Characterization of algebraic varieties by their groups of symmetries

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Joint work with Andriy Regeta and Christian Urech

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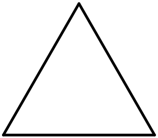
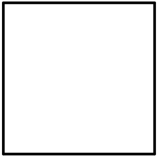
Is a geometric object uniquely determined
by its symmetry group?

Regular polygons





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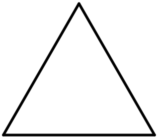
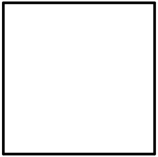


$$\mathrm{Sym}(T) = D_6 = \langle r, s \mid r^3 = s^2 = 1,\ rs = sr^{-1} \rangle$$



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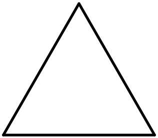

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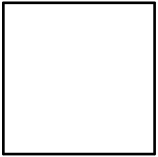
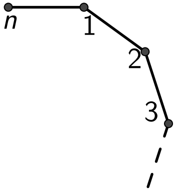


$$\mathrm{Sym}(S) = D_8 = \langle r, s \mid r^4 = s^2 = 1,\ rs = sr^{-1} \rangle$$

Regular polygons

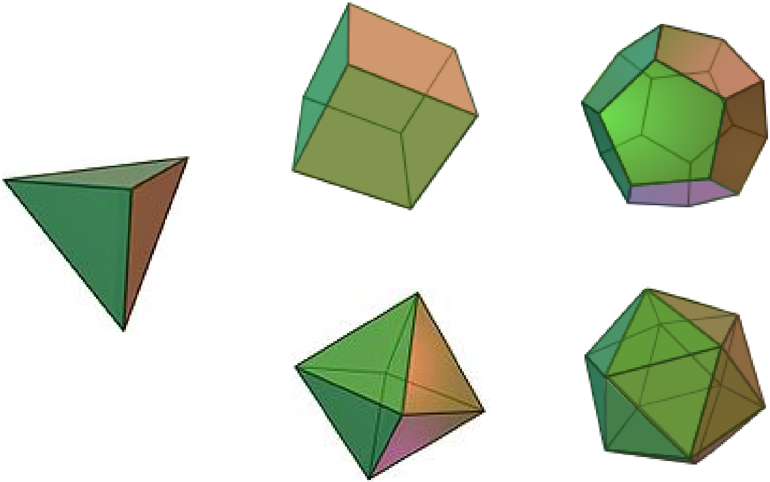


$$\mathrm{Sym}(P_n) = D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle$$

$$|\mathrm{Sym}(P_n)| = 2n$$

Regular polygons are uniquely determined by their symmetry group

Regular polytopes



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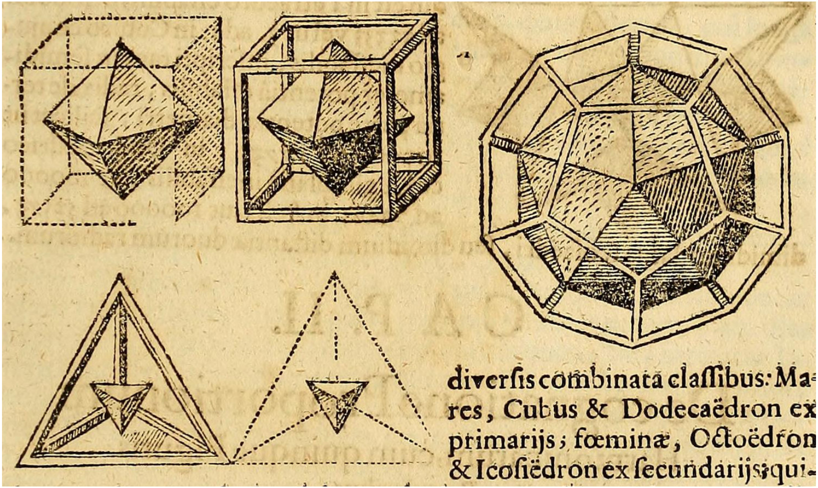

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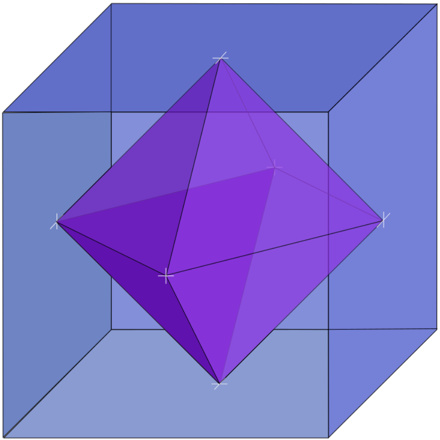
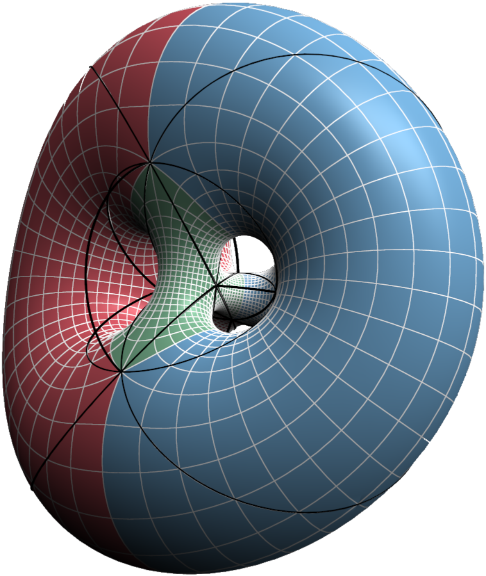


$$\mathrm{Sym}(C) \simeq \mathrm{Sym}(O)$$

Regular polytopes are not uniquely determined
by their symmetry group

Differentiable manifolds



Theorem (Filipkiewicz, 1982)

Any group isomorphism between the diffeomorphism groups of two differentiable manifolds is induced by a diffeomorphism between the manifolds.

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Differential manifolds are uniquely determined
by their symmetry group

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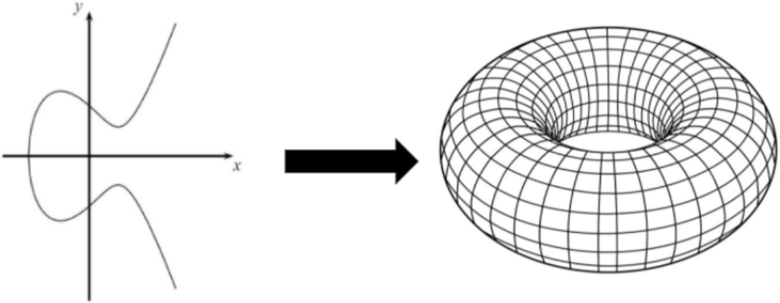
Algebraic varieties



Affine real curve | Projective complex curve

$$y^2 = x^3 - x + 1$$

Morphisms: two options

Regular morphisms: polynomial map $\varphi : X \to Y$

Ex: $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$, $(x, y) \mapsto (x^2y - x, x - y^3)$

Rational morphism: regular morphisms defined only in an open set (in the Zariski topology).

Ex: $\varphi : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$, $(x, y) \mapsto \left(\frac{x}{y}, \frac{y}{x}\right)$

Hence, in algebraic geometry there are two possibilities for the symmetry group:

Regular automorphism group $\rightsquigarrow \mathrm{Aut}(X)$

Ex: $\mathbb{C}^2 \to \mathbb{C}^2$ given by $(x, y) \mapsto (x, y + p(x))$

Birational automorphism group $\rightsquigarrow \mathrm{Bir}(X)$

Ex: $\mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ given by $(x, y) \mapsto (1/y, 1/x)$

In general, $\mathrm{Aut}(X) \subseteq \mathrm{Bir}(X)$

Sometimes, these groups are monstrosities

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$\mathrm{Aut}(\mathbb{C}^2)$: Joung- Van der Kulk Theorem

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$\mathrm{Aut}(\mathbb{C}^2)$: Joung- Van der Kulk Theorem

$\mathrm{Aut}(\mathbb{C}^3)$

Theorem (Hurwitz)

The order of $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ *for a smooth compact algebraic curve of genus* $g \geq 2$ *is bounded by* $84 \cdot (g - 1)$.

Algebraic curves are not uniquely determined
by their symmetry group

Theorem (Hacon, McKernan, Xu)

Let n be a positive integer.
Then there is a constant $C = C(n)$ such that for any projective n-dimensional variety X of general type, the order of $\mathrm{Bir}(X)$ is bounded by $C \cdot \mathrm{Vol}(X, K_X)$.

Algebraic varieties are not uniquely determined by their symmetry group

Theorem (Cantat)

$\mathrm{Bir}(\mathbb{P}^n) \simeq \mathrm{Bir}(\mathbb{P}^m)$ *if and only if* $n = m$.

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Let X *be an* n*-dimensional variety.*
If $\mathrm{Bir}(X)$ *is isomorphic to* $\mathrm{Bir}(\mathbb{P}^n)$, *then* X *is rational.*

Rational varieties are uniquely determined (up to birational equivalence) among n-dimensional varieties by their birational automorphism group

A toric variety is a normal algebraic variety endowed with with a faithful action of an algebraic torus $T = (\mathbb{C}^*)^n$ having an open dense orbit.

Ex: $(\mathbb{C}^*)^2 \times \mathbb{C}^2 \to \mathbb{C}^2$, $((t, s), (x, y)) \mapsto (tx, sy)$

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They may be non-normal as well

Ex: $X = \{x^2 - y^3 = 0\} \subset \mathbb{C}^2$

$\mathbb{C}^* \times X \to X$, $(t,(x,y)) \mapsto (t^3x, t^2y)$

Demazure (and later Cox) gave a description of $\mathrm{Aut}(X)$ for a complete toric variety.

For most complete toric varieties, we have $\mathrm{Aut}(X) = T$.

Toric varieties are not uniquely determined
by their automorphism group

Theorem (Regeta, Urech, L.)

Let X and X' be normal affine surfaces with S toric.
If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(X')$ *then* $X \simeq X'$.

Affine toric surfaces are uniquely determined among normal affine surfaces by their regular automorphism group

Proposition (Regeta, Urech, L.)

Let X be an affine surface. If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{C}^2)$ *then* $X \simeq \mathbb{C}^2$.

Also Regeta gave an example of two toric surfaces, one normal and another one non-normal with the same automorphism group

Proposition (Díaz, L.)

Let X be a toric surface.
There exits a non-normal toric surface X' such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(X')$ *if and only if X is different from* $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$ *and* $(\mathbb{C}^*)^2$

Idea of proof

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If $X = \mathbb{C}^2$, proven in Regeta, Urech, L.
If $X = (\mathbb{C}^*)^2$ there are no non-normal models of X
If $X = \mathbb{C} \times \mathbb{C}^*$, then $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X')$ are easy to compute

In any another case, letting $X = \operatorname{Spec} \mathbb{C}[S]$ with S a semigroup.
Take the set $\mathcal{H}$ of irreducible elements of S (Hilbert basis)
Now $X' = \operatorname{Spec} \mathbb{C}[S \setminus \mathcal{H}]$ we have

$$\mathrm{Aut}(X) = \mathrm{Aut}(X')$$

We use Arzhantsev - Zaidenberg description of $\mathrm{Aut}(X)$

Root subgroups

Definition

Let $T \subset \text{Aut}(X)$ be a maximal torus in $\text{Aut}(X)$. An algebraic subgroup $U \subset \text{Aut}(X)$ isomorphic to $\mathbb{G}_\mathrm{a}$ is called a root subgroup with respect to T if the normalizer of U in $\text{Aut}(X)$ contains T.

This is equivalent to saying that T and U span an algebraic group isomorphic to $\mathbb{G}_\mathrm{a} \rtimes_\chi T$ with $\chi : T \to \mathbb{G}_\mathrm{m}$ character.

A root subgroup is also uniquely determined by a homogeneous derivation of ∂ of $\mathcal{O}_X$ (with some integrability conditions, LND).

Demazure's description of $\text{Aut}^0(X)$ is based on a description of root subgroups of non-necessarily complete toric variety.

Let S be a f.g. cancellative monoid and
Let M be a minimal group where S is embedded
We also let $N = \mathrm{Hom}(M, \mathbb{Z})$ and S^* be the dual monoid

An element $\alpha \in M$ is called a *Demazure root of S* if

(*i*) There exists $\rho \in S^*(1)$ such that $\rho(\alpha) = -1$, and

(*ii*) The element $m + \alpha$ belongs to S for all $m \in S$ such that $\rho(m) > 0$.

Theorem

Let $\partial\colon k[S] \to k[S]$ be a homogeneous locally nilpotent derivation of degree α and $\partial \neq 0$, then α is a Demazure root of S and $\partial = \lambda\partial_\alpha$ for some $\lambda \in k^*$.

Idea of proof

Theorem (Regeta, Urech, L.)

Let X and X' be normal affine surfaces with X toric.
If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(X')$ then $X \simeq X'$.

Topology on Bir(X)

Let A be a variety and $f\colon A \times X \dashrightarrow A \times X$ be an A-birational map, i.e.,

- f is the identity in the first factor, and
- induces an isomorphism between open subsets U and V of $A \times X$ such that the projections from U and from V to A are both surjective.

This yields a map $A \to \mathrm{Bir}(X)$ that we call a morphism.

The Zariski topology on $\mathrm{Bir}(X)$ is the finest topology making all such morphisms continuous.

Algebraic elements in $\mathrm{Bir}(X)$

Definition

An algebraic subgroup of $\mathrm{Bir}(X)$ is the image of a morphism $G \to \mathrm{Bir}(X)$ that is also an homomorphism.

An element $g \in \mathrm{Bir}(X)$ is called algebraic if it is contained in an algebraic subgroup.

Divisibility in Bir(X)

Definition

Let G be a group.

- An element f in is called divisible by n if there exists an element $g \in G$ such that $g^n = f$.
- An element is called divisible if it is divisible by all $n \in \mathbb{Z}_{>0}$.

Lemma

Let X be a surface and $f \in \mathrm{Bir}(X)$.
Then the following two conditions are equivalent:

- *There exists a $k > 0$ such that f^k is divisible; and*
- *f is algebraic.*

Algebraic elements in $\mathrm{Aut}(S)$

Definition

An algebraic subgroup of $\mathrm{Aut}(X)$ is the image of a regular action $G \to \mathrm{Aut}(X)$ of an algebraic group.

An element $g \in \mathrm{Aut}(X)$ is called algebraic if it is contained in an algebraic subgroup.

Lemma

Let X be a normal affine surface and
let $g \in \mathrm{Aut}(X)$ be an automorphism. Then

g is an algebraic element in $\mathrm{Bir}(X)$ if and only if
g is an algebraic element in $\mathrm{Aut}(X)$.

Algebraic elements are preserved

Proposition

Let X and X' be normal affine surfaces,
$\varphi\colon \operatorname{Aut}(X) \to \operatorname{Aut}(X')$ a group homomorphism, and
$g \in \operatorname{Aut}(X)$ an algebraic element.

Then $\varphi(g)$ is an algebraic element in $\operatorname{Aut}(X')$.

Torus goes to a 2-dimensional torus

Lemma

Let X and X' be normal affine surfaces with X toric, and $\varphi\colon \operatorname{Aut}(X) \to \operatorname{Aut}(X')$ a group isomorphism.

Then $\varphi(T)$ is a 2-dimensional torus in $\operatorname{Aut}(X')$.

Root subgroups go to root subgroups

Lemma

Let X and X' be normal affine surfaces with X toric,
$\varphi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(X')$ *a group isomorphism, and*
$U \subset \mathrm{Aut}(X)$ *a root subgroup*

Then $\varphi(U)$ is a root subgroup in $\mathrm{Aut}(X')$ with respect to $\varphi(T)$.

End of the proof

We know now that X' is a toric surface and we have a bijection on the root subgroups of X and X' with respect to T and $\varphi(T)$.

Hence, to conclude the proof, it is enough to show that we can recover a toric surface X from the abstract group structure of its root subgroups and their relationship with the torus.

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Cooking techniques

¡Gracias!