INITIAL-BOUNDARY VALUE PROBLEMS FOR THE GENERALIZED KAWAHARA–ZAKHAROV–KUZNETSOV EQUATION

Andrei V. Faminskii (Moscow, Russia)

2nd Conference of Mathematical Centers of Russia

November 10, 2022

The famous Korteweg–de Vries equation (KdV)

$$u_t + u_{xxx} + bu_x + uu_x = 0$$

is the model equation describing propagation of one dimensional nonlinear waves in dispersive media. As well as the KdV equation the modified Korteweg–de Vries equation (mKdV)

$$u_t + u_{xxx} + bu_x + au^2 u_x = 0$$

is used in real physical situations. Moreover, one can consider wave processes with more complicated nonlinear effect and for its description use the generalized KdV equation

$$u_t + u_{xxx} + bu_x + g'(u)u_x = 0.$$

In the case of dispersive effects of higher orders the KdV equation can be substituted, for example, by the Kawahara equation

$$u_t - u_{xxxxx} + u_{xxx} + bu_x + uu_x = 0$$

or its generalized analogue

$$u_t - u_{xxxxx} + u_{xxx} + bu_x + g'(u)u_x = 0.$$

The Zakharov–Kuznetsov equation (ZK) in the case of two spatial dimensions

$$u_t + u_{xxx} + u_{xyy} + bu_x + uu_x = 0$$

extends the KdV model to the situation, when the waves propagate in one preassigned (x) direction with deformations in the transverse (y) direction.

As well as the mKdV equation one can consider the modified Zakharov–Kuznetsov equation

$$u_t + u_{xxx} + u_{xyy} + bu_x + au^2 u_x = 0,$$

which also has the physical meaning or more generally the generalized ZK equation

$$u_t + u_{xxx} + u_{xyy} + bu_x + g'(u)u_x = 0.$$

The Kawahara–Zakharov–Kuznetsov equation (KZK)

$$u_t - u_{xxxxx} + u_{xxx} + u_{xyy} + bu_x + uu_x = 0$$

is the natural model for two-dimensional nonlinear waves propagating in the media with the higher order dispersion. It is realized for description of electron-acoustic waves in a magnetized collisionless plasma.

Here we consider the version of the Kawahara–Zakharov–Kuznetsov equation with more general nonlinearity:

$$u_t - u_{xxxxx} + u_{xxx} + u_{xyy} + bu_x + g'(u)u_x = 0.$$

The goal of the study is to establish global existence and uniqueness of solutions to initial-boundary value problems for such equation as well as large-time decay under small input data. The obtained results are valid both for the KZK equation itself ($g'(u) = u$) and for the modified one ($g'(u) = au^2$).
Results on global existence are bases on estimates which are the analogues of the following conservation laws for the initial value problem

$$\iint_{\mathbb{R}^2} u^2\, dxdy = \text{const}, \quad \iint_{\mathbb{R}^2} \big(u_{xx}^2 + u_x^2 + u_y^2 - g^*(u)\big)\, dxdy = \text{const},$$

where $g^*(u) \equiv \int_0^u g(\theta)\, d\theta$. Further we differ the situations when only the analogue of the first conservation law is used and when all two of them are used and call solutions from the first class weak solutions, while from the second class – strong solutions.

Consider an initial-boundary value problem on a half-strip $\Sigma_+ = \mathbb{R}_+ \times (0, L) = \{(x, y) : x > 0, 0 < y < L\}$ $L > 0$ – arbitrary. We set an initial condition

$$u\big|_{t=0} = u_0(x, y), \qquad (x, y) \in \Sigma_+,$$

homogeneous boundary condition on the left boundary

$$u\big|_{x=0} = u_x\big|_{x=0} = 0, \qquad (t, y) \in B_T = (0, T) \times (0, L),$$

and homogeneous boundary conditions on the horizontal boundaries for $(t, x) \in (0, T) \times \mathbb{R}_+$ of two different types: either Dirichlet

$$u\big|_{y=0} = u\big|_{y=L} = 0$$

or Neumann

$$u_y\big|_{y=0} = u_y\big|_{y=L} = 0$$

(the results for periodic with respect to y are similar to the Neumann one and we'll not speak about them).

The choice of such a domain can be verified by the physical nature of the considered model, that is wave propagation in a channel of a finite width from the left boundary. Dirichlet boundary condition from the physical point of view corresponds to wave propagation processes with the absence of deformations on the boundaries of the channel, Neumann boundary condition is the no-flow condition through this boundaries, periodic boundary condition describes wave propagation in the media with periodic structure. Note that results on existence and uniqueness are the same for all types of boundary conditions.

The results are obtained in the weighted at $+\infty$ with respect to x anisotropic Sobolev spaces. The use of weighted spaces is crucial for the developed theory. Both power and exponential weights are allowed. Special interpolating inequalities in anisotropic weighted Sobolev spaces are established. In comparison with the Zakharov–Kuznetsov equation itself (or its modified analogue) the presence of the fifth derivative with respect to x provides more smooth solutions and allows to enlarge the growth rate of the nonlinearity, however, needs certain special technique.

For an arbitrary $T > 0$ we consider solutions to these problems in a domain $\Pi_T^+ = (0, T) \times \Sigma_+$. Let either $\psi(x) \equiv (1+x)^{2\alpha}$ or $\psi(x) \equiv e^{2\alpha x}$ for $\alpha > 0$,

$$L_2^{\psi(x)}(\Sigma_+) = \{\varphi(x,y) : \varphi\psi^{1/2} \in L_2(\Sigma_+)\}.$$

Introduce special function spaces $\widetilde{H}^{k,\psi(x)}(\Sigma_+)$ taking into account boundary conditions and the anisotropic character of the considered equation. For $k \geq 1$ let the space $\widetilde{H}^{k,\psi(x)}(\Sigma_+)$ consists of functions $\varphi(x,y)$ such that $\partial_x^{\nu_1}\partial_y^{\nu_2}\varphi \in L_2^{\psi(x)}(\Sigma_+)$ if $\nu_1 + 2\nu_2 \leq 2k$ and in the the Dirichlet case

$$\partial_y^{2m}\varphi|_{y=0} = \partial_y^{2m}\varphi|_{y=L} = 0 \quad \forall m < k/2,$$

while in the Neumann case

$$\partial_y^{2m+1}\varphi|_{y=0} = \partial_y^{2m+1}\varphi|_{y=L} = 0 \quad \forall m < (k-1)/2.$$

First consider weak solutions. We seek such solutions in the space $X_w^{\psi(x)}(\Pi_T^+)$, consisting of functions $u(t,x,y)$, such that

$$u \in C_w([0,T]; L_2^{\psi(x)}(\Sigma_+)) \cap L_2(0,T; \widetilde{H}^{1,\psi'(x)})$$

(the subscript w means the weak continuity). Subject the function $g \in C^1(\mathbb{R})$ to the following growth condition:

$$|g'(u)| \leq c|u|^p, \quad p \in (0,2].$$

Let $u_0 \in L_2^{\psi(x)}(\Sigma_+)$, then there exists a unique weak solution $u(t,x,y)$ from the class $X_w^{\psi(x)}(\Pi_T^+)$, where $\alpha > 0$ is arbitrary in the exponential case, and $\alpha \geq (3p+2)/(8p)$ in the power case. For example, if $g'(u) = u$ then $\alpha \geq 5/8$. For the ZK equation itself the analogous result was established in the power case for $\alpha \geq 1$, while for the mZK equation the satisfactory theory of weak solutions is absent in both cases.

Note that this result is irrelevant to the type of boundary conditions. Existence of global weak solutions is established also if $p \in [0, 8/3)$ and $\psi(x) \equiv 1$.

Besides the analog of the conservation law in L_2 which for the considered problems is written as follows:

$$\frac{d}{dt}\iint_{\Sigma_+} u^2\,dxdy + \int_0^L u_{xx}^2\big|_{x=0}\,dy = 0,$$

and provides an estimate

$$\|u(t,\cdot,\cdot)\|_{L_2(\Sigma_+)} \le \|u_0\|_{L_2(\Sigma_+)},$$

the global solubility in the class of weak solutions is based on the local-smoothing effect. Multiplying the ZK equation by $2u(t,x,y)\psi(x)$, where ψ is smooth positive increasing function, and integrating over Σ_+, one can easily derive an equality

$$\frac{d}{dt}\iint_{\Sigma_+} u^2\psi\,dxdy + \psi(0)\int_0^L u_{xx}^2\big|_{x=0}\,dy$$
$$+\iint_{\Sigma_+}(5u_{xx}^2+3u_x^2+u_y^2-bu^2)\psi'\,dxdy - \iint_{\Sigma_+}(5u_x^2+u^2)\psi'''\,dxdy$$
$$+\iint_{\Sigma_+} u^2\psi^{(5)}\,dxdy = \iint_{\Sigma_+}\big(g'(u)u\big)^*\psi'\,dxdy.$$

The weight function ψ is assumed to satisfy the following property:

$$|\psi^{(j)}(x)| \leq c(j)\psi(x) \quad \forall j \in \mathbb{N},\ \forall x \geq 0.$$

We call such functions admissible weight functions. Note that $\psi(x) \equiv e^{2\alpha x}$ and $\psi(x) \equiv (1+x)^{2\alpha}$ for $\alpha > 0$ are so, as well as their derivatives.

Then the terms in the left-hand side of this equality, containing ψ''' and $\psi^{(5)}$ can be subjected to the terms

$$\iint_{\Sigma_+} u^2\psi\, dxdy, \quad \iint_{\Sigma_+} (5u_{xx}^2 + 3u_x^2 + u_y^2)\psi'\, dxdy$$

in a proper sense. The main problem is to estimate the nonlinear term. Note that

$$\left|\left(g'(u)u\right)^*\right| \leq c|u|^{p+2}.$$

To this end we prove the following interpolating ienequality for the anisotropic Sobolev spaces. Let $\psi_1(x)$, $\psi_2(x)$ be two admissible weight functions, in addition verifying the following property: $\psi_j(x) \le c\psi_j(x')$ if $0 \le x \le x' \le x+1$ (of course, it holds for the exponential $e^{2\alpha x}$ and power $(1+x)^{2\alpha}$ weights). Let either $m=0$, $q \in [2,+\infty]$ or $m=1$, $q \in [2,6]$,

$$s = s(m,q) = \frac{2m+3}{8} - \frac{3}{4q}.$$

Then if $\varphi(0,y) \equiv 0$

$$\|\partial_x^m \varphi \psi_1^s \psi_2^{1/2-s}\|_{L_q(\Sigma_+)} \le c\big\|\big(|\varphi_{xx}| + |\varphi_y| + |\varphi|\big)\psi_1^{1/2}\|_{L_2(\Sigma_+)}^{2s} \|\varphi\psi_2^{1/2}\|_{L_2(\Sigma_+)}^{1-2s}.$$

If $\psi_1 = \psi_2 \equiv 1$ such inequality can be found in the book by O.V. Besov, V.P. Il'in and S.M. Nikolskii.

Let $\psi_1 \equiv \psi'$ (where either $\psi(x) \equiv e^{2\alpha x}$ or $\psi(x) \equiv (1+x)^{2\alpha}$ for $\alpha > 0$), $\psi_2 \equiv 1$, then with the use of this interpolating inequality we derive that for $q = p+2$, $s = s(0,q)$

$$\iint_{\Sigma_+} |u|^{p+2}\psi'\,dxdy \le c\Big(\iint_{\Sigma_+} (u_{xx}^2 + u_y^2 + u^2)\psi'\,dxdy\Big)^{qs} \times \Big(\iint_{\Sigma_+} u^2\psi'\,dxdy\Big)^{1-qs}\Big(\iint_{\Sigma_+} u^2\,dxdy\Big)^{p/2}.$$

Since the norm of the solution in $L_2(\Sigma_+)$ is already estimated, $qs = (3q-6)/8 < 1$ (remind that $q < 14/3$), we obtain that for an arbitrary small $\varepsilon > 0$

$$\Big|\iint_{\Sigma_+} (g'(u)u)^*\psi'\,dxdy\Big| \le \varepsilon \iint_{\Sigma_+} (u_{xx}^2 + u_y^2)\psi'\,dxdy + c(\varepsilon)\iint_{\Sigma_+} u^2\psi\,dxdy.$$

Therefore, the nonlinear term can be also subjected to the terms

$$\iint_{\Sigma_+} u^2\psi\,dxdy, \quad \iint_{\Sigma_+} (5u_{xx}^2 + 3u_x^2 + u_y^2)\psi'\,dxdy.$$

As a result we obtain the main global a priori estimate

$$\|u\|_{X_w^{\psi(x)}(\Pi_T^+)} = \sup_{t\in(0,T)} \|u(t,\cdot,\cdot)\psi^{1/2}\|_{L_2(\Sigma_+)} + \big\|\big(|u_{xx}| + |u_y|\big)(\psi')^{1/2}\big\|_{L_2(\Pi_T^+)} \le c.$$

Note that the result on existence of global weak solutions holds for any weight function $\psi(x) = e^{2\alpha x}$ and $\psi(x) = (1+x)^{2\alpha}$, $\alpha > 0$, and if

$$|g'(u)| \le c|u|^p, \quad p \in [0, 8/3).$$

Moreover the result on existence of global weak solutions also holds if $u_0 \in L_2(\Sigma_+)$.

Now consider uniqueness of weak solutions. Let $u, \widetilde{u} \in X_w^{\psi(x)}(\Pi_T^+)$ be two weak solutions to the considered problem, $w \equiv u - \widetilde{u}$, then multiplying the corresponding equation for the function w by $2w(t,x,y)\psi(x)$ and integrating over Σ_+ with use of the properties of admissible weight function, one derive an inequality

$$\frac{d}{dt}\iint_{\Sigma_+} w^2\psi\,dxdy + \iint_{\Sigma_+}(4w_{xx}^2 + 3w_x^2 + w_y^2)\psi'\,dxdy \\ \leq c\iint_{\Sigma_+} w^2\psi\,dxdy + 2\iint_{\Sigma_+}\big(g(u) - g(\widetilde{u})\big)(w\psi)_x\,dxdy.$$

Here $\big|g(u) - g(\widetilde{u})\big| \leq c\big(|u|^p + |\widetilde{u}|^p\big)|w|$. Assume that

$$\big(\psi'(x)\big)^{3p+2}\psi^{p-2}(x) \geq c_0 > 0 \quad \forall x \geq 0.$$

Then $(\psi/\psi') \leq c(\psi')^{3p/8}\psi^{p/8}$, therefore,

$$|u|^p|ww_x|\psi = |u|^p\big(\psi/(\psi')\big)^{1/4}\cdot|w_x|(\psi')^{1/4}\psi^{1/4}\cdot|w|\psi^{1/2} \\ \leq c\big(|u|(\psi')^{3/8}\psi^{1/8}\big)^p\cdot|w_x|(\psi')^{1/4}\psi^{1/4}\cdot|w|\psi^{1/2}.$$

Then with use of the interpolating inequality for the anisotropic Sobolev space one can derive that

$$\iint_{\Sigma_+} |u|^p|ww_x|\psi\,dxdy \le c\big\||u|(\psi')^{3/8}\psi^{1/8}\big\|^p_{L_\infty(\Sigma_+)}$$
$$\times\Big(\iint_{\Sigma_+} w_x^2(\psi')^{1/2}\psi^{1/2}\,dxdy \iint_{\Sigma_+} w^2\psi\,dxdy\Big)^{1/2}$$
$$\le \varepsilon\iint_{\Sigma_+}(w_{xx}^2+w_y^2+w^2)\psi'\,dxdy$$
$$+c(\varepsilon)\Big(\iint_{\Sigma_+}(u_{xx}^2+u_y^2+u^2)\psi'\,dxdy\Big)^{p/2}\iint_{\Sigma_+} w^2\psi\,dxdy,$$

where $\varepsilon > 0$ can be chosen arbitrarily small. Since $p/2 \le 1$ the the factor in the right-hand side, depending on u, belongs to $L_1(0,T)$ and the Gronwall inequality provides uniqueness.

Note that the crucial property of the admissible weight function

$$\big(\psi'(x)\big)^{3p+2}\psi^{p-2}(x) \ge c_0 > 0 \quad \forall x \ge 0.$$

holds in the exponential case $\psi(x) = e^{2\alpha x}$ for any $p \ge 0$ and all $\alpha > 0$, while in the power case $\psi(x) = (1+x)^{2\alpha}$ if $p > 0$ and $\alpha \ge (3p+8)/(8p)$.

Now let's summarize and formulate results on global exisrence and uniqueness of weak solutions in the cases of exponential and power weights.

Theorem (Exponential case, weak solutions)

Let $\psi(x) \equiv e^{2\alpha x}$ for certain $\alpha > 0$, $u_0 \in L_2^{\psi(x)}(\Sigma_+)$, $g \in C^1(\mathbb{R})$ and for certain $p \in [0, 8/3)$

$$|g'(u)| \le c|u|^p \quad \forall u \in \mathbb{R}.$$

Then for any $T > 0$ there exists a weak solution $u \in X_w^{\psi(x)}(\Pi_T^+)$. If, in addition, $p \le 2$ such a solution is unique.

Theorem (Power case, weak solutions)

Let $\psi(x) \equiv (1+x)^{2\alpha}$ *for certain* $\alpha > 0$, $u_0 \in L_2^{\psi(x)}(\Sigma_+)$, $g \in C^1(\mathbb{R})$ *and for certain* $p \in [0, 8/3)$

$$|g'(u)| \le c|u|^p \quad \forall u \in \mathbb{R}.$$

Then for any $T > 0$ *there exists a weak solution* $u \in X_w^{\psi(x)}(\Pi_T^+)$. *If, in addition,* $p \in (0, 2]$ *and* $\alpha \ge (3p+8)/(8p)$ *such a solution is unique.*

In the case of the Dirichlet boundary conditions

$$u\Big|_{y=0} = u\Big|_{y=L} = 0$$

and exponential weights we also establish a result on large-time decay of small weak solutions.

Theorem (Large-time decay of weak solutions, Dirichlet boundary conditions)

Let for $p \in (0,2]$

$$|g'(u)| \leq c|u|^p \quad \forall u \in \mathbb{R}.$$

Let $L_0 = +\infty$ *if* $b \leq 0$, *while for* $b > 0$ *there exists* $L_0 \in (0,+\infty)$ *such that in both cases for any* $L \in (0, L_0)$ *there exist* $\alpha_0 > 0$, $\epsilon > 0$ *and* $\beta > 0$ *such that if* $u_0 \in L_2^{\psi(x)}(\Sigma_+)$ *for* $\psi(x) \equiv e^{2\alpha x}$, $\alpha \in (0, \alpha_0]$, $\|u_0\|_{L_2(\Sigma_+)} \leq \epsilon$, *then for the unique weak solution* $u \in X_w^{\psi(x)}(\Pi_T^+)$ $\forall T > 0$

$$\|e^{\alpha x} u(t,\cdot,\cdot)\|_{L_2(\Sigma_+)} \leq e^{-\alpha\beta t} \|e^{\alpha x} u_0\|_{L_2(\Sigma_+)} \quad \forall t \geq 0.$$

The idea of the here goes back to N.A. Larkin and is based on the following argument. As usual multiply the equation by $2u(t,x,y)\psi(x)$, where $\psi(x) \equiv e^{2\alpha x}$, and integrate over Σ_+, then

$$\frac{d}{dt}\iint_{\Sigma_+} u^2\psi\,dxdy + 2\alpha\iint_{\Sigma_+}(5u_{xx}^2 + 3u_x^2 + u_y^2)\psi\,dxdy$$

$$+2\alpha(2-20\alpha^2)\iint_{\Sigma_+} u_x^2\psi\,dxdy + 2\alpha(16\alpha^4-4\alpha^2-b)\iint_{\Sigma_+} u^2\psi\,dxdy$$

$$\leq 2\alpha\iint_{\Sigma_+}\big(g'(u)u\big)^*\psi\,dxdy.$$

We know that

$$\|u(t,\cdot,\cdot)\|_{L_2(\Sigma_+)} \leq \|u_0\|_{L_2(\Sigma_+)}.$$

Then the nonlinear term can be estimated as follows:

$$2\iint_{\Sigma_+}\big(g'(u)u\big)^*\psi\,dxdy \leq \frac{1}{2}\iint_{\Sigma_+}(u_{xx}^2 + u_y^2)\psi\,dxdy$$

$$+ c^*\Big(\|u_0\|_{L_2(\Sigma_+)}^{8p/(8-3p)} + \|u_0\|_{L_2(\Sigma_+)}^{p}\Big)\iint_{\Sigma_+} u^2\psi\,dxdy.$$

We also apply the Steklov inequality

$$\frac{\pi^2}{L^2}\int_0^L u^2\,dy \le \int_0^L u_y^2\,dy,$$

and this is the reason, why the argument can be carried out only in the case of the Dirichlet boundary conditions.
Thus, uniformly with respect to L and small α

$$\frac{d}{dt}\iint_{\Sigma_+} u^2\psi\,dxdy + \alpha\iint_{\Sigma_+}(u_{xx}^2+u_x^2+u_y^2)\psi\,dxdy$$
$$+\alpha\Big(\frac{\pi^2}{2L^2}-2b-8\alpha^2-32\alpha^4-c^*\big(\|u_0\|_{L_2(\Sigma_+)}^{8p/(8-3p)}+\|u_0\|_{L_2(\Sigma_+)}^{p}\big)\Big)$$
$$\times\iint_{\Sigma_+} u^2\psi\,dxdy \le 0,$$

whence for small u_0 the result on exponential decay follows.

Now in brief consider strong solutions. We seek such solutions in the space $X_w^{1,\psi(x)}(\Pi_T^+)$, consisting of functions $u(t,x,y)$, such that

$$u \in C_w([0,T]; \widetilde{H}^{1,\psi(x)}(\Sigma_+)) \cap L_2(0,T; \widetilde{H}^{2,\psi'(x)}(\Sigma_+)),$$

that is, in particular,

$$u, u_{xx}, u_y \in C_w([0,T]; L_2^{\psi(x)}(\Sigma_+)),$$
$$u_{xxxx}, u_{xxy}, u_{yy} \in L_2(0,T; L_2^{\psi'(x)}(\Sigma_+)).$$

Theorem (Exponential case, strong solutions)

Let $\psi(x) \equiv e^{2\alpha x}$ *for certain* $\alpha > 0$, $u_0 \in \widetilde{H}^{1,\psi(x)}(\Sigma_+))$, $u(0,y) \equiv 0$, $g \in C^2(\mathbb{R})$ *and for certain* $p \in [1, 8/3)$

$$|g''(u)| \le c|u|^{p-1} \quad \forall u \in \mathbb{R}.$$

Then for any $T > 0$ *there exists a unique strong solution* $u \in X_w^{1,\psi(x)}(\Pi_T^+)$.

Theorem (Power case, strong solutions)

Let $\psi(x) \equiv (1+x)^{2\alpha}$ *for certain* $\alpha > 0$, $u_0 \in \widetilde{H}^{1,\psi(x)}(\Sigma_+))$, $u(0,y) \equiv 0$, $g \in C^2(\mathbb{R})$ *and for certain* $p \in [1, 8/3)$

$$|g''(u)| \leq c|u|^{p-1} \quad \forall u \in \mathbb{R}.$$

Then for any $T > 0$ *there exists a strong solution* $u \in X_w^{1,\psi(x)}(\Pi_T^+)$. *If, in addition,* $\alpha \geq 1/(8p)$ *such a solution is unique.*

The corresponding result on global existence of strong solutions is established also for $\psi(x) \equiv 1$ (that is if $u_0 \in \widetilde{H}^1(\Sigma_+)$).

The argument for the considered equation in comparison with, for example, the ZK equation is facilitated by the embedding $\widetilde{H}^1(\Sigma_+) \subset L_\infty(\Sigma_+)$.

In comparison with the previously established results for the similar problems for the Zakharov–Kuznetsov equation itself ($g'(u) = u$), the uniqueness assumption for the power weights $\alpha \geq 1/2$ is weakened here to the assumption $\alpha \geq 1/8$. For the exponential weights the result is the same.

In comparison with the previously established results for the similar problems for the modified Zakharov–Kuznetsov equation ($g'(u) = au^2$, $a = \pm 1$), the uniqueness assumption for the power weights $\alpha \geq 3/8$ is weakened here to the assumption $\alpha \geq 1/16$. For the exponential weights the uniqueness result is the same. However, the global existence result for the modified ZK equation in the focusing case $a = 1$ was established only for small input data both for the exponential and power cases.

In the case of the Dirichlet boundary conditions and exponential weights a result on large-time decay of small strong solutions is also valid.

Theorem (Large-time decay of strong solutions, Dirichlet boundary conditions)

Let for $p \in [1, 8/3)$

$$|g''(u)| \le c|u|^{p-1} \quad \forall u \in \mathbb{R}.$$

Let $L_0 = +\infty$ *if* $b \le 0$ *, while for* $b > 0$ *there exists* $L_0 \in (0, +\infty)$ *such that in both cases for any* $L \in (0, L_0)$ *there exist* $\alpha_0 > 0$ *,* $\epsilon > 0$ *and* $\beta > 0$ *such that if* $u_0 \in \widetilde{H}^{1,\psi(x)}(\Sigma_+)$ *for* $\psi(x) \equiv e^{2\alpha x}$ *,* $\alpha \in (0, \alpha_0]$ *,* $\|u_0\|_{L_2(\Sigma_+)} \le \epsilon$ *,* $u_0(0, y) \equiv 0$ *, then for the unique strong solution* $u \in X_w^{1,\psi(x)}(\Pi_T^+)$ $\forall T > 0$

$$\|e^{\alpha x} u(t, \cdot, \cdot)\|_{\widetilde{H}^1(\Sigma_+)} \le c e^{-\alpha\beta t} \quad \forall t \ge 0.$$

where the constant c *depends on* b *,* α *,* β *,* $\|u_0\|_{\widetilde{H}^{1,\psi(x)}(\Sigma_+)}$ *and the properties of the function* g *.*

Previously the initial-boundary value problem on the half-strip Σ_+ with the Dirichlet boundary conditions was considered by N.A. Larkin in 2014 for the Kawahara–Zakharov–Kuznetsov equation

$$u_t - u_{xxxxx} + u_{xxx} + u_{xyy} + u_x + uu_x = 0.$$

The weighted at $+\infty$ spaces with the exponential weights $\psi(x) = e^{2\alpha x}$, $0 < \alpha < \sqrt{3/20}$, were used. It was assumed that u_0, u_{0xx}, u_{0yy}, $\Delta u_{0x} - \partial_x^5 u_0$ lied in the space $L_2^{\psi(x)}(\Sigma_+)$. Then existence and uniqueness of global regular solutions (more regular than the ones, considered here) were established. Results on large-time decay of small solutions were also obtained.
No anisotropic Sobolev spaces were used by him.

The results were published in the following paper.

Faminskii A.V. Initial-boundary value problems on a half-strip for the generalized Kawahara–Zakharov–Kuznetsov equation / A.V. Faminskii // Zeitschrift für angewandte Mathematik und Physik. – 2022. – V. 73. – Art. 93.

THANK YOU FOR YOUR ATTENTION!