Multiscale Analysis of Stationary Thermoelastic Vibrations of a Composite Material

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Intoduction

Models of bonded structures in Elasticity: Dumont, Lebon, and Rizzoni (2018), Serpilli, Rizzoni, and Lebon (2019), Benveniste and Miloh (2001), Geymonat, Krasucki, and Lenci (1999), Schmidt (2008), Bonaldi, Geymonat, Krasucki, and Serpilli (2017), Serpilli (2018)
Models of thin inclusions in Elasticity: Simonenko (1974, 1975), Caillerie and Nedelec (1980), Khludnev, Rudoy, Shcherbakov, etc. (2014-2020), Baranova, Mogilevskaya, Mantic, and Jimenez-Alfaro (2020)
Homogenization of fiber composites: Panasenko G. (1990), Serpilli, M., Rizzoni, R., Dumont, S., Lebon, F. (2018, 2022), Charef H., Sili A. (2013), Bellieud M., Bouchitté G. (2002)

Thermoelastic wave model

By $u = (u_1, u_2)$ and $\sigma = \{\sigma_{ij}\}$ denote displacement vectors and strain tensors respectively. Then the equation of motion has the form

$$- \operatorname{div} \sigma(u, \theta) + \rho\, \partial_{tt} u = \rho F, \tag{1}$$

where $F = (F_1, F_2)$ is a given mass force, ρ is a media density. Linear constitutive equation (Duhamel–Neumann's Law) has the form

$$\sigma(u, \theta) = Ae(u) - B\theta, \tag{2}$$

where $e(u) = \{e_{ij}(u)\}$ is the strain tensor, $e_{ij}(u) = 1/2(\partial_{x_i} u_j + \partial_{x_j} u_i)$, θ is the temperature variation.
The heat conduction is described by the equations

$$- \operatorname{div} q(\theta) + c\, \partial_t \theta + TB : \partial_t e(u) = Q, \tag{3}$$

$$q(\theta) = K \nabla \theta, \tag{4}$$

where Q is the heat source.

We consider oscillations of the medium in time. We suppose that

$$\begin{aligned} F = F(x,t) &= F_c(x)\cos\omega t + F_s(x)\sin\omega t,\\ Q = Q(x,t) &= Q_c(x)\cos\omega t + Q_s(x)\sin\omega t,\\ u = u(x,t) &= u_c(x)\cos\omega t + u_s(x)\sin\omega t,\\ \theta = \theta(x,t) &= \theta_c(x)\cos\omega t + \theta_s(x)\sin\omega t, \end{aligned} \tag{5}$$

where $\omega \in \mathbb{R}$ is the oscillation frequency. Then equations (1)–(3) reduce to

$$\begin{aligned} -\operatorname{div}\sigma(u_c,\theta_c) - \rho\omega^2 u_c = \rho F_c, &\quad -\operatorname{div} q(\theta_c) + c\omega\theta_s + T\omega B : e(u_s) = Q_c,\\ -\operatorname{div}\sigma(u_s,\theta_s) - \rho\omega^2 u_s = \rho F_s, &\quad -\operatorname{div} q(\theta_s) - c\omega\theta_c - T\omega B : e(u_c) = Q_s, \end{aligned} \tag{6}$$

These are the basic relations of stationary vibrations of thermoelasticity.

Let the body occupy domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\Gamma = \partial\Omega$. Let Γ be separated into two parts Γ_D, Γ_N with the nonzero one-dimensional Hausdorff measure, on which the body edge is fixed and free. Also suppose that the edge Γ_D is isothermal, and there are no external heat sources on Γ_N. In such a case, the following boundary conditions are fulfilled

$$\begin{aligned} u_c = 0, \quad u_s = 0, \quad \theta_c = 0, \quad \theta_s = 0 \quad \text{on} \quad \Gamma_D, \\ \sigma(u_c, \theta_c)n = 0, \quad \sigma(u_s, \theta_s)n = 0, \quad q(\theta_c)n = 0, \quad q(\theta_s)n = 0 \quad \text{on} \quad \Gamma_N, \end{aligned} \tag{7}$$

where n is the unit outward normal vector to Γ. If F, Q, A, B, ρ, ω, K, c and T are given, equations (6) and (7) together provide the boundary value problem of finding u_c, u_s, θ_c, θ_s.
This problem is well-posed for some restrictions on initial data.

Model with a thin inclusion

In the two-dimensional space $\mathbb{R}^2$ referred to the coordinate system Oy_1y_2, let Ω be a bounded domain with a Lipschitz boundary $\partial\Omega$. Let $\gamma = \big(\Omega \cap \{y_2 = 0\}\big)$ be the intersection of Ω with axis Oy_2, which is supposed to have a positive one-dimensional measure:

$$\gamma = \big\{(y_1, y_2) \in \mathbb{R}^2 : \ 0 < y_1 < l^*, \ y_2 = 0\big\}, \quad l^* = \text{const} > 0.$$

Line segment γ divides Ω into subdomains $\Omega_\pm$ with Lipschitz boundaries $\partial\Omega_\pm$.

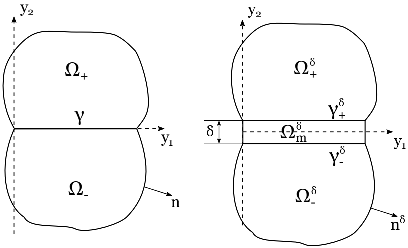

Figure: Left: γ divides Ω into two parts. Right: the assembly of thermoelastic bodies; $\Omega_\pm^\delta$ are adherents, Ω_m^δ is adhesive, parameter δ characterizes thickness of the adhesive

Let us introduce a small real parameter $\delta > 0$. We consider the assembly consisting of the three thermoelastic isotropic bodies. To describe this assembly, we introduce the following geometric objects:

$$\Omega_m^\delta = \{0 < y_1 < l^*\} \times \{-\delta/2 < y_2 < \delta/2\},$$
$$\Omega_\pm^\delta = \big\{(y_1, y_2) \in \mathbb{R}^2 \colon\ (y_1, y_2 \mp \delta/2) \in \Omega_\pm\big\},$$
$$\gamma_\pm^\delta = \gamma \pm (0, \delta/2),$$

which depend on the fixed parameter δ. Also, to the entire region occupied by the material, we relate the domain $\Omega^\delta = \Omega_+^\delta \cup \Omega_-^\delta \cup \Omega_m^\delta \cup \gamma_+^\delta \cup \gamma_-^\delta$ and introduce the sets $\Gamma_\pm^\delta = \partial\Omega^\delta \cap \partial\Omega_\pm^\delta$, which are the parts of the outer boundary $\partial\Omega^\delta$.

Problem A-diff

Consider the formulation of the problem that describes stationary thermoelastic waves in the bonded adhesive and adherents. The state of the composite body is described by the displacements functions u_c^δ, u_s^δ and the temperature variation functions θ_c^δ, θ_s^δ, defined in Ω^δ. The equations of the boundary value problem describing thermoelastic waves are

$$\left.\begin{array}{l} -\operatorname{div}\sigma(u_c^\delta,\theta_c^\delta)-\rho\omega^2 u_c^\delta=\rho F_c,\\ -\operatorname{div} q(\theta_c^\delta)+c\omega\theta_s^\delta+T\omega B:e(u_s^\delta)=Q_c,\\ -\operatorname{div}\sigma(u_s^\delta,\theta_s^\delta)-\rho\omega^2 u_s^\delta=\rho F_s,\\ -\operatorname{div} q(\theta_s^\delta)-c\omega\theta_c^\delta-T\omega B:e(u_c^\delta)=Q_s \end{array}\right. \quad \text{in } \Omega^\delta, \tag{8}$$

$$u_c^\delta=0,\quad u_s^\delta=0,\quad \theta_c^\delta=0,\quad \theta_s^\delta=0 \quad \text{on } \Gamma_D^\delta=\Gamma_+^\delta\cup\Gamma_-^\delta, \tag{9}$$

$$\sigma(u_c^\delta,\theta_c^\delta)n^\delta=0,\ \sigma(u_s^\delta,\theta_s^\delta)n^\delta=0,\ q(\theta_c^\delta)n^\delta=0,\ q(\theta_s^\delta)n^\delta=0 \text{ on } \partial\Omega^\delta\backslash\overline{\Gamma_D^\delta}, \tag{10}$$

where n^δ is the unit outward normal to $\partial\Omega^\delta$.

Problem B-diff

Suppose that the adhesive layer is hard with high conductivity:

$$\lambda, \mu, k, \rho, c \sim \frac{1}{\delta} \quad \text{in} \quad \Omega_m^\delta.$$

This is related with the desire to obtain a hard thin inclusion as a limit instead of the adhesive.

Find functions u_c, u_s, θ_c, θ_s, defined in Ω, such that

$$\begin{array}{ll} -\mathrm{div}\,\sigma(u_c,\theta_c) - \rho\omega^2 u_c = \rho\bar{F}_c, & -\mathrm{div}\,q(\theta_c) + c\omega\theta_s + T\omega B : e(u_s) = \bar{Q}_c, \\ -\mathrm{div}\,\sigma(u_s,\theta_s) - \rho\omega^2 u_s = \rho\bar{F}_s, & -\mathrm{div}\,q(\theta_s) - c\omega\theta_c - T\omega B : e(u_c) = \bar{Q}_s \end{array}, \tag{11}$$

$$u_c = 0, \quad u_s = 0, \quad \theta_c = 0, \quad \theta_s = 0 \ \text{ on } \ \partial\Omega, \tag{12}$$

$$\left.\begin{array}{r} [\sigma(u_c,\theta_c)\nu] = -\partial_{x_1}(a^{in}\partial_{x_1}u_c - b^{in}\theta_c) - \rho\omega^2 u_c, \\ [\sigma(u_s,\theta_s)\nu] = -\partial_{x_1}(a^{in}\partial_{x_1}u_s - b^{in}\theta_s) - \rho\omega^2 u_s, \\ [q(\theta_c)\nu] = -\partial_{x_1}(k^{in}\partial_{x_1}\theta_c) + d^{in}\omega\theta_s + T\omega b^{in}\partial_{x_1}u_s, \\ [q(\theta_s)\nu] = -\partial_{x_1}(k^{in}\partial_{x_1}\theta_s) - d^{in}\omega\theta_c - T\omega b^{in}\partial_{x_1}u_c \end{array}\right. \quad \text{on} \quad \gamma, \tag{13}$$

where $\nu = (0, 1)$, bracket $[\,\cdot\,]$ designates a jump on γ, tensor σ and vector q, as before, are calculated according to the constitutive relations, and scalars k^{in} and d^{in} and the elements of matrix a^{in} and vector b^{in} are the new coefficients, describing the material properties of the thin inclusion γ. We calculate:

$$a^{in} = \begin{pmatrix} 4\mu(\lambda+\mu)/(\lambda+2\mu) & 0 \\ 0 & 0 \end{pmatrix}, \quad b^{in} = \begin{pmatrix} 2\beta\mu/(\lambda+2\mu) \\ 0 \end{pmatrix}, \quad k^{in} = k, \quad d^{in} = c + T\beta^2/(\lambda+2\mu). \tag{14}$$

Generalization to any finite number of thin inclusions

Using the similar arguments (with natural modifications), as in the previous sections, we construct the well-posed model of stationary vibrations of thermoelastic body incorporating a family of thin inclusions $\gamma^{\varepsilon} = \Omega \cap \{x_2 = j\varepsilon,\ j \in \mathbb{Z}\}$, which are parallel to each other and spaced apart from each other at a distance of $\varepsilon > 0$, as on Figure 2. In this case, the essential geometric requirement is only that each of the subdomains, into which the domain Ω is divided by the set γ^{ε}, has a Lipschitz boundary.

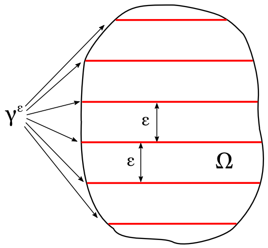

Figure: Several thin inclusions

Homogenization by the number of thin inclusions

Let thin inclusions occupy the set of physical positions

$$\gamma^\varepsilon = \Omega \cap \{x_2 = j\varepsilon,\, j \in \mathbb{Z}\}, \tag{15}$$

where $\varepsilon > 0$ is a dimensionless parameter characterizing the distance between two neighboring inclusions.
Introduce into considerations the space

$$V^\varepsilon = \Big\{(u,\theta) \in H^1_0(\Omega) \times H^1_0(\Omega) \colon\; u\big|_{\gamma^\varepsilon} \in H^1_0(\gamma^\varepsilon),\; \theta\big|_{\gamma^\varepsilon} \in H^1_0(\gamma^\varepsilon)\Big\}$$

with the standard norm and scalar product.
We consider that the coefficients in the model also depend on ε:

$$\begin{array}{c} \mu^\varepsilon := \mu(x, \frac{x_2}{\varepsilon}), \quad \lambda^\varepsilon := \lambda(x, \frac{x_2}{\varepsilon}), \quad \beta^\varepsilon := \beta(x, \frac{x_2}{\varepsilon}), \\ \rho^\varepsilon := \rho(x, \frac{x_2}{\varepsilon}), \quad k^\varepsilon := k(x, \frac{x_2}{\varepsilon}), \quad c^\varepsilon := c(x, \frac{x_2}{\varepsilon}) \end{array} \quad \text{in} \quad \Omega,$$

$$\begin{array}{r} a^{in}_\varepsilon := \varepsilon^p a^{in}(x_1), \quad b^{in}_\varepsilon := \varepsilon^p b^{in}(x_1), \quad \rho^\varepsilon := \varepsilon^p \rho^{in}(x_1), \\ k^{in}_\varepsilon := \varepsilon^p k^{in}(x_1), \quad d^{in}_\varepsilon := \varepsilon^q d^{in}(x_1) \end{array} \quad \text{on} \quad \gamma^\varepsilon,$$

where $\mu = \mu(x,\xi_2)$, $\lambda = \lambda(x,\xi_2)$, $\beta = \beta(x,\xi_2)$, $\rho = \rho(x,\xi_2)$, $k = k(x,\xi_2)$, and $c = c(x,\xi_2)$ are 1-periodic in ξ_2; parameters $p, q \in \mathbb{Z}$ are given.

Problem B_ε

For any fixed $\varepsilon > 0$ and $p, q \in \mathbb{Z}$, find a quadruple of functions $(u_c^\varepsilon, \theta_c^\varepsilon, u_s^\varepsilon, \theta_s^\varepsilon) \in V^\varepsilon \times V^\varepsilon$ satisfying the integral equalities

$$\int_\Omega \Big(2\mu(x, \frac{x_2}{\varepsilon})\mathbb{D}_x(u_r^\varepsilon) : \mathbb{D}_x(v_r) + \big(\lambda(x, \frac{x_2}{\varepsilon})\mathrm{div}_x u_r^\varepsilon - \beta(x, \frac{x_2}{\varepsilon})\theta_r^\varepsilon\big)\mathrm{div}_x v_r - \omega^2 \rho(x$$

$$+\varepsilon^p \int_{\gamma^\varepsilon} \Big(a_{11}^{in}(x_1)\partial_{x_1} u_{r1}^\varepsilon \partial_{x_1} v_{r1} - b_1^{in}(x_1)\theta_r^\varepsilon \partial_{x_1} v_{r1} - \omega^2 \rho^{in}(x_1) u_r^\varepsilon \cdot v_r\Big)\, d\nu^\varepsilon(x) =$$

$$= \int_\Omega \rho(x, \frac{x_2}{\varepsilon}) F_r(x) \cdot v_r(x)\, dx, \quad \text{`}r\text{' stands for `}c\text{' and `}s\text{'}, \quad (16)$$

$$\int_\Omega \Big(k(x, \frac{x_2}{\varepsilon})\nabla_x \theta_c^\varepsilon \cdot \nabla_x \vartheta_c + \omega c(x, \frac{x_2}{\varepsilon})\theta_s^\varepsilon \vartheta_c + T\omega\beta(x, \frac{x_2}{\varepsilon})(\mathrm{div}_x u_s^\varepsilon)\vartheta_c\Big)\, dx$$

$$+ \varepsilon^p \int_{\gamma^\varepsilon} \Big(k^{in}(x_1)\partial_{x_1}\theta_c^\varepsilon \partial_{x_1}\vartheta_c + T\omega b_1^{in}(x_1)(\partial_{x_1} u_{s1}^\varepsilon)\vartheta_c\Big)\, d\nu^\varepsilon(x)$$

$$+ \varepsilon^q \int \omega d^{in}(x_1)\theta_s^\varepsilon \vartheta_c\, d\nu^\varepsilon(x) = \int Q_c \vartheta_c(x)\, dx \quad (17)$$

$$\int_\Omega \Big(k(x,\frac{x_2}{\varepsilon})\nabla_x\theta_s^\varepsilon\cdot\nabla_x\vartheta_s-\omega c(x,\frac{x_2}{\varepsilon})\theta_c^\varepsilon\vartheta_s-T\omega\beta(x,\frac{x_2}{\varepsilon})(\mathrm{div}_x u_c^\varepsilon)\vartheta_s\Big)\,dx$$

$$+\,\varepsilon^p\int_{\gamma^\varepsilon}\Big(k^{in}(x_1)\partial_{x_1}\theta_s^\varepsilon\partial_{x_1}\vartheta_s-T\omega b_1^{in}(x_1)(\partial_{x_1}u_{c1}^\varepsilon)\vartheta_s\Big)\,d\nu^\varepsilon(x)$$

$$-\,\varepsilon^q\int_{\gamma^\varepsilon}\omega d^{in}(x_1)\theta_c^\varepsilon\vartheta_s\,d\nu^\varepsilon(x)=\int_\Omega Q_s\vartheta_s(x)\,dx \quad (18)$$

for all quadruples of test functions $(v_c,\vartheta_c,v_s,\vartheta_s)\in V^\varepsilon\times V^\varepsilon$. In (16) and further, $\mathbb{D}_x$ is the symmetric part of the gradient (i.e., $\mathbb{D}_x(\phi)=(1/2)(\nabla_x\phi+(\nabla_x\phi)^T)$) for any admissible vector-function ϕ), a_{11}^{in} and b_1^{in} are the only nonzero components of matrix a^{in} and vector b^{in}, respectively.

Some assumptions

We restrict ourselves to considering the case $p = q = 1$, so we set $p = q = 1$ everywhere further.

Conditions W. **1.** Functions $\mu, \lambda, \beta, \rho, k$, and c are differentiable in (x, ξ_2) and 1-periodic in ξ_2. Functions $a_{11}^{in}, b_1^{in}, \rho^{in}, k^{in}$, and d^{in} are differentiable in x_1.

2. There exist positive constants $\mu_*, \mu^*, \lambda_*, \lambda^*, \beta_*, \beta^*, \rho_*, \rho^*, k_*, k^*, c_*$, and c^* such that the bounds

$$\mu_* \le \mu(x, \xi_2) \le \mu^*, \quad \lambda_* \le \lambda(x, \xi_2) \le \lambda^*, \quad \beta_* \le \beta(x, \xi_2) \le \beta^*,$$
$$\rho_* \le \rho(x, \xi_2) \le \rho^*, \quad k_* \le k(x, \xi_2) \le k^*, \quad c_* \le c(x, \xi_2) \le c^*$$

hold for all $x \in \Omega, \xi_2 \in \mathbb{R}$ and the bounds

$$\mu_* \le a_{11}^{in}(x_1) \le \mu^*, \; \beta_* \le b_1^{in}(x_1) \le \beta^*, \; \rho_* \le \rho^{in}(x_1) \le \rho^*, \; k_* \le k^{in}(x_1) \le k^*$$

hold for all $x_1 \in [-l_1, l_1]$, where $l_1 = \max\limits_{(x_1', x_2') \in \overline{\Omega}} |x_1'|$.

3. The elastic stiffness and thermal conductivity properties dominate over frequency of vibrations and the linear thermal extension property in the following sense.
There exist positive constants c_1 and c_2 such that

$$\mu_* > \max\left\{\frac{1}{2}C_{Korn}^2\omega^2\rho^*, \frac{1}{2}C_{Korn}^2\omega^2\rho^* C_T, \frac{\beta^*}{2}\left(\frac{1}{c_1}+\frac{T\omega}{c_2}\right)+2C_{PF}^2\omega^2\rho^*\right\},$$

$$\lambda_* > \left(\frac{1}{c_1}+\frac{T\omega}{c_2}\right)\frac{\beta^*}{2}, \quad k_* > (c_1+T\omega c_2)C_{PF}^2\beta^*,$$

where C_{Korn} is the constant from Korn's inequality on Ω, C_{PF} is the constant from Poincare–Friedrichs inequality on Ω, and C_T is the constant from the estimate of trace on γ^ε. We note that C_{Korn}, $C_{PF} = \sqrt{2}l_1$ and $C_T = 4l_2^2$ do not depend on ε. Here, $l_2 = \max\limits_{(x_1', x_2')\in\overline{\Omega}} |x_2'|$.

Theorem 1. *Let Conditions W hold. Assume that* $F_c, F_s \in L^2(\Omega)^2, Q_c, Q_s \in L^2(\Omega)$. *Then for any fixed* $\varepsilon \in (0, \varepsilon_0]$ *there exists a unique solution* $(u_c^\varepsilon, \theta_c^\varepsilon, u_s^\varepsilon, \theta_s^\varepsilon) \in V^\varepsilon \times V^\varepsilon$ *to Problem* B_ε*. Moreover, there is a constant* $c_0 > 0$ *independent of* ε *such that the set of the uniform (in* ε*) estimates holds true:*

$$\|u_r^\varepsilon\|_{H_0^1(\Omega)^2} \leq c_0, \quad \|\theta_r^\varepsilon\|_{H_0^1(\Omega)^2} \leq c_0, \quad \varepsilon^{\frac{1}{2}}\|\partial_{x_1} u_{r1}^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \leq c_0,$$
$$\varepsilon^{\frac{1}{2}}\|u_r^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \leq c_0, \quad \varepsilon^{\frac{1}{2}}\|\partial_{x_1} \theta_r^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \leq c_0, \quad \varepsilon^{\frac{1}{2}}\|\theta_r^\varepsilon\|_{L^2(\gamma^\varepsilon)^2} \leq c_0, \tag{19}$$

where 'r*' stands for '*c*' and '*s*'.*

Theorem 2. *Assume that* $F_c, F_s \in L^2(\Omega)^2, Q_c, Q_s \in L^2(\Omega)$. *Then the family* $\{(u_c^\varepsilon, \theta_c^\varepsilon, u_s^\varepsilon, \theta_s^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]}$ *of solutions to Problem* B_ε *as* $\varepsilon \to 0+$ *tends to the solution* $(u_c^*, \theta_c^*, u_s^*, \theta_s^*)$ *of Problem H, stated below, in the sense of the limiting relations*

$$u_c^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} u_c^* \quad u_s^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} u_s^* \ \textit{weakly in}\ \ H_0^1(\Omega)^2,\ \textit{strongly in}\ \ L^2(\Omega)^2, \tag{20}$$

$$\theta_c^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} \theta_c^* \quad \theta_s^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} \theta_s^* \ \textit{weakly in}\ \ H_0^1(\Omega),\ \textit{strongly in}\ \ L^2(\Omega). \tag{21}$$

Stationary vibrations of the homogenized composite

Find amplitudes $u_c^*, u_s^* \in H_0^1(\Omega)^2$ of displacements and amplitudes $\theta_c^*, \theta_s^* \in H_0^1(\Omega)$ of temperature of periodic in time thermomechanical oscillations of a composite satisfying the system of integral equalities

$$\int\limits_{\Omega} \left\{\mathcal{A}_\mu(x) : \nabla_x u_r^*(x) - \mathbb{A}_\gamma(x)\theta_r^*(x)\right\} : \nabla_x v_r(x)\, dx -$$

$$- \int\limits_{\Omega} \omega^2 \big(\hat{\rho}(x) + \rho^{in}(x_1)\big) u_r^*(x) \cdot v_r(x)\, dx = \int\limits_{\Omega} \hat{\rho}(x) F_r(x) \cdot v_r(x)\, dx,$$

for all $v_r \in H_0^1(\Omega)^2$, 'r' stands for 'c' and 's', (22)

$$\int\limits_{\Omega} \Big\{\mathbb{A}_k(x)\nabla_x\theta_c^*(x) \cdot \nabla_x\vartheta_c(x) + \omega\big(T\mathbb{A}_\gamma(x) : \nabla_x u_s^*(x) +$$

$$+ m_{cv}(x)\theta_s^*(x)\big)\vartheta_c(x)\Big\}\, dx = \int\limits_{\Omega} Q_c(x)\vartheta_c(x)\, dx \quad \text{for all } \vartheta_c \in H_0^1(\Omega),$$

(23)

$$
\int_{\Omega} \Big\{ \mathbb{A}_k(x)\nabla_x\theta_s^*(x)\cdot\nabla_x\vartheta_s(x) - \omega\Big(T\mathbb{A}_\gamma(x):\nabla_x u_c^*(x) +
$$

$$
+ m_{cv}(x)\theta_c^*(x)\Big)\vartheta_s(x)\Big\}\, dx = \int_{\Omega} Q_s(x)\vartheta_s(x)\, dx, \quad \text{for all} \;\; \vartheta_s \in H_0^1(\Omega). \tag{24}
$$

Thus, the distributions of displacement and temperature in Ω for $t \in \mathbb{R}$ have the forms

$$
u^*(x,t) = u_c^* \cos\omega t + u_s^* \sin\omega t, \tag{25}
$$

$$
\theta^*(x,t) = \theta_c^* \cos\omega t + \theta_s^* \sin\omega t. \tag{26}
$$

Variational formulation

$$\int_\Omega \Big\{ \big(\hat\rho(x) + \rho^{in}(x_1)\big) \frac{\partial^2 u^*(x,t)}{\partial t^2} \cdot v(x) + \\ + \big(\mathcal{A}_\mu(x) : \nabla_x u^*(x,t) - \mathbb{A}_\gamma(x)\theta^*(x,t)\big) : \nabla_x v(x) \Big\}\, dx = \\ = \int_\Omega \hat\rho(x) F(x,t) \cdot v(x)\, dx \quad \text{for all} \quad v \in H_0^1(\Omega)^2, \quad t \in \mathbb{R}. \quad (27)$$

$$\int_\Omega \Big\{ m_{cv}(x) \frac{\partial \theta^*(x,t)}{\partial t} \vartheta(x) + \\ + \mathbb{A}_k(x) \nabla_x \theta^*(x,t) \cdot \nabla_x \vartheta(x) + T\mathbb{A}_\gamma(x) \nabla_x \frac{\partial u^*(x,t)}{\partial t} \vartheta(x) \Big\}\, dx = \\ = \int_\Omega Q(x,t) \vartheta(x)\, dx \quad \text{for all} \quad \vartheta \in H_0^1(\Omega), \quad t \in \mathbb{R}. \quad (28)$$

Differential formulation

In the sense of distributions, these two integral equalities are equivalent to the partial differential equations

$$(\hat{\rho} + \rho^{in}) \frac{\partial^2 u^*}{\partial t^2} - \mathrm{div}_x(\mathcal{A}_\mu : \nabla_x u^* - \mathbb{A}_\gamma \theta^*) = \hat{\rho} F, \quad x \in \Omega, \quad t \in \mathbb{R}, \tag{29}$$

$$m_{cv} \frac{\partial \theta^*}{\partial t} - \mathrm{div}_x(\mathbb{A}_k \nabla_x \theta^*) + T \mathbb{A}_\gamma \nabla_x \frac{\partial u^*}{\partial t} = Q, \quad x \in \Omega, \quad t \in \mathbb{R}, \tag{30}$$

respectively.

Here we use the following notation:
For a 1-periodic in ξ_2 function $\phi = \phi(x, \xi)$ by $\hat{\phi}$ we denote its mean in ξ_2 over the period $[0, 1)$:

$$\hat{\phi}(x, \xi_1) := \int_0^1 \phi(x, \xi) \, d\xi_2 \tag{31}$$

We denote

$$\Lambda_{\lambda^2}(x) := \int_0^1 \frac{\lambda^2(x, \xi_2) \, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}, \quad \Lambda_{\lambda}(x) := \int_0^1 \frac{\lambda(x, \xi_2) \, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)},$$

$$\Lambda_1(x) := \int_0^1 \frac{d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}, \quad \Lambda_{\lambda\gamma}(x) := \int_0^1 \frac{\lambda(x, \xi_2)\beta(x, \xi_2) \, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)},$$

$$\Lambda_{\gamma}(x) := \int_0^1 \frac{\beta(x, \xi_2) \, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}, \quad \Lambda_{\gamma^2}(x) := \int_0^1 \frac{\beta^2(x, \xi_2) \, d\xi_2}{2\mu(x, \xi_2) + \lambda(x, \xi_2)}.$$

With these notations, the components of tensor $\mathcal{A}_\mu = \left(a^\mu_{ijkl}(x)\right)_{i,j,k,l=1,2}$ are

$$
\begin{gathered}
a^\mu_{1111}(x) = 2\hat{\mu}(x) + \hat{\lambda}(x) + a^{in}_{11}(x_1) - \Lambda_{\lambda^2}(x) + \Lambda^2_\lambda \Lambda_1^{-1}(x), \\
a^\mu_{1122}(x) = a^\mu_{2211}(x) = \Lambda_\lambda(x)\Lambda_1^{-1}(x), \\
a^\mu_{1212}(x) = a^\mu_{2121}(x) = a^\mu_{1221}(x) = a^\mu_{2112}(x) = \widehat{(\mu^{-1})}(x), \\
a^\mu_{2222}(x) = \Lambda_1^{-1}(x), \\
a^\mu_{2111}(x) = a^\mu_{1211}(x) = a^\mu_{1112}(x) = a^\mu_{2212}(x) = a^\mu_{1121}(x) = \\
= a^\mu_{2221}(x) = a^\mu_{1222}(x) = a^\mu_{2122}(x) \equiv 0, \quad (32)
\end{gathered}
$$

the matrix-valued functions $\mathbb{A}_\gamma$ and $\mathbb{A}_k$ have the representations

$$
\mathbb{A}_\gamma = \begin{pmatrix} \hat{\beta}(x) + b^{in}_1(x_1) + \Lambda_\lambda(x)\Lambda_\gamma(x)\Lambda_1^{-1}(x) - \Lambda_{\lambda\gamma}(x) & 0 \\ 0 & \Lambda_\gamma(x)\Lambda_1^{-1}(x) \end{pmatrix}, \quad (33)
$$

$$
\mathbb{A}_k = \begin{pmatrix} \hat{k}(x) + k^{in}(x_1) & 0 \\ 0 & \widehat{(k^{-1})}(x) \end{pmatrix}, \quad (34)
$$

and the scalar functions $\hat{\rho}$ and m_{cv} have the forms

$$\hat{\rho}(x) = \int_0^1 \rho(x, \xi_2)\, d\xi_2, \qquad (35)$$

$$m_{cv}(x) = \hat{c}(x) + d^{in}(x_1) + T\Lambda_{\gamma^2}(x) - T\Lambda_\gamma^2(x)\Lambda_1^{-1}(x). \qquad (36)$$

Theorem

(i) Let Conditions W hold and $\mathcal{A}_\mu, \mathbb{A}_\gamma, \mathbb{A}_k$, $\hat{\rho}$, *and* m_{cv} *be defined by formulas* (32)-(36). *Assume that* $F_c, F_s \in L^2(\Omega)^2, Q_c, Q_s \in L^2(\Omega)$. *Then there exists a unique solution* $(u_c^*, \theta_c^*, u_s^*, \theta_s^*) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H_0^1(\Omega)$ *to Problem H. Consequently, the displacement field* $u^* = u^*(x,t)$ *and the temperature field* $\theta^* = \theta^*(x,t)$ *are uniquely defined by formulas* (25) *and* (26), *respectively.*

(ii) Let Conditions W hold and $\mathcal{A}_\mu, \mathbb{A}_\gamma, \mathbb{A}_k$, $\hat{\rho}$, *and* m_{cv} *be defined by formulas* (32)-(36). *Then,*

(iia) the fourth-rank tensor $\mathcal{A}_\mu$ *satisfies the symmetry condition*

$$a^\mu_{ijkl} = a^\mu_{ijlk} = a^\mu_{klij} \tag{37}$$

and the positive-definiteness condition

$$(\mathcal{A}_\mu : \mathbb{X}) : \mathbb{X} \geq 0 \quad \text{in} \quad \Omega, \quad \text{for all} \quad \mathbb{X} = (X_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times 2}, \tag{38}$$

$$(\mathcal{A}_\mu : \mathbb{X}) : \mathbb{X} = 0 \quad \text{if and only if} \quad X_{kl} + X_{lk} = 0,\ k,l = 1,2; \tag{39}$$

Theorem

(iib) the 2×2*-matrices* $\mathbb{A}_\gamma$ *and* $\mathbb{A}_k$ *are uniformly positive definite, i.e., there are constants* $c_\gamma > 0$ *and* $c_k > 0$ *such that*

$$\mathbb{A}_\gamma \chi \cdot \chi \geq c_\gamma |\chi|^2 \quad \textit{in} \quad \Omega, \quad \textit{for all} \quad \chi \in \mathbb{R}^2; \tag{40}$$

$$\mathbb{A}_k \chi \cdot \chi \geq c_k |\chi|^2 \quad \textit{in} \quad \Omega, \quad \textit{for all} \quad \chi \in \mathbb{R}^2; \tag{41}$$

(iic) functions $\hat{\rho}$ *and* m_{cv} *are strictly positive, i.e., there are constants* $c_\rho > 0$ *and* $c_m > 0$ *such that*

$$\hat{\rho} \geq c_\rho \; (\textit{in fact, } c_\rho = \rho_*), \quad m_{cv} \geq c_m \quad \textit{in} \quad \Omega. \tag{42}$$

The toolbox of the method of two-scale convergence

The homogenization procedure for Problem B_ε as $\varepsilon \to 0+$, i.e., the limiting passage in the integral equalities (16)-(18), is based on implementation of the standard Allaire-Nguetseng method of two-scale convergence and its modification for homogenization on manifolds of minor dimension, proposed by G. Allaire, A. Damlamian, and U. Hornung.

Definition

Let $\{v^\varepsilon\}_{\varepsilon\to 0+}$ be a sequence in $L^2(\Omega)$. We say that $\{v^\varepsilon\}_{\varepsilon\to 0+}$ *two-scale converges* to a function $v_0 \in L^2(\Omega \times \Xi)$ if the limiting relation

$$\int\limits_{\Omega} v^\varepsilon(x)\,\varphi\Big(x, \frac{x}{\varepsilon}\Big)\,dx \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_{\Omega}\int\limits_{\Xi} v_0(x,\xi)\,\varphi(x,\xi)\,d\xi\,dx$$

holds for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.

Proposition

(Existence of two-scale convergent sequences.) *Assume* $\{v^\varepsilon\}_{\varepsilon>0}$ *is a bounded family in* $L^2(\Omega)$*; then there is a sequence* $\{v^{\varepsilon'}\}$ *and a function* $v_0 \in L^2(\Omega \times \Xi)$ *such that* $\{v^{\varepsilon'}\}$ *two-scale converges to* v_0 *as* $\varepsilon' \to 0+$.

Proposition

(Two-scale convergence of gradients.) *Assume* $\{v^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $H^1(\Omega)$ *such that* $v^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} v_0$ *weakly in* $H^1(\Omega)$*; then*

- **(i)** $\{v^\varepsilon\}$ *two-scale converges to* v_0*;*
- **(ii)** *there exist a subsequence* $\{\varepsilon' \to 0+\}$ *and a function* $v_1 = v_1(x, \xi)$ *belonging to* $L^2(\Omega; H^1_\sharp(\Xi))$ *such that*

$$\nabla v^{\varepsilon'} \underset{\varepsilon'\to 0+}{\longrightarrow} \nabla_x v_0 + \nabla_\xi v_1.$$

Give a description of thin inclusions in a form suitable for using the two-scale convergence toolbox, and then present the necessary concepts and results on two-scale convergence on thin inclusions.

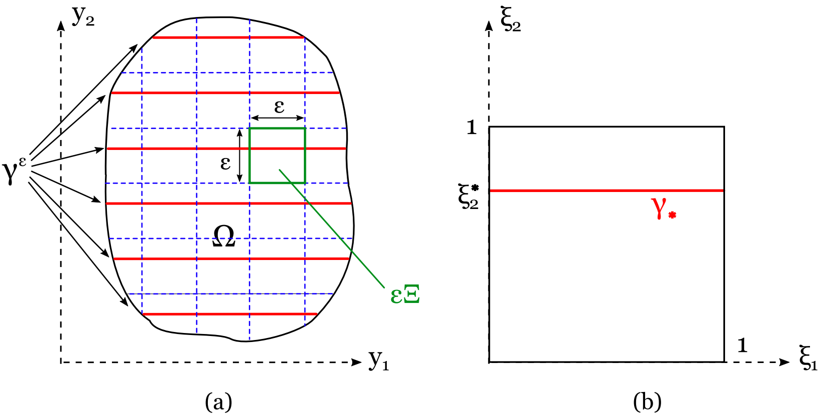

Figure: The regular ε-net and the periodicity cell

Definition

Let $\{w^\varepsilon\}_{\varepsilon\to 0+}$ be a sequence in $L^2(\gamma^\varepsilon)$. We call it *two-scale convergent to* $w_0 \in L^2(\Omega\times\gamma_*)$ (we have $w_0 = w_0(x,\xi_1,\xi_2^*)$) if the limiting relation

$$\varepsilon \int\limits_{\gamma^\varepsilon} w^\varepsilon(x)\varphi\Big(x,\frac{x}{\varepsilon}\Big)\, d\sigma^\varepsilon(x) \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_{\Omega}\int\limits_{\gamma_*} w_0(x,\xi_1,\xi_2^*)\,\varphi(x,\xi_1,\xi_2^*)\, d\xi_1\, dx$$

holds for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.

Proposition

Assume $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $L^2(\gamma^\varepsilon)$ *such that*

$$\varepsilon^{1/2}\|w^\varepsilon\|_{L^2(\gamma^\varepsilon)} \le c_*,$$

where $c_* > 0$ *is independent of* ε*; then there exist a subsequence from* $\{\varepsilon \to 0+\}$, *still labeled by* ε, *and a limiting function* $w_0 \in L^2(\Omega\times\gamma_*)$ *(*$w_0 = w_0(x,\xi_1,\xi_2^*)$*) such that the limiting relation*

$$w^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} w_0$$

two-scale converges.

Proposition

(i) *Assume* $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $H^1(\Omega)$ *such that*

$$\|w^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla_x w^\varepsilon\|_{L^2(\Omega)} \le c_{**},$$

where $c_{**} > 0$ *is independent of* ε*; then, for* $\varepsilon > 0$*, the trace of* w^ε *on* γ^ε *does exist and satisfies the bound*

$$\varepsilon \int\limits_{\gamma^\varepsilon} |w^\varepsilon(x)|^2 d\sigma^\varepsilon(x) \le c_{***},$$

where $c_{***} > 0$ *is independent of* ε*.*

(ii) *Let, in addition to hypotheses of item (i), the limiting relation*

$$\int\limits_{\Omega} w^\varepsilon(x)\,\varphi\Big(x, \frac{x}{\varepsilon}\Big)\,dx \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_{\Omega}\int\limits_{\Xi} w_0(x,\xi)\,\varphi(x,\xi)d\xi\,dx, \ \ \forall\,\varphi \in C(\overline{\Omega}; C_\sharp(\Xi)),$$

hold true with some function $w_0 \in L^2(\Omega; H^1_\sharp(\Xi))$*.*

Proposition

Then there exists a subsequence $\{\varepsilon' \to 0+\}$ *of* $\{\varepsilon \to 0+\}$ *such that the sequence of traces of* $w^{\varepsilon'}$ *on* $\gamma^{\varepsilon'}$ *converges to the trace of* w_0 *on* γ *two-scale as* $\varepsilon' \to 0+$*, i.e.,*

$$\varepsilon' \int_{\gamma^{\varepsilon'}} w^{\varepsilon'}(x)\, \varphi\Big(x, \frac{x}{\varepsilon'}\Big)\, d\sigma^{\varepsilon'}(x) \underset{\varepsilon' \to 0+}{\longrightarrow} \int_{\Omega} \int_{\gamma_*} w_0(x, \xi_1, \xi_2^*)\, \varphi(x, \xi_1, \xi_2^*)\, d\xi_1\, dx,$$

for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.

(iii) *Furthermore, in hypotheses of items (i) and (ii), the limiting relation for the gradients holds true:*

$$\varepsilon' \int_{\Omega} \nabla_x w^{\varepsilon'}(x) \cdot \Phi\Big(x, \frac{x}{\varepsilon'}\Big)\, dx \underset{\varepsilon' \to 0+}{\longrightarrow} \int_{\Omega} \int_{\Xi} \nabla_\xi w_0(x, \xi) \cdot \Phi(x, \xi)\, d\xi\, dx,$$

for all $\Phi \in C(\overline{\Omega}; C_\sharp(\Xi))^2$.