Spin^c-STRUCTURES AND SEIBERG–WITTEN EQUATIONS

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Introduction

In the study of Riemann surfaces, i.e. Riemannian manifolds of dimension 2, a key role is played by the complex structure, compatible with Riemannian metric, existing on any such manifold and closely related Cauchy–Riemann $\bar{\partial}$-operator. However, if we consider the 4-dimensional Riemannian manifolds then the subclass of manifolds, having the complex structure, is already sufficiently narrow and it is hard to understand general differential-geometric properties of 4-dimensional Riemannian manifolds relying on the study of this subclass. So in the investigation of 4-dimensional Riemannian manifolds two natural questions arise: the first one — which structure can serve as the replacement of the complex structure on Riemannian 4-manifolds and the second one — which linear differential operator can play the role of $\bar{\partial}$-operator in the 4-dimensional case.

To answer the first question it is proposed to replace the complex structure with the $Spin^c$-structure existing on any 4-dimensional Riemannian manifold. To answer the second question we propose to replace the $\bar{\partial}$-operator on the 4-dimensional Riemannian manifold with the Dirac operator associated with the given $Spin^c$-structure.

Having the Spin^c-structure, one can introduce the Seiberg–Witten action functional. We are especially interested in the local minima of this functional. They satisfy to the Seiberg–Witten equations which are one of the main subjects of this talk.

These equations, found at the end of XXth century, are one of the principal inventions in topology and geometry of 4-dimensional Riemannian manifolds. The Seiberg–Witten equations, as also Yang–Mills equations, are the limiting case of more general supersymmetric Yang–Mills equations. However, unlike the conformally invariant Yang–Mills equations, they are not invariant under the change of the scale. Therefore, in order to draw a "useful information" from them, it is necessary to introduce a scale parameter λ and then take the limit for $\lambda \to \infty$. This limit is called adiabatic and it is another main subject of this talk.

As we have pointed out above, we are especially interested in the local minima of Seiberg–Witten action functional being solutions of the Seiberg–Witten equations. These equations acquire a more accessible form in the symplectic case. In this case it is possible to describe in more detail the aforementioned adiabatic limit. It is also useful to consider separately the case of Kähler surfaces. In this case there is no need to take the limit for $\lambda \to \infty$, it is enough to take λ sufficiently large.

I. Spinor geometry

As we have pointed out in the introduction, a key role in the study of 4-dimensional Riemannian manifolds is played by the Spin^c-structure which exists on any Riemannian 4-manifold. We refer for the general definition of Spin^c-structure to the book by Lawson and Michelsohn while here give only the description of the properties of this structure used in the Seiberg–Witten theory.

Let (X, g) be a compact oriented Riemannian 4-manifold provided with the Levi-Civita connection. We can introduce the Clifford multiplication ρ by differential forms on X. This multiplication is defined by representing such forms by linear endomorphisms acting on the smooth sections of the spinor bundle $W \to X$.

It is a complex Hermitian vector bundle of rank 4 decomposing into the direct sum

$$W = W^{+} \oplus W^{-}$$

of complex semispinor bundles $W^{\pm}$ of rank 2 over X.

The spinor bundle $W \to X$ may be provided with the spinor connection ∇ being an extension of the Levi-Civita connection to a connection on the bundle W.

The Dirac operator D, acting on smooth sections of the bundle W, is given by the composition $\rho \circ \nabla$ of Clifford multiplication with spinor connection.

In the case when the manifold (X, g) is symplectic, i.e. has the symplectic form ω compatible with the metric g, it may be provided with an almost complex structure J compatible both with ω and g.

In this case there is a canonical construction of the spinor bundle W identified with

$$W_{\mathrm{can}} = \Lambda^{0,*}(T^*X) = \bigoplus_{q=0}^{2} \Lambda^{0,q}(T^*X).$$

Accordingly, the semispinor bundles are given by

$$W^+_{\mathrm{can}} = \Lambda^{0,0}(T^*X) \oplus \Lambda^{0,2}(T^*X)\,, \quad W^-_{\mathrm{can}} = \Lambda^{0,1}(T^*X).$$

In this case there is also a canonical spinor connection A_{can} on W_{can} and an explicit formula for the Clifford multiplication.

Moreover, for any Hermitian line bundle $E \to X$, provided with a Hermitian connection B, it is possible to construct an associated spinor bundle $W_E := W_{\text{can}} \otimes E$ and introduce the spinor connection ∇_A on W_E such that $A = A_E$ is the tensor product of the canonical spinor connection A_{can} on W_{can} and the given Hermitian connection B on E.

The Dirac operator

$$D_A = \rho \circ \nabla_A : \Gamma(X, W^+) \longrightarrow \Gamma(X, W^-)$$

coincides with $\bar{\partial}_B + \bar{\partial}_B^*$ where $\bar{\partial}_B^*$ is the operator which is L^2-conjugate to the operator $\bar{\partial}_B$.

II. Seiberg–Witten equations on 4-dimensional Riemannian manifolds

Let (X, g) be a compact oriented Riemannian 4-manifold provided with the Spin^c-structure.

Consider the following Seiberg–Witten action functional

$$S(A,\Phi)=\frac{1}{2}\int_X \left\{|F_A|^2+|\nabla_A\Phi|^2+\left(\mathrm{s}(g)+|\Phi|^2\right)\frac{|\Phi|^2}{4}\right\}\mathrm{vol}$$

where Φ is a smooth section of the bundle W^+, F_A is the curvature of the connection ∇_A, $\mathrm{s}(g)$ is the scalar curvature of the manifold (X, g), vol is the volume element of the manifold (X, g).

The local minima of the introduced functional satisfy the following Seiberg–Witten equations

$$\begin{cases} D_A\Phi = 0, \\ F_A^+ = \Phi\otimes\Phi^* - \frac{1}{2}|\Phi|^2\cdot \mathrm{Id} \end{cases} \tag{1}$$

where $\Phi\otimes\Phi^* - \frac{1}{2}|\Phi|^2\cdot \mathrm{Id}$ is the traceless Hermitian endomorphism of the bundle W^+, associated with Φ, and F_A^+ is the selfdual component of the curvature F_A (with respect to the Hodge $\star$-operator of the metric g). In the second equation the selfdual form of the curvature F_A^+ is interpreted as the endomorphism of the spinor bundle W^+ given by the Clifford multiplication by this form. The smooth section Φ of the bundle W^+ is given by the pair of forms (φ_0, φ_2) where $\varphi_0 \in \Omega^0(X, E)$, $\varphi_2 \in \Omega^{0.2}(X, E)$.

The Seiberg–Witten equations, as well as the Seiberg–Witten functional $S(A, \Phi)$, are invariant under the gauge transforms given by the formula

$$A \longmapsto A + u^{-1}du, \;\; \Phi \longmapsto u^{-1}\Phi$$

where $u = e^{i\chi}$ and χ is a smooth real-valued function so that $u \in C^{\infty}(X, \mathrm{U}(1))$.

Along with Seiberg–Witten equations (1) we shall consider their perturbed version given by

$$\begin{cases} D_A\Phi = 0, \\ F_A^+ + \eta = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \cdot \mathrm{Id} \end{cases} \tag{2}$$

where η is a selfdual 2-form on X. This perturbation is necessary to introduce in order to guarantee the existence of a solution of Seiberg–Witten equations.

III. Seiberg–Witten equations on a Kähler surface

Suppose now that X is a compact Kähler surface, i.e. a smooth compact 2-dimensional complex manifold provided with the Kähler form ω. In this case the complexified bundle $\Lambda^2_+ \otimes \mathbb{C}$ of selfdual 2-forms on X is decomposed into the direct sum of subbundles

$$\Lambda^2_+ \otimes \mathbb{C} = \Lambda^{2,0} \oplus \mathbb{C}[\omega] \oplus \Lambda^{0,2}.$$

Accordingly, the second Seiberg–Witten equation (2) for the curvature is decomposed into the sum of three equations — one equation for the component of the curvature, parallel to ω, another one for the $(0,2)$-component and the third one for the $(2,0)$-component, which is conjugate to the equation for the $(0,2)$-component and by this reason is omitted below.

Taking this into account the Seiberg–Witten equations take the form

$$\begin{cases} \bar{\partial}_B\varphi_0 + \bar{\partial}_B^*\varphi_2 = 0, \\ F_B^{0,2} + \eta^{0,2} = \dfrac{\bar{\varphi}_0\varphi_2}{2}, \\ F_{A_{\text{can}}}^{\omega} + F_B^{\omega} = \dfrac{i}{4}(|\varphi_0|^2 - |\varphi_2|^2) - \eta^{\omega}. \end{cases} \tag{3}$$

The first equation is the Dirac equation, the second one corresponds to the (0,2)-component of the curvature and the third one corresponds to the component of the curvature, parallel to ω.

In order to guarantee the existence of a solution of (3) it is necessary to impose on the bundle E the following topological condition: there exists $\lambda > 0$ such that

$$0 \leq c_1(E) \cdot [\omega] < \frac{c_1(K) \cdot [\omega]}{2} + \lambda \mathrm{Vol}(X)$$

where $c_1(E)$ is the first Chern class of the bundle E, $K = K(X)$ is the canonical bundle of the manifold X. This is the condition of stability of the bundle E which is analogous to the necessary solvability condition for the vortex equations on a compact Riemann surface, found by Bradlow.

The proof of the existence of a solution of equation (3) under the stability condition proceeds as in Bradlow's paper.

Namely, we take the perturbation η in the form $\eta = \pi i \lambda \omega$ and choose λ sufficiently large to ensure the fulfillment of the stability condition. Then, repeating the Bradlow's argument, it is possible to prove that the moduli space of solutions of the equation (3) is identified with the space of effective divisors of degree $c_1(E)$ on X.

IV. Seiberg–Witten equations on a 4-dimensional symplectic manifold

Suppose now that $X = (X, \omega, J, g)$ is a compact symplectic 4-dimensional manifold provided with symplectic 2-form ω and compatible almost complex structure J. Let $E \to X$ be a Hermitian line bundle with Hermitian connection B and $W_E := W_{\mathrm{can}} \otimes E$ is the associated spinor bundle.

Take the perturbation η in the form $\eta = -F^+_{A_{\mathrm{can}}} + \pi i \lambda \omega$ with $\lambda > 0$. The parameter λ plays the role of the scale parameter in the adiabatic limit $\lambda \to \infty$. To take this limit we introduce the renormalized sections

$$\alpha := \frac{\varphi_0}{\sqrt{\lambda}}, \ \beta := \frac{\varphi_2}{\sqrt{\lambda}}.$$

In terms of renormalized sections the perturbed Seiberg–Witten equations take the form

$$\begin{cases} \bar{\partial}_B \alpha + \bar{\partial}_B^* \beta = 0, \\ \dfrac{2}{\lambda} F_B^{0,2} = \bar{\alpha}\beta, \\ \dfrac{4i}{\lambda} F_B^{\omega} = 4\pi + |\beta|^2 - |\alpha|^2. \end{cases} \tag{4}$$

Note that all variables in these equations, such as B, α, β, depend on λ and, in order to underline this, we denote them sometimes by B_λ, α_λ, β_λ.

V. Adiabatic limit in Seiberg–Witten equations

According to Taubes result, solutions $(\alpha_\lambda, \beta_\lambda)$ of the perturbed equations (4) have the following behavior for $\lambda \to \infty$:

(1) $|\alpha_\lambda| \to 1$ everywhere outside the zeros $\alpha_\lambda^{-1}(0)$;

(2) $|\beta_\lambda| \to 0$ everywhere together with the first order derivatives.

Denote by $C_\lambda := \alpha_\lambda^{-1}(0)$ the zero set of the solution α_λ. The curves C_λ converge in the sense of currents to some pseudoholomorphic divisor which is Poincaré-dual to $c_1(E)$. This divisor is given by the chain of the form $\sum d_k C_k$, consisting of connected pseudoholomorphic curves C_k taken with multiplicities d_k.

Consider one of these curves C_k and denote it temporarily by C. In other words, C is a compact pseudoholomorphic curve in the compact symplectic 4-dimensional manifold (X, ω, J, g). Let

$$\pi : N \longrightarrow C$$

be the normal bundle of the curve C with the fibre N_z at $z \in C$ identified with the orthogonal complement to the space T_zC in T_zX. Since the operator of the almost complex structure J preserves T_zC, it preserves also N_z. So $\pi : N \to C$ is a complex line bundle.

In the adiabatic limit the original Seiberg–Witten equations reduce to the family of Ginzburg–Landau equations in the complex planes N_z normal to the tangent planes $T_z(C_k)$.

Recall that Ginzburg–Landau equations in the complex plane are the 2-dimensional analogue of Seiberg–Witten equations. Solutions of these equations, called the Ginzburg–Landau vortices, are parameterized by effective divisors, i.e. points of the complex plane taken with multiplicities.

Conversely, in order to reconstruct a solution of Seiberg–Witten equations from such family of vortex solutions in normal planes N_z this family should satisfy a nonlinear equation of $\bar{\partial}$-type.

So we have for Seiberg–Witten equations on 4-dimensional symplectic manifolds the following correspondence established by the transition to the adiabatic limit:

$$\left\{\text{solutions of Seiberg–Witten equations}\right\} \longmapsto \left\{\text{families of vortex solutions in normal planes to pseudoholomorphic divisors}\right\}$$