MÖBIUS TRANSFORMATIONS AND QUANTUM STOCHASTIC MODELS

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New Trends In Mathematical Physics,
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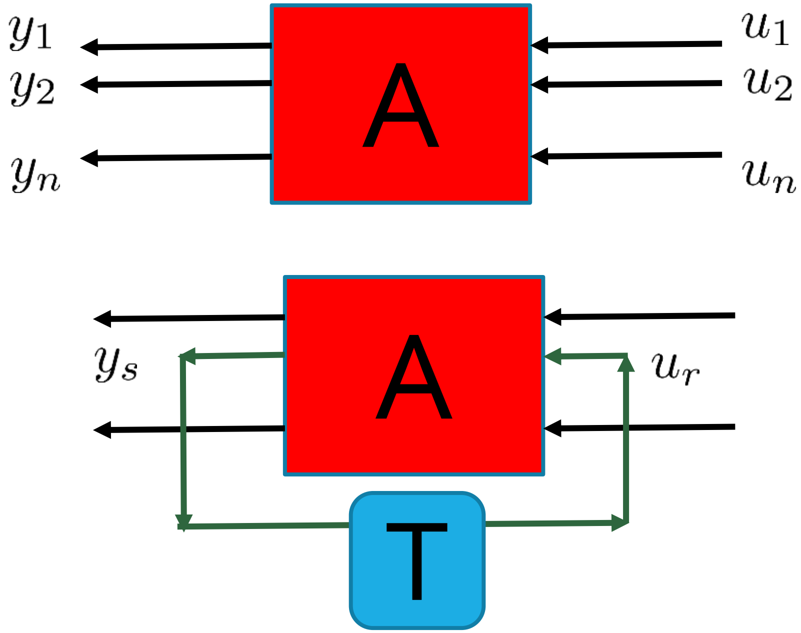
Möbius (Fractional Linear) Transformations in Quantum Stochastic Models

- Stratonovich (Symmetric) to Ito
- Feedback Networks
- Adiabatic Elimination

Fractional Linear Transformations

- Shorting

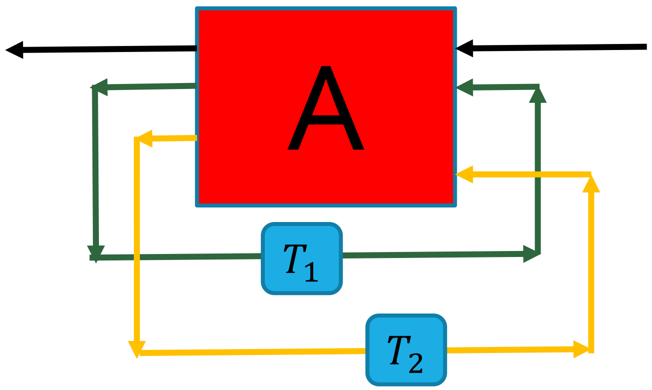


$$y_j = A_{jk} u_k$$

$$u_r = T y_s$$

$$[\Phi_{r,s}(A,T)]_{\alpha\beta} = A_{\alpha\beta} + A_{\alpha r} T (1 - A_{rs} T)^{-1} A_{s\beta}, \quad \alpha \neq s, \beta \neq r.$$

Commutativity of Shorting



$$\Phi_{r_1,s_2}(\cdot,T_1)\circ\Phi_{r_2,s_2}(\cdot,T_2)=\Phi_{r_2,s_2}(\cdot,T_2)\circ\Phi_{r_1,s_2}(\cdot,T_1)$$

$$=\Phi_{(r_1,r_2),(s_1,s_2)}\left(\cdot,\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}\right).$$

Siegel's Theorem

Theorem (C.L. Siegel, 1930's) Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be unitary on the direct sum of Hilbert spaces $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\|A_{22}\| < 1$.

Let T be unitary on $\mathfrak{h}_2$.

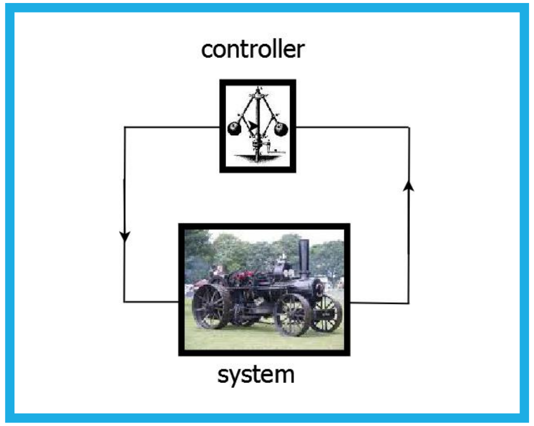
Then

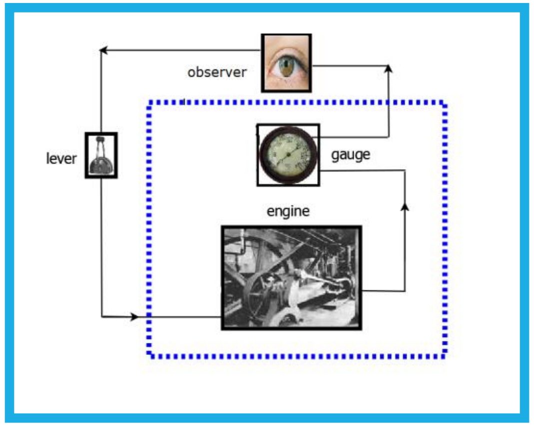
$$\Phi_{2,2}(A, T) = A_{11} + A_{12}T(I - A_{22}T)^{-1}A_{21}$$

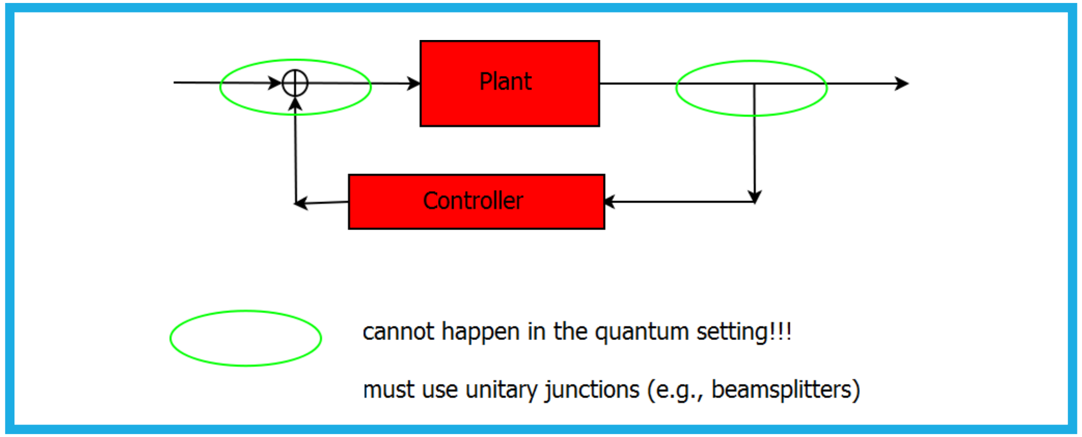
will be unitary on $\mathfrak{h}_1$.

Networks and Feedback Control

- (Left) Coherent Feedback Control
- (Right) Measurement-Based Feedback Control
- Classical Feedback Control System

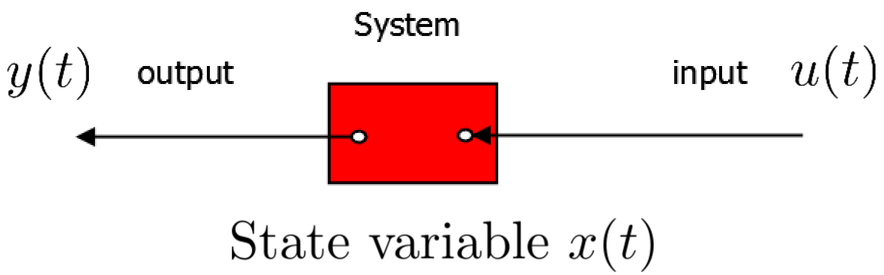






Transfer Functions

- Widely used for linear time-invariant systems in engineering.



$$\dot{x} = A\,x + B\,y,$$
$$y = C\,x + D\,u.$$

- In the Laplace domain

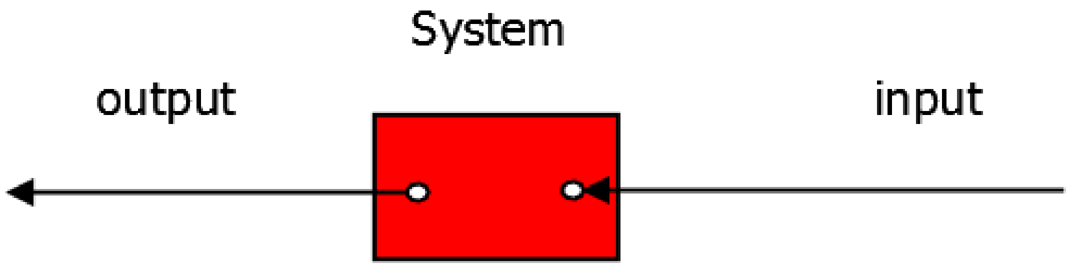
$$Y(s) = T(s)\,U(s),$$

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = D + C\,(sI - A)^{-1}\,B.$$

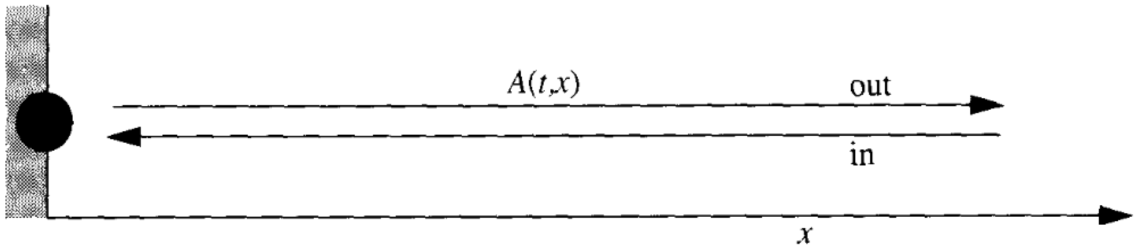
Quantum Input-Output Systems

Hudson-Parthasarathy (1984)

V.P. Belavkin (1979+)

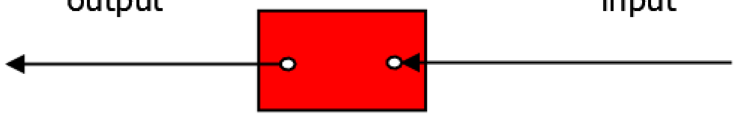


Gardiner-Collett (1985)



Quantum Input Processes

The “wires” are quantum fields!



- Field quanta of type *k* annihilated at the system at time *t*: $b_{\mathrm{in},k}(t)$

$$[b_{\mathrm{in},j}(t), b_{\mathrm{in},k}(s)^*] = \delta_{jk}\, \delta(t-s).$$

- Hudson-Parthasarathy (Fock Space):

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} \left(\otimes^{n}_{\mathrm{symm.}} \mathfrak{h}_{1\mathrm{P}} \right), \qquad \mathfrak{h}_{1\mathrm{P}} = \oplus_k L^2[0,\infty).$$

- Default state is the (Fock) vacuum $|\Omega\rangle$

$$b_{\mathrm{in},k}(t)\, |\Omega\rangle \equiv 0.$$

Quantum Input Processes

- Creation/annihilation processes

$$B_k^*(t) = \int_0^t b_k(s)^* ds, \qquad B_k(t) = \int_0^t b_k(s) ds$$

- Scattering processes

$$\Lambda_{jk}^*(t) = \int_0^t b_j(s)^* b_k(s) ds$$

Quantum Ito Table

- Table

$$dB_j dB_k^* = \delta_{jk} dt$$

$$d\Lambda_{jl} dB_k^* = \delta_{lk} dB_j^*$$
$$dB_j d\Lambda_{kl} = \delta_{jk} dB_l$$
$$d\Lambda_{jl} d\Lambda_{ki} = \delta_{lk} d\Lambda_{ji}$$

- Product Rule

$$d(XY) = dX(t)\,Y(t) + X(t)\,dY(t) + dX(t)\,dY(t).$$

Quantum Stochastic Models

Single input – Emission/Absorption Interaction

- Hudson-Parthasarathy equation:

$$dU(t) = L \otimes dB^*(t)U(t) - L^* \otimes dB(t)U(t) - (\tfrac{1}{2}L^*L + iH) \otimes dt\, U(t)$$

- Heisenberg Picture

$$j_t(X) = U(t)^*(X \otimes I)U(t)$$

$$dj_t(X) = j_t([L, X]) \otimes dB^*(t) + j_t([X, L^*]) \otimes dB(t) + j_t(\mathscr{L}X) \otimes dt$$

- Lindblad Generator

$$\mathscr{L}X = \frac{1}{2}L^*[X, L] + \frac{1}{2}[L^*, X]L - i[X, H]$$

Quantum Stochastic Models

- Output field:

$$B_{\text{out}}(t) = U(t)^* I \otimes B(t)\, U(t).$$

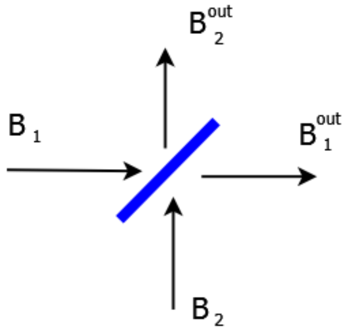
- Input-Output Relations

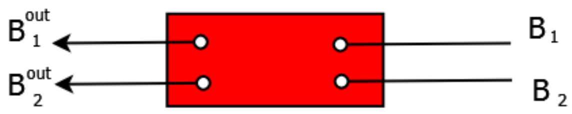
$$dB_{\text{out}}(t) = I \otimes dB(t) + j_t(L)dt.$$

$$b_{\text{out}}(t) = I \otimes b_{\text{in}}(t) + j_t(L)$$

Quantum Stochastic Models

- Two inputs – pure scattering





$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \qquad S^{-1} = S^*$$

Hudson-Parthasarathy form: $$dU(t) = \sum_{j,k}(S_{jk} - \delta_{jk}) \otimes d\Lambda_{jk}(t)U(t).$$

Heisenberg Picture $$dj_t(X) = \sum_{j,k} j_t\left(\sum_l S^*_{lk} X S_{lk} - \delta_{jk} X\right) \otimes d\Lambda_{jk}(t).$$

Input-Output Relations $$b_{\text{out},j}(t) = \sum_k j_t(S_{jk})\, b_{\text{in},k}(t)$$

SLH Formalism

- Hamiltonian *H* $$H^* = H$$
- Coupling/Collapse Operators *L* $$L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$
- Scattering Operator *S* $$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \qquad S^{-1} = S^*$$

Quantum Stochastic Models

- General (S , L , H) case

Hudson-Parthasarathy unitary evolution - QSDE (quantum Ito stochastic calculus)

$$
\begin{aligned}
dU(t) \quad &= \quad \Big\{ (S_{jk} - \delta_{jk} I) \otimes d\Lambda_{jk}(t) + L_j \otimes dB_j^*(t) \\
&\quad - L_j^* S_{jk} \otimes dB_k(t) - (\frac{1}{2} L_k^* L_k + iH) \otimes dt \Big\} U(t)
\end{aligned}
$$

Quantum Stochastic Models

Heisenberg Picture

$$dj_t(X) = j_t(S_{lj}^* X S_{lk} - \delta_{jk} X) d\Lambda_{jk}(t) + j_t(S_{lj}^* [L_l, X]) dB_{\text{in},j}^*(t) + j_t([X, L_l^*] S_{lk}) dB_{\text{in},k(t)} + j_t(\mathscr{L} X) dt.$$

Lindblad Generator

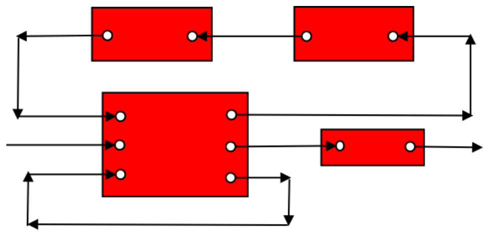
$$\mathscr{L} X = \frac{1}{2} L_k^* [X, L_k] + \frac{1}{2} [L_k^*, X] L_k - i[X, H]$$

Input-Output Relations

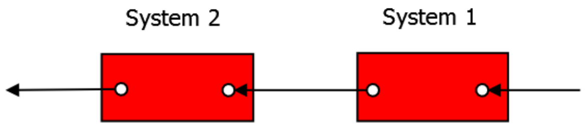
$$dB_{\text{out},j}(t) = j_t(S_{jk}) dB_{\text{in},k}(t) + j_t(L_j) dt$$

Quantum Networks

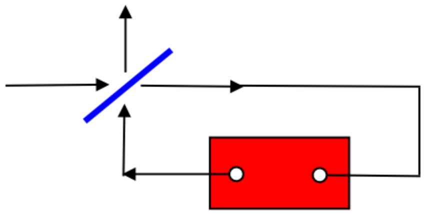
- How to connect models?



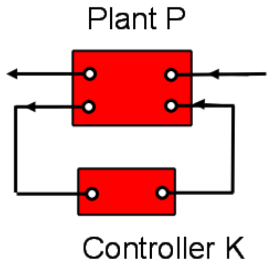
- Cascaded models



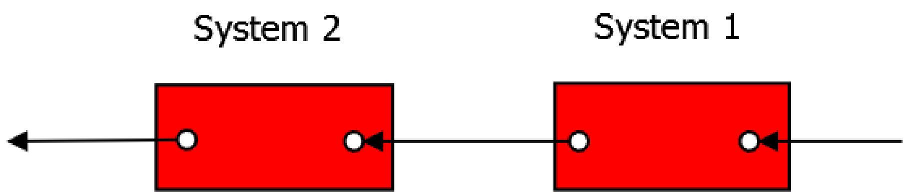
- Algebraic loops



- Feedback Control



The Series Product



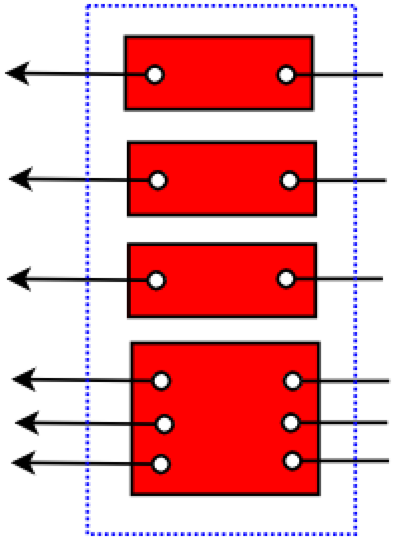
The cascaded system in the **instantaneous feedforward** limit is equivalent to the single component

$$(S_2, L_2, H_2) \triangleleft (S_1, L_1, H_1) = \left(S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \mathrm{Im}\left\{L_2^\dagger S_2 L_1\right\}\right).$$

J. G., M.R. James, *The Series Product and Its Application to Quantum Feedforward and Feedback Networks* IEEE Transactions on Automatic Control, 2009.

Network Rule # 1

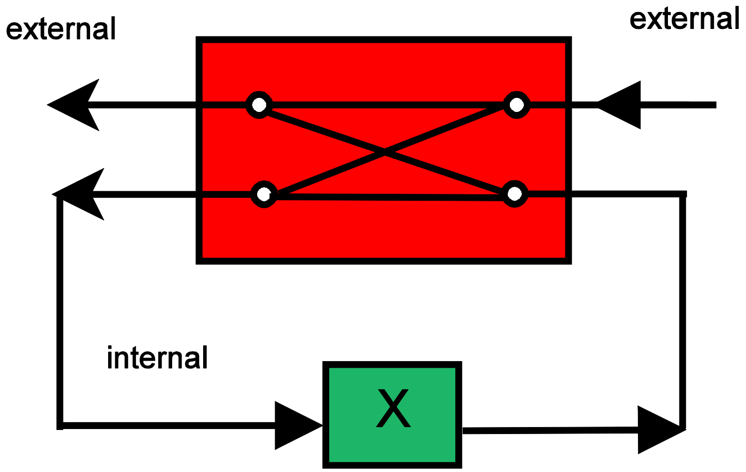
Open loop systems in parallel



Models $(S_j, L_j, H_j)_{j=1}^{n}$ in parallel

$$\left(\begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H_1 + \cdots + H_n \right).$$

Network Rule # 2
Feedback Reduction Formula



$$S = \begin{bmatrix} \mathsf{S}_{\mathtt{ii}} & \mathsf{S}_{\mathtt{ie}} \\ \mathsf{S}_{\mathtt{ei}} & \mathsf{S}_{\mathtt{ee}} \end{bmatrix}, \ \mathsf{L} = \begin{bmatrix} \mathsf{L}_{\mathtt{i}} \\ \mathsf{L}_{\mathtt{e}} \end{bmatrix}$$

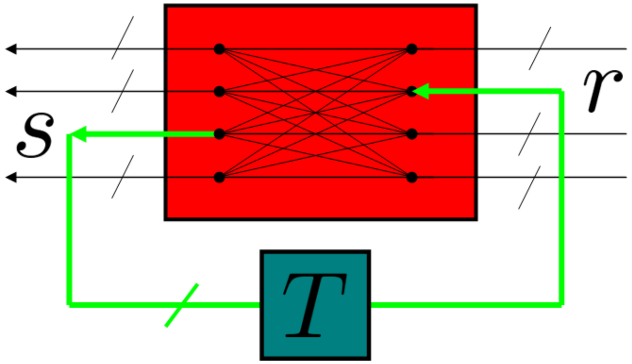
The reduced model obtained by eliminating all the internal channels (instantaneous feedback) is determined by the operators $(\mathsf{S}^{\mathrm{fb}}, \mathsf{L}^{\mathrm{fb}}, \mathsf{H}^{\mathrm{fb}})$ given by

$$\begin{aligned} \mathsf{S}^{\mathrm{fb}} &= \mathsf{S}_{\mathtt{ee}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{S}_{\mathtt{ie}}, \\ \mathsf{L}^{\mathrm{fb}} &= \mathsf{L}_{\mathtt{e}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}, \\ \mathsf{H}^{\mathrm{fb}} &= \mathsf{H} + \sum_{i=\mathtt{i},\mathtt{e}} \mathrm{Im} \mathsf{L}_j^{\dagger} X \mathsf{S}_{j\mathtt{i}} \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}. \end{aligned}$$

J. G., M.R. James, *Quantum Feedback Networks: Hamiltonian Formulation*, Commun. Math. Phys., 1109-1132, Volume 287, Number 3 / May, 2009.

$$\mathsf{V} = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sum_j L_j^* L_j - iH & -\sum_j L_j^* S_{j1} & \cdots & -\sum_j L_j^* S_{jm} \\ L_1 & S_{11} & \cdots & S_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ L_n & S_{n1} & \cdots & S_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \mathsf{V}_{00} & \mathsf{V}_{01} & \cdots & \mathsf{V}_{0m} \\ \mathsf{V}_{10} & \mathsf{V}_{11} & \cdots & \mathsf{V}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{V}_{n0} & \mathsf{V}_{n1} & \cdots & \mathsf{V}_{nn} \end{bmatrix}.$$



The feedback reduction formula is

$$\left[\mathscr{F}_{(r,s)}(\mathsf{V},T)\right]_{\alpha\beta} = \mathsf{V}_{\alpha\beta} - \mathsf{V}_{\alpha r} T \left(1 - \mathsf{V}_{rs} T\right)^{-1} \mathsf{V}_{s\beta}$$

Properties of the Feedback Reduction Formula

- Mathematically a Schur complement of the matrix of coefficient operators:

$$\mathbf{G} = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S - I \end{bmatrix}.$$

- Equivalently formulated as a fractional linear transformation.
- Independent of the order of edge-elimination.

Transfer Operator

- Quantum Transfer function

For a fixed set of coupling parameters (S, L, H) we define the corresponding *transfer operator*

$$\mathcal{T}(s) = \left[\begin{array}{c|c} -\frac{1}{2}L^\dagger L - iH & -L^* S \\ \hline L & S \end{array}\right](s)$$
$$= S - L(sI + \frac{1}{2}L^* L + iH)^{-1} L^\dagger S$$

- Properties

The transfer operator $\mathcal{T}(s)$ is well-defined for $\operatorname{Re} s > 0$. For all $\omega \in \mathbb{R}$, such that $i\omega \in \rho(\frac{1}{2}L^* L + iH)$, we have $\mathcal{T}(\mathrm{i}\omega)$ well-defined and unitary:

$$\mathcal{T}(i\omega)^\dagger \mathcal{T}(i\omega) = \mathcal{T}(i\omega)\mathcal{T}(i\omega)^\dagger = I.$$

(All-Pass Representation) *The characteristic operator admits the following "all-pass" representation:*

$$\mathcal{T}(s) = \frac{1 - \frac{1}{2}\Sigma(s)}{1 + \frac{1}{2}\Sigma(s)} S,$$

where $\Sigma(s) = L(s + iH)^{-1} L^*$.

Corollary *Suppose that the model parameters satisfy the condition* $[L, H] \equiv 0$, *then the characteristic operator takes the form*

$$\mathcal{T}(s) = \frac{s - \frac{1}{2}LL^* + iH}{s + \frac{1}{2}LL^* + iH} S.$$

Proposition *For any unitary* V *on the plant Hilbert space, the HP parameters* (S, LV, V^*HV) *generate the same characteristic operator as* (S, L, H). *More generally we have the following invariance property of the characteristic function:*

$$\left[\begin{array}{c||c} A & B \\ \hline\hline C & D \end{array}\right] = \left[\begin{array}{c||c} V^*AV & V^*B \\ \hline\hline CV & D \end{array}\right].$$

Model Reduction/Singular Perturbations

- Adiabatic Elimination

- Cavity mode *b* strongly coupled to input field: $\mathfrak{h}_{\text{sys}} \equiv \mathfrak{h}_{\text{atom}} \otimes \mathfrak{h}_{\text{mode}}$

$$H = H_{\text{atom}} \otimes I_{\text{mode}} + E \otimes b^* b + F \otimes b^* + F^* \otimes b, \qquad L = k\sqrt{\gamma} b. \ (k \to \infty)$$

- Limit QSDE on reduced system space (atom only!)

$$\tilde{S} = \frac{\gamma/2 - iE}{\gamma/2 + iE} \qquad \tilde{L} = \frac{i\sqrt{\gamma}}{\gamma/2 + iE} F, \qquad \tilde{H} = H_{\text{atom}} + F^* \text{Im}\{\frac{1}{\gamma/2 + iE}\} F.$$

JG, R. van Handel, Journ. Stat. Phys. 127, no.3 (2007)

Singular Perturbations

- General Case $\quad \mathfrak{h} = \mathfrak{h}_{\text{slow}} \oplus \mathfrak{h}_{\text{fast}}$ $\quad$ Decomposition $\quad X = \begin{bmatrix} X_{\mathtt{ss}} & X_{\mathtt{sf}} \\ X_{\mathtt{fs}} & X_{\mathtt{ff}} \end{bmatrix}.$
- Scaling

$$L(k) \equiv k \begin{bmatrix} 0 & L_{\mathtt{sf}}^{(1)} \\ 0 & L_{\mathtt{ff}}^{(1)} \end{bmatrix} + L^{(0)}$$

$$H(k) \equiv \begin{bmatrix} H_{\mathtt{ss}}^{(0)}, & H_{\mathtt{sf}}^{(0)} + kH_{\mathtt{sf}}^{(1)} \\ H_{\mathtt{fs}}^{(0)} + kH_{\mathtt{fs}}^{(1)}, & H_{\mathtt{ff}}^{(0)} + kH_{\mathtt{ff}}^{(1)} + k^2 H_{\mathtt{ff}}^{(2)} \end{bmatrix};$$

- Leads to

$$-\frac{1}{2}L(k)^* L(k) - iH(k) \equiv k^2 \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathtt{ff}} \end{bmatrix} + k \begin{bmatrix} 0 & Z_{\mathtt{sf}} \\ Z_{\mathtt{fs}} & * \end{bmatrix} + \begin{bmatrix} R_{\mathtt{ss}} & * \\ * & * \end{bmatrix}$$

Critical Assumption: $A_{\mathtt{ff}}$ is invertible on $\mathfrak{h}_{\mathtt{f}}$.

Singular Perturbation

- In terms of Transfer Operators

$$\lim_{k\to\infty} \mathcal{T}_{(S,L(k),H(k))}(s) = \left[\begin{array}{c||c} -\frac{1}{2}\hat{L}^*\hat{L} - i\hat{H} & -\hat{L}^\dagger\hat{S} \\ \hline\hline \hat{L} & \hat{S} \end{array}\right]$$

for $\mathrm{Re}\, s > 0$

- Limit Coefficients

$$\hat{S} = \left[\begin{array}{cc} \hat{S}_{\mathtt{ss}} & \hat{S}_{\mathtt{sf}} \\ \hat{S}_{\mathtt{fs}} & \hat{S}_{\mathtt{ff}} \end{array}\right], \; \hat{L} = \left[\begin{array}{cc} \hat{L}_{\mathtt{s}} & 0 \\ \hat{L}_{\mathtt{f}} & 0 \end{array}\right], \; \hat{H} = \left[\begin{array}{cc} \hat{H}_{\mathtt{ss}} & 0 \\ 0 & 0 \end{array}\right],$$

$$\hat{S}_{\mathtt{ab}} = \sum_{\mathtt{c}} \left\{ \delta_{\mathtt{ac}} + L^{(1)}_{\mathtt{af}} \frac{1}{A_{\mathtt{ff}}} L^{(1)*}_{\mathtt{cf}} \right\} S_{\mathtt{cb}}, \quad \hat{L}_{\mathtt{a}} = L^{(0)}_{\mathtt{as}} - L^{(1)}_{\mathtt{af}} \frac{1}{A_{\mathtt{ff}}} Z_{\mathtt{fs}}, \quad \hat{H}_{\mathtt{ss}} = \mathrm{Im}\{R_{\mathtt{ss}}\} + \mathrm{Im}\left\{ Z_{\mathtt{sf}} \frac{1}{A_{\mathtt{ff}}} Z_{\mathtt{fs}} \right\}.$$

Singular Perturbation

- Further assume $\hat{L}_{\mathtt{f}} = \hat{S}_{\mathtt{sf}} = \hat{S}_{\mathtt{fs}} = 0$ so that

$$\hat{S} = \begin{bmatrix} \hat{S}_{\mathtt{ss}} & 0 \\ 0 & \hat{S}_{\mathtt{ff}} \end{bmatrix}, \quad \hat{L} = \begin{bmatrix} \hat{L}_{\mathtt{s}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} \hat{H}_{\mathtt{ss}} & 0 \\ 0 & 0 \end{bmatrix},$$

- Limit is

$$\hat{\mathcal{T}}(s) = \begin{bmatrix} \hat{\mathcal{T}}_{\mathtt{ss}}(s) & 0 \\ 0 & \hat{S}_{\mathtt{ff}} \end{bmatrix}, \qquad \hat{\mathcal{T}}_{\mathtt{ss}}(s) = \left[\begin{array}{c||c} -\frac{1}{2}\hat{L}_{\mathtt{s}}^{\dagger}\hat{L}_{\mathtt{s}} - i\hat{H}_{\mathtt{ss}} & -\hat{L}_{\mathtt{s}}^{\dagger}\hat{S}_{\mathtt{ss}} \\ \hline\hline \hat{L}_{\mathtt{s}} & \hat{S}_{\mathtt{ss}} \end{array}\right].$$

- Theorem (Bouten-Silberfarb)

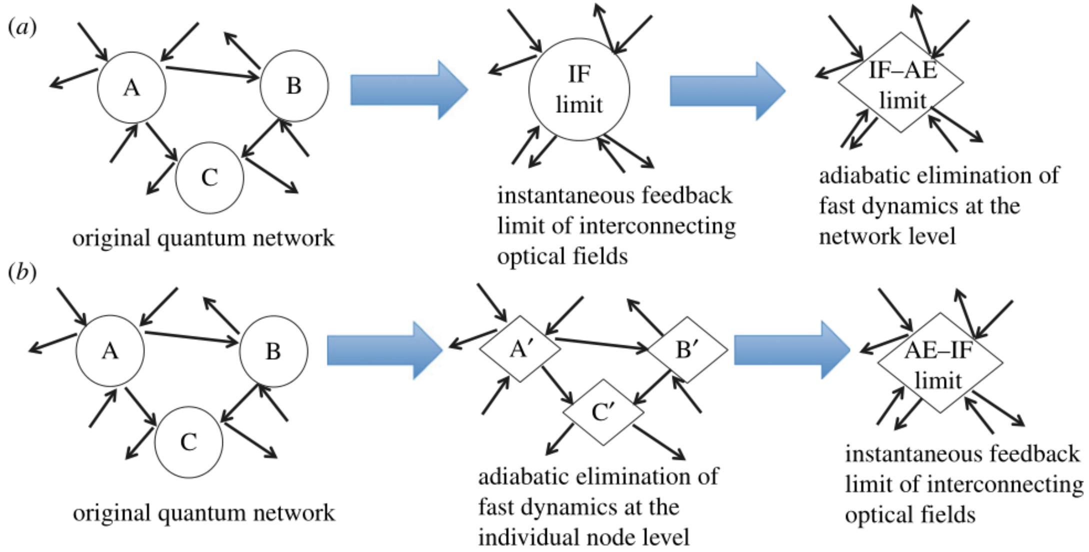
Let $(S, L(k), H(k))$ be a sequence of bounded parameters satisfying all the assumptions. Then $U_k(t) P_{\mathtt{s}}$ converges strongly to $U(t) P_{\mathtt{s}}$, that is

$$\lim_{k\to\infty} \| U_k(t)\psi - U(t)\psi \| = 0$$

for all $\psi \in \mathfrak{h} \otimes \mathfrak{F}$ with $P_{\mathtt{f}} \otimes I\psi = 0$.

Adiabatic Elimination

- An important model simplification split the systems into slow and fast subspaces
- Mathematical this is also a Schur complement of the model matrix ***G***



- It commutes with feedback reduction!
- JG, H. Nurdin, S. Wildfeuer, *J. Math. Phys., 51, 123518 (2010);* H. Nurdin, JG, *Phil. Trans. R. Soc., A 370, 5422-36 (2012)*

Спасибо
за
ваше внимание!