas* (32)-(36). *Assume that* $F_c, F_s \in L^2(\Omega)^2, Q_c, Q_s \in L^2(\Omega)$. *Then there exists a unique solution* $(u_c^*, \theta_c^*, u_s^*, \theta_s^*) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H_0^1(\Omega)$ *to Problem H. Consequently, the displacement field* $u^* = u^*(x,t)$ *and the temperature field* $\theta^* = \theta^*(x,t)$ *are uniquely defined by formulas* (25) *and* (26), *respectively.*

(ii) Let Conditions W hold and $\mathcal{A}_\mu, \mathbb{A}_\gamma, \mathbb{A}_k$, $\hat{\rho}$, *and* m_{cv} *be defined by formulas* (32)-(36). *Then,*

(iia) the fourth-rank tensor $\mathcal{A}_\mu$ *satisfies the symmetry condition*

$$a^\mu_{ijkl} = a^\mu_{ijlk} = a^\mu_{klij} \tag{37}$$

and the positive-definiteness condition

$$(\mathcal{A}_\mu : \mathbb{X}) : \mathbb{X} \geq 0 \quad \text{in} \;\; \Omega, \quad \text{for all} \;\; \mathbb{X} = (X_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times 2}, \tag{38}$$

$$(\mathcal{A}_\mu : \mathbb{X}) : \mathbb{X} = 0 \quad \text{if and only if} \quad X_{kl} + X_{lk} = 0, \;\; k,l = 1,2; \tag{39}$$

Theorem

(iib) the 2×2*-matrices* $\mathbb{A}_\gamma$ *and* $\mathbb{A}_k$ *are uniformly positive definite, i.e., there are constants* $c_\gamma > 0$ *and* $c_k > 0$ *such that*

$$\mathbb{A}_\gamma \chi \cdot \chi \geq c_\gamma |\chi|^2 \quad \text{in} \quad \Omega, \quad \text{for all} \quad \chi \in \mathbb{R}^2; \tag{40}$$

$$\mathbb{A}_k \chi \cdot \chi \geq c_k |\chi|^2 \quad \text{in} \quad \Omega, \quad \text{for all} \quad \chi \in \mathbb{R}^2; \tag{41}$$

(iic) functions $\hat{\rho}$ *and* m_{cv} *are strictly positive, i.e., there are constants* $c_\rho > 0$ *and* $c_m > 0$ *such that*

$$\hat{\rho} \geq c_\rho \ (\text{in fact, } c_\rho = \rho_*), \quad m_{cv} \geq c_m \quad \text{in} \quad \Omega. \tag{42}$$

The toolbox of the method of two-scale convergence

The homogenization procedure for Problem B_ε as $\varepsilon \to 0+$, i.e., the limiting passage in the integral equalities (16)-(18), is based on implementation of the standard Allaire-Nguetseng method of two-scale convergence and its modification for homogenization on manifolds of minor dimension, proposed by G. Allaire, A. Damlamian, and U. Hornung.

Definition

Let $\{v^\varepsilon\}_{\varepsilon\to 0+}$ be a sequence in $L^2(\Omega)$. We say that $\{v^\varepsilon\}_{\varepsilon\to 0+}$ *two-scale converges* to a function $v_0 \in L^2(\Omega\times\Xi)$ if the limiting relation

$$\int\limits_\Omega v^\varepsilon(x)\,\varphi\Big(x,\frac{x}{\varepsilon}\Big)\,dx \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_\Omega\int\limits_\Xi v_0(x,\xi)\,\varphi(x,\xi)\,d\xi\,dx$$

holds for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.

Proposition

(Existence of two-scale convergent sequences.) *Assume* $\{v^\varepsilon\}_{\varepsilon>0}$ *is a bounded family in* $L^2(\Omega)$*; then there is a sequence* $\{v^{\varepsilon'}\}$ *and a function* $v_0 \in L^2(\Omega \times \Xi)$ *such that* $\{v^{\varepsilon'}\}$ *two-scale converges to* v_0 *as* $\varepsilon' \to 0+$*.*

Proposition

(Two-scale convergence of gradients.) *Assume* $\{v^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $H^1(\Omega)$ *such that* $v^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} v_0$ *weakly in* $H^1(\Omega)$*; then*

- **(i)** $\{v^\varepsilon\}$ *two-scale converges to* v_0*;*
- **(ii)** *there exist a subsequence* $\{\varepsilon' \to 0+\}$ *and a function* $v_1 = v_1(x, \xi)$ *belonging to* $L^2(\Omega; H^1_\sharp(\Xi))$ *such that*

$$\nabla v^{\varepsilon'} \underset{\varepsilon'\to 0+}{\longrightarrow} \nabla_x v_0 + \nabla_\xi v_1 .$$

Give a description of thin inclusions in a form suitable for using the two-scale convergence toolbox, and then present the necessary concepts and results on two-scale convergence on thin inclusions.

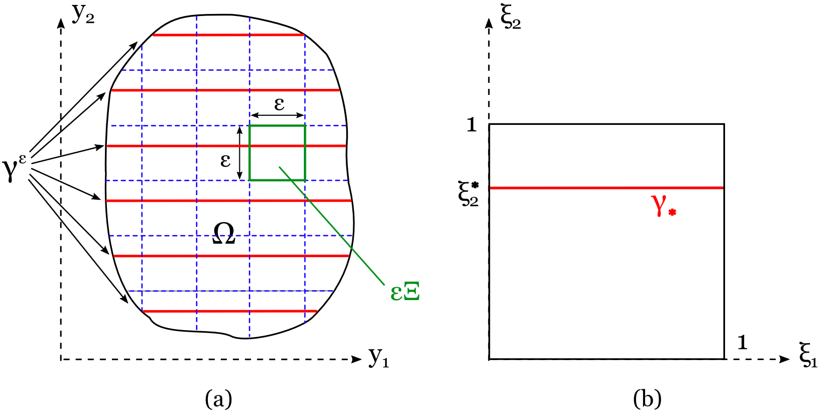

Figure: The regular ε-net and the periodicity cell

Definition

Let $\{w^\varepsilon\}_{\varepsilon\to 0+}$ be a sequence in $L^2(\gamma^\varepsilon)$. We call it *two-scale convergent to* $w_0 \in L^2(\Omega\times\gamma_*)$ (we have $w_0 = w_0(x,\xi_1,\xi_2^*)$) if the limiting relation

$$\varepsilon\int\limits_{\gamma^\varepsilon} w^\varepsilon(x)\varphi\Big(x,\frac{x}{\varepsilon}\Big)\,d\sigma^\varepsilon(x) \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_{\Omega}\int\limits_{\gamma_*} w_0(x,\xi_1,\xi_2^*)\,\varphi(x,\xi_1,\xi_2^*)\,d\xi_1\,dx$$

holds for all $\varphi\in C(\overline{\Omega};C_\sharp(\Xi))$.

Proposition

Assume $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $L^2(\gamma^\varepsilon)$ *such that*

$$\varepsilon^{1/2}\|w^\varepsilon\|_{L^2(\gamma^\varepsilon)} \le c_*,$$

where $c_* > 0$ *is independent of* ε*; then there exist a subsequence from* $\{\varepsilon\to 0+\}$*, still labeled by* ε*, and a limiting function* $w_0\in L^2(\Omega\times\gamma_*)$ *(*$w_0 = w_0(x,\xi_1,\xi_2^*)$*) such that the limiting relation*

$$w^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} w_0$$

two-scale converges.

Proposition

(i) *Assume* $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $H^1(\Omega)$ *such that*

$$\|w^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla_x w^\varepsilon\|_{L^2(\Omega)} \le c_{**},$$

where $c_{**} > 0$ *is independent of* ε*; then, for* $\varepsilon > 0$*, the trace of* w^ε *on* γ^ε *does exist and satisfies the bound*

$$\varepsilon \int\limits_{\gamma^\varepsilon} |w^\varepsilon(x)|^2 d\sigma^\varepsilon(x) \le c_{***},$$

where $c_{***} > 0$ *is independent of* ε*.*

(ii) *Let, in addition to hypotheses of item (i), the limiting relation*

$$\int\limits_{\Omega} w^\varepsilon(x)\,\varphi\Big(x, \frac{x}{\varepsilon}\Big)\,dx \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_{\Omega}\int\limits_{\Xi} w_0(x,\xi)\,\varphi(x,\xi)d\xi\,dx, \ \ \forall\,\varphi \in C(\overline{\Omega}; C_\sharp(\Xi)),$$

hold true with some function $w_0 \in L^2(\Omega; H^1_\sharp(\Xi))$*.*

Proposition

Then there exists a subsequence $\{\varepsilon' \to 0+\}$ *of* $\{\varepsilon \to 0+\}$ *such that the sequence of traces of* $w^{\varepsilon'}$ *on* $\gamma^{\varepsilon'}$ *converges to the trace of* w_0 *on* γ *two-scale as* $\varepsilon' \to 0+$, *i.e.,*

$$\varepsilon' \int\limits_{\gamma^{\varepsilon'}} w^{\varepsilon'}(x)\, \varphi\Big(x, \frac{x}{\varepsilon'}\Big)\, d\sigma^{\varepsilon'}(x) \underset{\varepsilon' \to 0+}{\longrightarrow} \int\limits_{\Omega} \int\limits_{\gamma_*} w_0(x, \xi_1, \xi_2^*)\, \varphi(x, \xi_1, \xi_2^*)\, d\xi_1\, dx,$$

for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.
(iii) *Furthermore, in hypotheses of items (i) and (ii), the limiting relation for the gradients holds true:*

$$\varepsilon' \int\limits_{\Omega} \nabla_x w^{\varepsilon'}(x) \cdot \Phi\Big(x, \frac{x}{\varepsilon'}\Big)\, dx \underset{\varepsilon' \to 0+}{\longrightarrow} \int\limits_{\Omega} \int\limits_{\Xi} \nabla_\xi w_0(x, \xi) \cdot \Phi(x, \xi)\, d\xi\, dx,$$

for all $\Phi \in C(\overline{\Omega}; C_\sharp(\Xi))^2$.