On self-similar solutions of a multi-phase Stefan problem

Evgeny Panov (Veliky Novgorod)

In a half-plane $t > 0$, $x \in \mathbb{R}$ we consider the Stefan problem for the heat equation

$$u_t = a_k^2 u_{xx}, \quad u_k < u < u_{k+1}, \tag{1}$$

where $u_- = u_0 < u_1 < \cdots < u_n < u_{n+1} = u_+$, u_k, $k = 1, \dots, n$ being the temperatures of phase transitions, $a_k > 0$, $k = 0, \dots, n$. On the unknown lines $x = x_k(t)$ of phase transitions the following Stefan condition

$$d_k x_k'(t) + a_{k+1}^2 u_x(t, x_k(t)+) - a_k^2 u_x(t, x_k(t)-) = 0, \quad d_k \geq 0, \tag{2}$$

is postulated. We will study the Cauchy problem with the Riemann initial data

$$u(0,x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{3}$$

After the change $w = u + \sum_{k=1}^n d_k \theta(u - u_k)$, where $\theta(z)$ is the Heaviside function, a solution of problem (1), (2), (3) transforms into a weak solution of a degenerate parabolic equation

$$w_t = (a^2(w) w_x)_x = A(w)_{xx}, \tag{4}$$

where $a(w) = a_k$ on intervals $(u_k + h_k, u_{k+1} + h_k)$, $a(w) = 0$ on $(u_{k+1} + h_k, u_{k+1} + h_{k+1})$, $h_k = \sum_{i=1}^k d_i$, $k = 0, \dots, n$ (we agree that $d_0 = d_{n+1} = 0$).

Moreover, the initial data (3) transforms into the Riemann data

$$w(0,x) = \begin{cases} u_-, & x < 0, \\ u_+ + h_n, & x > 0. \end{cases} \quad (5)$$

This reduction implies the existence and uniqueness of a solution to our Riemann-Stefan problem, cf

O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, Linear and Quasi-Linear Equations of Parabolic Type, AMS, Providence, RI, 1968.

(Chapter 5,§ 9). By the uniqueness and the invariance of our problem under the transformation group $(t,x) \to (\lambda^2 t, \lambda x)$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, a solution of problem (1), (2), (3) is self-similar: $u(t,x) = v(\xi)$, $\xi = x/\sqrt{t}$. For the heat equation $u_t = a^2 u_{xx}$ a self-similar solution must satisfy the linear ODE $a^2 v'' = -\xi v'/2$, the general solution of which is

$$v = C_1 F(\xi/a) + C_2,\ \ C_1, C_2 = \mathrm{const},\ \ \text{where } F(\xi) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\xi} e^{-s^2/4} ds.$$

This allows to write our solution in the form

$$v(\xi) = u_k + \frac{u_{k+1} - u_k}{F(\xi_{k+1}/a_k) - F(\xi_k/a_k)} (F(\xi/a_k) - F(\xi_k/a_k)), \quad (6)$$

$$\xi_k < \xi < \xi_{k+1},\ \ k = 0, \dots, n,$$

where $-\infty = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = +\infty$ and we agree that $F(-\infty) = 0$, $F(+\infty) = 1$. The parabolas $\xi = \xi_k$, $k = 1, \dots, n$, are free boundaries, which must be determined by conditions (2). In the variable ξ these conditions have the form

$$d_k\xi_k/2 + \frac{a_k(u_{k+1} - u_k)F'(\xi_k/a_k)}{F(\xi_{k+1}/a_k) - F(\xi_k/a_k)} - \frac{a_{k-1}(u_k - u_{k-1})F'(\xi_k/a_{k-1})}{F(\xi_k/a_{k-1}) - F(\xi_{k-1}/a_{k-1})} = 0, \ k = 1, \dots, n. \tag{7}$$

To prove that this nonlinear system has a solution, we notice that (7) coincides with the condition $\nabla E(\bar{\xi}) = 0$, where the function

$$E(\bar{\xi}) = -\sum_{k=0}^{n} (a_k)^2 (u_{k+1} - u_k) \ln(F(\xi_{k+1}/a_k) - F(\xi_k/a_k)) + \sum_{k=1}^{n} d_k \xi_k^2/4, \tag{8}$$
$$\bar{\xi} = (\xi_1, \dots, \xi_n) \in \Omega,$$

the open convex domain Ω is given by the inequalities $\xi_1 < \cdots < \xi_n$. Observe that $E(\bar{\xi}) \in C^\infty(\Omega)$. Since for all $k = 0, \dots, n$

$$\ln(F(\xi_{k+1}/a_k) - F(\xi_k/a_k)) < 0, \tag{9}$$

we find that $E(\bar{\xi}) > 0$. We will call the function $E(\bar{\xi})$ the entropy.

Lemma 1.

The sets $E(\bar{\xi}) \le c$ are compact for each $c \in \mathbb{R}$. In particular, the function $E(\bar{\xi})$ reaches its minimal value.

If $E(\bar{\xi}) \le c$ then it follows from (8) and (9) that for all $k = 0, \dots, n$

$$-(a_k)^2(u_{k+1} - u_k) \ln(F(\xi_{k+1}/a_k) - F(\xi_k/a_k)) \le E(\bar{\xi}) \le c,$$

and therefore

$$F(\xi_{k+1}/a_k) - F(\xi_k/a_k) \ge \delta \doteq \exp(-c/m) > 0, \tag{10}$$

where $m = \min\limits_{k=0,\dots,n} (a_k)^2(u_{k+1} - u_k) > 0$. In the case $k = 0, n$ it follows from this inequality that $F(\xi_1/a_0) \ge \delta$, $F(-\xi_n/a_n) = 1 - F(\xi_n/a_n) \ge \delta$. Hence, $-r \le \xi_1 < \xi_n \le r$, where a constant $r > 0$ satisfies the condition $\max(F(-r/a_0), F(-r/a_n)) \le \delta$. Since the remaining coordinates of the vector $\bar{\xi}$ lie between ξ_1 and ξ_n, we conclude that $|\bar{\xi}|_\infty \le r$. Further, since $F'(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4} < 1$, the function $F(x)$ is Lipschitz with constant 1, and it follows from (10) that

$$(\xi_{k+1} - \xi_k)/a_k \ge F(\xi_{k+1}/a_k) - F(\xi_k/a_k) \ge \delta, \quad k = 1, \dots, n-1,$$

and we obtain the estimates $\xi_{k+1} - \xi_k \geq \delta_1 = \delta \min a_k$. Thus, the set $E(\bar{\xi}) \leq c$ is contained in a compact

$$K = \{ \ \bar{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid |\bar{\xi}|_\infty \leq r, \ \xi_{k+1} - \xi_k \geq \delta_1 \ \forall k = 1, \ldots, n-1 \ \}.$$

Since $E(\bar{\xi})$ is continuous on K, the set $E(\bar{\xi}) \leq c$ is a closed subset of K and therefore is compact. For $c > N \doteq \inf E(\bar{\xi})$, this set is not empty and the entropy $E(\bar{\xi})$ reaches on it a minimal value, which is evidently equal N. $\square$

We have established the existence of minimal value $E(\bar{\xi}_0) = \min E(\bar{\xi})$. At the point $\bar{\xi}_0$ the required condition $\nabla E(\bar{\xi}_0) = 0$ is satisfied, and $\bar{\xi}_0$ is a solution of system (7).

The uniqueness of this solution follows from uniqueness of a solution of our Stefan problem. Alternatively, this uniqueness can be derived from the strict convexity of the entropy (then the entropy can have at most one critical point in Ω). In view of representation (8), the strict convexity of the entropy directly follows from the below lemma.

Lemma 2

The function $P(x, y) = -\ln(F(x) - F(y))$ is strictly convex in a half-plane $x > y$.

To prove the lemma, we must establish that the Hessian D^2P is positive definite at any point. By direct calculations we find

$$\frac{\partial^2}{\partial x^2}P(x,y) = \frac{(F'(x))^2 - F''(x)(F(x)-F(y))}{(F(x)-F(y))^2}, \ \frac{\partial^2}{\partial y^2}P(x,y) = \frac{(F'(y))^2 - F''(y)(F(y)-F(x))}{(F(x)-F(y))^2}, \ \frac{\partial^2}{\partial x \partial y}P(x,y) = -\frac{F'(x)F'(y)}{(F(x)-F(y))^2},$$

so that we have to prove the positive definiteness of the matrix $Q = (F(x)-F(y))^2 D^2 P(x,y)$ with the components

$$Q_{11} = (F'(x))^2 - F''(x)(F(x)-F(y)),$$
$$Q_{22} = (F'(y))^2 - F''(y)(F(y)-F(x)), \ Q_{12} = Q_{21} = -F'(x)F'(y).$$

Since $F''(x) = -\frac{x}{2}F'(x)$, then the diagonal elements of this matrix can be written in the form

$$Q_{11} = F'(x)(\frac{x}{2}(F(x)-F(y)) + F'(x)) = F'(x)(\frac{x}{2}(F(x)-F(y)) + (F'(x)-F'(y))) + F'(x)F'(y),$$
$$Q_{22} = F'(y)(\frac{y}{2}(F(y)-F(x)) + (F'(y)-F'(x))) + F'(x)F'(y).$$

By the Cauchy mean value theorem there exists such a value $z \in (y, x)$ that

$$\frac{F'(x) - F'(y)}{F(x) - F(y)} = \frac{F''(z)}{F'(z)} = -z/2.$$

Thus,

$$Q_{11} = F'(x)(F(x) - F(y))(x - z)/2 + F'(x)F'(y),$$
$$Q_{22} = F'(y)(F(x) - F(y))(z - y)/2 + F'(x)F'(y)$$

and therefore $Q = R_1 + F'(x)F'(y)R_2$, where R_1 is a diagonal matrix with the positive diagonal elements $F'(x)(F(x) - F(y))(x - z)/2$, $F'(y)(F(x) - F(y))(z - y)/2$ while $R_2 = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Since $R_1 > 0$, $R_2 \geq 0$, then the matrix $Q > 0$ (strictly positive definite), as was to be proved. □

With the help of Lemma 2 we establish that the entropy is indeed strictly convex and therefore has only one critical point, which is a minimum point. We conclude that free boundaries $\xi = \xi_k$, $k = 1, \dots, n$, are uniquely defined, and the problem (1), (2), (3) has a unique solution (6).

Adding to the entropy $E(\bar\xi)$ the constant

$$\sum_{k=0}^{n}(a_k)^2(u_{k+1}-u_k)\ln((u_{k+1}-u_k)/a_k),$$

we obtain the following variant of this function

$$E_1(\bar\xi) = -\sum_{k=0}^{n}(a_k)^2(u_{k+1}-u_k)\ln\left(\frac{F(\xi_{k+1}/a_k)-F(\xi_k/a_k)}{(u_{k+1}-u_k)/a_k}\right)+\sum_{k=1}^{n}d_k\xi_k^2/4$$
$$\approx -\sum_{k=0}^{n}(a_k)^2(u_{k+1}-u_k)\ln\left(F'(\xi_k/a_k)\frac{\xi_{k+1}-\xi_k}{u_{k+1}-u_k}\right)+\sum_{k=1}^{n}d_k\xi_k^2/4. \quad (11)$$

Now we consider the case when $a_k\approx a(u_k)$, $d_k\approx b(u_k)(u_{k+1}-u_k)$, $a(u), b(u)\in L^\infty(\mathbb{R})$, $a(u)>0, b(u)\ge 0$. Then, passing in the entropy function (11) to the limit as $\max(u_{k+1}-u_k)\to 0$, we obtain (at least formally) the integral functional

$$J(\xi)=\int_{u_-}^{u_+}[b(u)(\xi(u))^2/4-(a(u))^2\ln(F'(\xi(u)/a(u))\xi'(u))]du,$$

where $\xi(u)$ is expected to be the inverse function to the self-similar solution $u=u(\xi)$.

Taking into account that

$$\ln(F'(\xi(u)/a(u))\xi'(u)) = \ln F'(\xi(u)/a(u)) + \ln \xi'(u) = -\frac{(\xi(u))^2}{4(a(u))^2} + \ln \xi'(u),$$

we may write our functional in the form

$$J(\xi) = \int_{u_-}^{u_+} [(b(u)+1)(\xi(u))^2/4 - (a(u))^2 \ln \xi'(u)]du. \quad (12)$$

Let us write the corresponding Euler-Lagrange equation

$$(a^2/\xi')' + (b+1)\xi/2 = 0. \quad (13)$$

Notice that $1/\xi'(u) = u'(\xi)$. Therefore, after multiplication of (13) by $u'(\xi)$, we arrive at the relation

$$(a^2u')' + (b+1)\xi u'/2 = 0,$$

which is the semilinear diffusion equation $(b+1)u_t = (a^2(u)u_x)_x$, written in the self-similar variable $\xi = x/\sqrt{t}$.

Thank you for your attention!