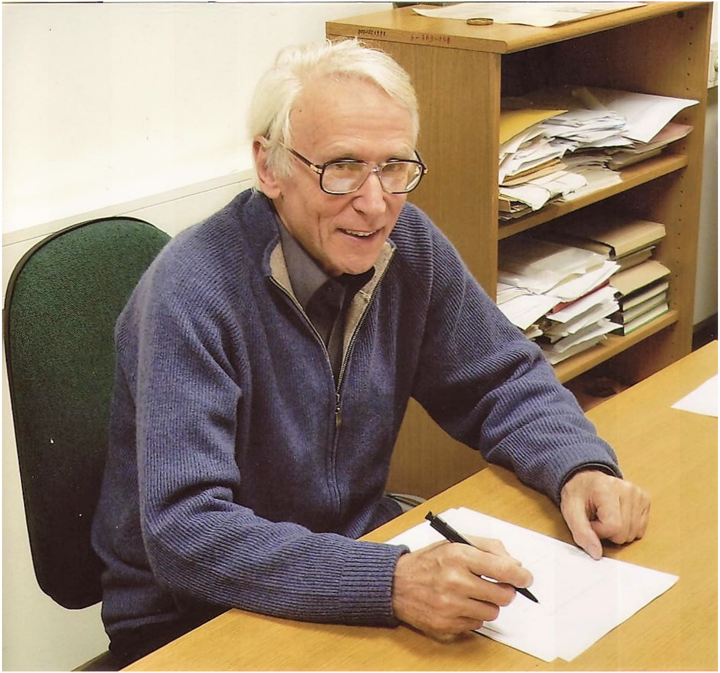

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The structure of quantum corrections and exact results in supersymmetric theories revealed by the higher covariant derivative regularization

The higher covariant derivative regularization

We will discuss the application of the higher covariant derivative method for the regularization of various supersymmetric theories. This regularization allows to reveal some interesting features of quantum corrections which cannot be seen in the case of using dimensional reduction.

The higher covariant derivative regularization was proposed by A.A.Slavnov

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301;
Theor.Math.Phys. **13** (1972) 1064.

By construction, it includes insertion of the Pauli–Villars determinants for removing residual one-loop divergencies

A.A.Slavnov, Theor.Math.Phys. **33**, (1977), 977.

Unlike dimensional reduction, this regularization is self-consistent. It can be formulated in a manifestly supersymmetric way in terms of $\mathcal{N}=1$ superfields

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

and, therefore, does not break supersymmetry in higher orders.

Moreover, the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

can naturally be obtained in the case of using the higher covariant derivative regularization. It relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mbox{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB};\qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$

$$f^{ACD} f^{BCD} \equiv C_2\delta^{AB};\qquad r \equiv \delta_{AA} = \mbox{dim}\, G.$$

Explicit calculations and the problem of constructing an NSVZ scheme

Three- and four-loop calculations in $\mathcal{N}=1$ supersymmetric theories made with dimensional reduction supplemented by modified minimal subtraction (i.e. in the so-called $\overline{\text{DR}}$-scheme)

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

revealed that the NSVZ relation in the $\overline{\text{DR}}$-scheme holds only in the one- and two-loop approximations, where the β-function is scheme independent.

However, in the three- and four-loop approximations it is possible to restore the NSVZ relation with the help of a specially tuned finite renormalization of the gauge coupling constant. Note that a possibility of making this finite renormalization is highly nontrivial.

This implies that the NSVZ relation holds only in some special renormalization schemes, which are usually called "NSVZ schemes", and the $\overline{\text{DR}}$-scheme is not NSVZ.

Now, let us discuss how one can derive the NSVZ equation in all orders and construct all-loop NSVZ schemes with the help of the higher covariant derivative regularization.

Renormalizable $\mathcal{N}=1$ supersymmetric gauge theories with matter superfields at the classical level are described by the action

$$S=\frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mbox{c.c.}\Big\}.$$

We assume that the gauge group is simple, and the chiral matter superfields ϕ_i lie in its representation R. The bare gauge and Yukawa coupling constants are denoted by e_0 and λ_0^{ijk}, respectively. The strength of the gauge superfield V is defined by the equation

$$W_a\equiv\frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big).$$

The theory under consideration is gauge invariant if the (bare) masses and Yukawa couplings satisfy the conditions

$$m_0^{im}(T^A)_m{}^j+m_0^{mj}(T^A)_m{}^i=0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k+\lambda_0^{imk}(T^A)_m{}^j+\lambda_0^{mjk}(T^A)_m{}^i=0.$$

For quantizing the theory it is convenient to use the background field method. Moreover, it is necessary to take into account nonlinear renormalization of the quantum gauge superfield

O. Piguet and K. Sibold, Nucl.Phys. **B197** (1982) 257; 272;
I.V.Tyutin, Yad.Fiz. **37** (1983) 761.

This can be done with the help of the replacement $e^{2V} \to e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}$, where $\boldsymbol{V}$ and V are the background and quantum gauge superfields, respectively, and the function $\mathcal{F}(V)$ includes an infinite set of parameters needed for describing the nonlinear renormalization. In the lowest order

J.W.Juer and D.Storey, Phys.Lett. **119B** (1982) 125; Nucl. Phys. **B216** (1983) 185.

$$\mathcal{F}(V)^A = V^A + e_0^2\, y_0\, G^{ABCD}\, V^B V^C V^D + \ldots,$$

where y_0 is one of the constants entering this set, and G^{ABCD} is a certain function of the structure constants.

The background gauge invariance

$$\phi_i \to (e^A)_i{}^j \phi_j; \qquad V \to e^{-A^+} V e^{A^+}; \qquad e^{2\boldsymbol{V}} \to e^{-A^+} e^{2\boldsymbol{V}} e^{-A}$$

parameterized by a chiral superfield A remains a manifest symmetry of the effective action.

For constructing the regularized theory we first add to its action terms with higher derivatives,

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\,d^2\theta\,W^a\left(e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}\right)_{Adj} R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\left(e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\right)_{Adj} W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$
$$+\Big[\int d^4x\,d^2\theta\left(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\right) + \text{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a; \qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\bar D_{\dot a}e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\text{tr}\int d^4x\,d^4\theta\,\boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj}\bar{\boldsymbol{\nabla}}^2 V.$$

Also it is necessary to introduce the Faddeev-Popov and Nielsen–Kalosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = K(0) = 1$.

For regularizing residual one-loop divergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \text{Det}(PV, M_\varphi)^{-1} \text{Det}(PV, M)^c \times \exp\Big\{ i\Big(S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} + S_{\text{sources}}\Big)\Big\},$$

where $D\mu$ is the functional integration measure, and

$$\text{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\text{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

Here we use chiral commuting Pauli–Villars superfields.
The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lie in a representation R_{PV} and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\text{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

It is important to distinguish the renormalization group functions (RGFs) defined in terms of the bare couplings α_0 and λ_0,

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad \gamma_x(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_x}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

and RGFs standardly defined in terms of the renormalized couplings α and λ,

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad \widetilde{\gamma}_x(\alpha,\lambda) \equiv \frac{d\ln Z_x}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459.

RGFs defined in terms of the bare couplings do not depend on a renormalization prescription for a fixed regularization, but depend on a regularization.

RGFs defined in terms of the renormalized couplings depend both on a regularization and on a renormalization prescription.

Both definitions of RGFs give the same functions in the HD+MSL-scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms. This means that the renormalization constants include only powers of $\ln\Lambda/\mu$, where μ is a renormalization point.

$$\widetilde{\beta}(\alpha,\lambda)\Big|_{\mathrm{HD+MSL}} = \beta(\alpha_0\to\alpha,\lambda_0\to\lambda);$$

$$\widetilde{\gamma}_x(\alpha,\lambda)\Big|_{\mathrm{HD+MSL}} = \gamma_x(\alpha_0\to\alpha,\lambda_0\to\lambda).$$

Let us briefly describe the proof of the following statements:

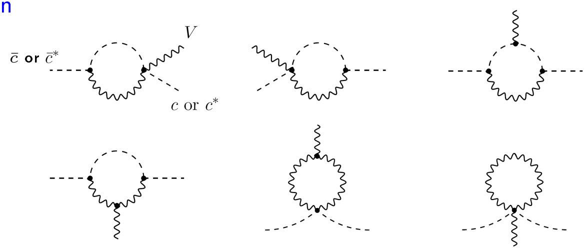
1. NSVZ equation is valid for RGFs defined in terms of the bare couplings in the case of using the higher covariant derivative regularization for an arbitrary renormalization prescription.

2. For RGFs defined in terms of the renormalized couplings some NSVZ schemes are given by the HD+MSL prescription. (MSL can supplement various versions of the higher covariant derivative regularization.)

The all-order ultraviolet finiteness of triple vertices in which two external lines correspond to the Faddeev–Popov ghosts and one external line corresponds to the quantum gauge superfield has been proved in the general ξ-gauge.

K.S., Nucl.Phys. **B909** (2016) 316.

The one-loop contribution to these vertices comes from the superdiagrams presented below. The finiteness of their sum has been verified by an explicit calculation



The two-loop finiteness of the considered vertices has been verified in

M.Kuzmichev, N.Meshcheriakov, S.Novgorodtsev, I.Shirokov, K.S., Phys. Rev. D **104** (2021) no.2, 025008; M.Kuzmichev, N.Meshcheriakov, S.Novgorodtsev, V.Shatalova, I.Shirokov, K.S., Eur. Phys. J. C **82** (2022) no.1, 69.

1. Non-renormalization of the triple gauge-ghost vertices

The all-loop finiteness of the triple gauge-ghost vertices has been proved with the help of the Slavnov–Taylor identities

A.A.Slavnov, Theor. Math. Phys. **10** (1972) 99;
J.C.Taylor, Nucl. Phys. **B 33** (1971) 436.

and rules for calculating supergraphs.

There are 4 vertices of the considered structure, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$. All of them have the same renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, due to their finiteness

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0,$$

where $\alpha = Z_\alpha\alpha_0$; $V = Z_VZ_\alpha^{-1/2}V_R$; $\bar{c}c = Z_cZ_\alpha^{-1}\bar{c}_Rc_R$. Using the finiteness of the triple gauge-ghost vertices we obtain

$$\beta(\alpha_0,\lambda_0) = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big),$$

where γ_c and γ_V are the anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare couplings), respectively.

Using the non-renormalization of the triple gauge-ghost vertices it is possible to rewrite the non-Abelian NSVZ equation in a new form

K.S., Nucl.Phys. **B909** (2016) 316.

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}$$
$$= -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

It relates the β-function in a certain loop to the anomalous dimensions of quantum superfields in the previous loop, because the right hand side does not contain a denominator depending on couplings.

(Note that the original NSVZ equation relates the β-function only to the anomalous dimension of the matter superfields, but in all previous orders.)

Using the higher covariant derivative regularization it is possible to derive the new form of the NSVZ equation in all orders for RGFs defined in terms of the bare couplings.

3. The β-function of supersymmetric theories as an integral of double total derivatives

A key observation needed for the derivation of the NSVZ relation is that in the case of using the higher covariant derivative regularization the integrals giving the β-function defined in terms of the bare couplings are integrals of double total derivatives in $\mathcal{N}=1$ supersymmetric gauge theories. This was first noted in

A.A.Soloshenko, K.S., ArXiv: hep-th/0304083v1 (the factorization into total derivatives);
A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445 (the factorization into double total derivatives).

The all-loop proof of this statement has been done in

K.S., Nucl. Phys. **852** (2011) 71.

for the Abelian case and in

K.S., JHEP **10** (2019) 011.

for general non-Abelian gauge theories.

As an example, at the next slide we present the three-loop expression for the β-function of $\mathcal{N}=1$ SQED with N_f flavors.

3. An example: the three-loop β-function of $\mathcal{N}=1$ SQED as an integral of double total derivatives

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = N_f \frac{d}{d\ln\Lambda}\Big\{ 2\pi \int \frac{d^4Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \frac{\ln(Q^2+M^2)}{Q^2} + 4\pi \int \frac{d^4Q}{(2\pi)^4} \frac{d^4K}{(2\pi)^4} \frac{e^2}{K^2 R_K^2}$$
$$\times \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \Big(\frac{1}{Q^2(K+Q)^2} - \frac{1}{(Q^2+M^2)((K+Q)^2+M^2)} \Big) \Big[R_K \Big(1 + \frac{e^2 N_f}{4\pi^2} \ln\frac{\Lambda}{\mu}\Big)$$
$$-2e^2 N_f \Big(\int \frac{d^4L}{(2\pi)^4} \frac{1}{L^2(K+L)^2} - \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2+M^2)((K+L)^2+M^2)} \Big) \Big]$$
$$+4\pi \int \frac{d^4Q}{(2\pi)^4} \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{e^4}{K^2 R_K L^2 R_L} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \Big\{ \Big(-\frac{2K^2}{Q^2(Q+K)^2(Q+K+L)^2}$$
$$\times \frac{1}{(Q+L)^2} + \frac{2}{Q^2(Q+K)^2(Q+L)^2} \Big) - \Big(-\frac{2(K^2+M^2)}{((Q+K)^2+M^2)((Q+L)^2+M^2)}$$
$$\times \frac{1}{(Q^2+M^2)((Q+K+L)^2+M^2)} + \frac{2}{(Q^2+M^2)((Q+K)^2+M^2)((Q+L)^2+M^2)}$$
$$- \frac{4M^2}{(Q^2+M^2)^2((Q+K)^2+M^2)((Q+L)^2+M^2)} \Big) + O(e^6) \Big\}$$

(Here the integration domains do not include small vicinities of singularities.)

3. Integrals of double total derivatives and a graphical interpretation of the NSVZ relation

The integrals of double total derivatives do not vanish due to singularities of the integrands. Really, if $f(Q^2)$ is a non-singular function which rapidly decrease at infinity, then

$$\int \frac{d^4Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu}\Big(\frac{f(Q^2)}{Q^2}\Big) = \int\limits_{S^3_\varepsilon} \frac{dS^\mu}{(2\pi)^4}\Big(-\frac{2Q_\mu}{Q^4} f(Q^2) + \frac{2Q_\mu}{Q^2} f'(Q^2)\Big)$$
$$= \frac{1}{4\pi^2} f(0) \ne 0.$$

Due to similar equations the double total derivatives effectively cut lines of quantum superfields. As a result, we obtain diagrams contributing to various anomalous dimensions, in which a number of loops is less by 1. For example, in the Abelian case this gives the NSVZ β-function

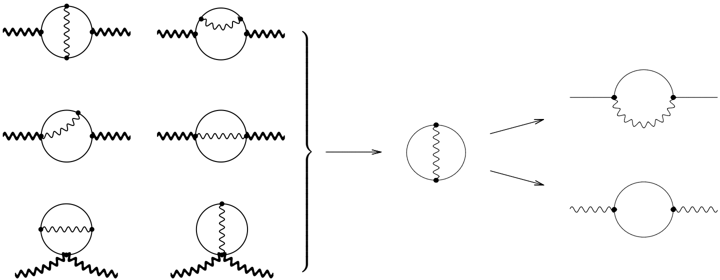
$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. **B 166** (1986) 334.

3. Graphical interpretation of the new form of the NSVZ relation

This allows to give a simple qualitative interpretation of the new form of the NSVZ equation:

For each vacuum supergraph the NSVZ equation relates a contribution to the β-function obtained by attaching two external lines of the background gauge superfield to the corresponding contribution to the anomalous dimension of quantum superfields obtained by all various cuts of internal lines:



In the non-Abelian case internal lines can correspond to the quantum gauge superfield, the Faddeev–Popov ghosts, and the matter superfields.

4. Deriving the new form of the exact NSVZ β-function by summing singular contributions

Due to the factorization into integrals of double total derivatives the function β/α^2 can be found by summing singular contributions. In all loops this has been done in

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

The result can be written as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} - \frac{\beta_{\text{1-loop}}(\alpha_0)}{\alpha_0^2}$$
$$= \frac{1}{\pi}C_2\gamma_V(\alpha_0,\lambda_0) + \frac{1}{\pi}C_2\gamma_c(\alpha_0,\lambda_0) - \frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0).$$

gauge propagators

Faddeev–Popov ghost propagators

matter propagators

5. Conditions required for validity of the NSVZ relation in all loops. An all-loop NSVZ scheme.

Thus, the NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big),$$

and, therefore, the NSVZ relation

$$\beta(\alpha_0,\lambda_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big)}{2\pi(1 - C_2\alpha_0/2\pi)}$$

are valid in all orders of the perturbation theory for RGFs defined in terms of the bare couplings if a theory is regularized by higher covariant derivatives.

Consequently, for RGFs defined in terms of the renormalized couplings, similar equations hold in the HD+MSL scheme in all orders of the perturbation theory.

Knowing the conditions for the validity of the NSVZ relation, it is possible to use it for calculating the β-function in higher orders of the perturbation theory.

The two-loop anomalous dimension of the matter superfields with the higher derivative regularization

For theories with a single gauge coupling the two-loop anomalous dimension defined in terms of the bare coupling constant for $\mathcal{N}=1$ supersymmetric theories regularized by higher derivatives has been calculated in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

$$(\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) = -\frac{\alpha_0}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{2\pi^2}\left[C(R)^2\right]_i{}^j - \frac{1}{16\pi^4}$$
$$\times\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde} - \frac{3\alpha_0^2}{2\pi^2}C_2C(R)_i{}^j\Big(\ln a_\varphi + 1 + \frac{A}{2}\Big) + \frac{\alpha_0^2}{2\pi^2}T(R)C(R)_i{}^j$$
$$\times\Big(\ln a + 1 + \frac{A}{2}\Big) - \frac{\alpha_0}{8\pi^3}\lambda^*_{0lmn}\lambda_0^{jmn}C(R)_i{}^l(1-B+A) + \frac{\alpha_0}{4\pi^3}\lambda^*_{0imn}\lambda_0^{jml}$$
$$\times C(R)_l{}^n(1-A+B) + O\Big(\alpha_0^3,\alpha_0^2\lambda_0^2,\alpha_0\lambda_0^4,\lambda_0^6\Big),$$

where

$$A = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{R(x)};\qquad B = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{F^2(x)}\qquad a = \frac{M}{\Lambda};\qquad a_\varphi = \frac{M_\varphi}{\Lambda}.$$

Obtaining the three-loop β-function from the NSVZ equation

If the anomalous dimension of the matter superfields defined in terms of the bare couplings has been calculated in L-loops with the higher derivative regularization, then it is possible to construct the $(L+1)$-loop β-function from the NSVZ equation without loop calculations. For example, in the three-loop approximation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha_0}{4\pi^2}\Big\{-3C_2^2 + \frac{1}{r}C_2\,\mbox{tr}\,C(R) + \frac{2}{r}\,\mbox{tr}\left[C(R)^2\right]\Big\}$$
$$-\frac{1}{8\pi^3 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{8\pi^3}\Big\{-3C_2^3 + \frac{1}{r}C_2^2\,\mbox{tr}\,C(R) - \frac{2}{r}\,\mbox{tr}\left[C(R)^3\right] + \frac{2}{r}$$
$$\times C_2\,\mbox{tr}\left[C(R)^2\right]\Big(3\ln a_\varphi + 4 + \frac{3A}{2}\Big) - \frac{2}{r^2}\,\mbox{tr}\,C(R)\,\mbox{tr}\left[C(R)^2\right]\Big(\ln a + 1 + \frac{A}{2}\Big)\Big\}$$
$$-\frac{\alpha_0 C_2}{16\pi^4 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0}{16\pi^4 r}\left[C(R)^2\right]_j{}^i\lambda^*_{0imn}\lambda_0^{jmn}\Big(1 + A - B\Big) - \frac{\alpha_0}{8\pi^4 r}$$
$$\times C(R)_j{}^i C(R)_l{}^n\lambda^*_{0imn}\lambda_0^{jml}\Big(1 - A + B\Big) + \frac{1}{32\pi^5 r}C(R)_j{}^i\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde}$$
$$+O\Big(\alpha_0^3,\,\alpha_0^2\lambda_0^2,\,\alpha_0\lambda_0^4,\,\lambda_0^6\Big).$$

This result has been confirmed by some explicit three-loop calculation, see, e.g.,

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345;
A.E.Kazantsev, V.Yu.Shakhmanov, K.S., JHEP 1804 (2018) 130.

Obtaining RGFs defined in terms of the renormalized couplings

To calculate RGFs defined in terms of the renormalized couplings, first, we integrate the equations

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad (\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d(\ln Z_\phi)_i{}^j}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}$$

and obtain the expressions for the renormalized gauge coupling constant and $(\ln Z_\phi)_i{}^j$. They depend on a set of finite constants which determine a subtraction scheme in the considered approximation. Next, we substitute the expressions obtained in this way into the equations

$$\widetilde\beta(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad (\widetilde\gamma_\phi)_i{}^j(\alpha,\lambda) \equiv \frac{d(\ln Z_\phi)_i{}^j}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

These RGFs will nontrivially depend on the finite constants due to the scheme dependence.

The results for the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings are rather large and are not presented here. They can be found in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

Finite constants fixing a renormalization prescription

The finite constants in the lowest approximation are defined by the equations

$$(\ln Z_\phi)_i{}^j(\alpha,\lambda) = \frac{\alpha}{\pi}C(R)_i{}^j\Big(\ln\frac{\Lambda}{\mu}+g_{11}\Big) - \frac{1}{4\pi^2}\lambda^*_{imn}\lambda^{jmn}\Big(\ln\frac{\Lambda}{\mu}+g_{12}\Big)$$
$$+O(\alpha^2,\alpha\lambda^2,\lambda^4);$$

$$\frac{1}{\alpha}-\frac{1}{\alpha_0} = -\frac{3}{2\pi}C_2\Big(\ln\frac{\Lambda}{\mu}+b_{11}\Big)+\frac{1}{2\pi}T(R)\Big(\ln\frac{\Lambda}{\mu}+b_{12}\Big)-\frac{3\alpha}{4\pi^2}C_2^2\Big(\ln\frac{\Lambda}{\mu}$$
$$+b_{21}\Big)+\frac{\alpha}{4\pi^2 r}C_2\,\mathrm{tr}\,C(R)\Big(\ln\frac{\Lambda}{\mu}+b_{22}\Big)+\frac{\alpha}{2\pi^2 r}\mathrm{tr}\left[C(R)^2\right]\Big(\ln\frac{\Lambda}{\mu}+b_{23}\Big)$$
$$-\frac{1}{8\pi^3 r}C(R)_j{}^i\lambda^*_{imn}\lambda^{jmn}\Big(\ln\frac{\Lambda}{\mu}+b_{24}\Big)+O(\alpha^2,\alpha\lambda^2,\lambda^4).$$

In the HD+MSL scheme all finite constants g_i and b_i are equal to 0, and the NSVZ relation is valid in the $O(\alpha^2,\alpha\lambda^2,\lambda^4)$ approximation. For other schemes this in general is not true.

Here (at the next slide) we present the result for considered RGFs only for a particular case, namely, for one-loop finite $\mathcal{N}=1$ supersymmetric theories, see

P.West, Phys.Lett. **B 137** (1984) 371;
A.Parkes, P.West, Phys.Lett. **B 138** (1984) 99.

An important particular case is theories finite in the one-loop approximation which satisfy the conditions

$$T(R) = 3C_2; \qquad \lambda^*_{imn}\lambda^{jmn} = 4\pi\alpha C(R)_i{}^j.$$

In this case the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings have the form

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) = -\frac{3\alpha^2}{2\pi^2}C_2 C(R)_i{}^j\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) - \frac{\alpha}{4\pi^2}\Big(\frac{1}{\pi}\lambda^*_{imn}\lambda^{jml}C(R)_l{}^n$$
$$+2\alpha\left[C(R)^2\right]_i{}^j\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big);$$
$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = \frac{3\alpha^2}{4\pi^3 r}C_2\,\mathrm{tr}\left[C(R)^2\right]\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) + \frac{\alpha}{8\pi^3 r}\Big(\frac{1}{\pi}C(R)_j{}^i C(R)_l{}^n$$
$$\times\lambda^*_{imn}\lambda^{jml} + 2\alpha\,\mathrm{tr}\left[C(R)^3\right]\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big).$$

We see that in this case the NSVZ equation is satisfied in the lowest nontrivial approximation for an arbitrary renormalization presription,

$$\frac{\beta(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda) + O(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6).$$

The NSVZ equation for theories finite in the lowest loops

For $\mathcal{N}=1$ supersymmetric theories finite in the one-loop approximation it is possible to tune a subtraction scheme so that the theory will be all-loop finite

D.I.Kazakov, Phys. Lett. B **179** (1986) 352; A.V.Ermushev, D.I.Kazakov, O.V.Tarasov, Nucl.Phys. B **281** (1987) 72; C.Lucchesi, O.Piguet, K.Sibold, Helv.Phys.Acta **61** (1988) 321; Phys.Lett. B **201** (1988) 241.

If a subtraction scheme is tuned in such a way that the β-function vanishes in the first L loops and the anomalous dimension of the matter superfields vanishes in the first $(L-1)$ loops, then

K.S., Eur.Phys.J. C **81** (2021) 571.

for an arbitrary renormalization prescription the $(L+1)$-loop gauge β-function satisfies the equation

$$\frac{\beta_{L+1}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r} C(R)_i{}^j (\gamma_{\phi,L})_j{}^i(\alpha,\lambda).$$

Therefore, if a theory is finite in a certain approximation, its β-function vanishes in the next order. This exactly agrees with the earlier known result of

A.J.Parkes, P.West, Nucl.Phys. B **256** (1985) 340;
M.T.Grisaru, B.Milewski and D.Zanon, Phys.Lett. **155B** (1985) 357.

The NSVZ equations can also be written for theories with multiple gauge couplings. In this case a number of the NSVZ equations is equal to a number of (simple or $U(1)$) factors in the gauge group $G = G_1 \times G_2 \times \ldots \times G_n$. They can be written in the form

D.Korneev, D.Plotnikov, K.S., N.Tereshina, JHEP **10** (2021) 046.

$$\frac{\beta_K(\alpha,\lambda)}{\alpha_K^2} = -\frac{1}{2\pi(1 - C_2(G_K)\alpha_K/2\pi)}\Big[3C_2(G_K) - \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K}\Big(1 - \gamma_{\mathsf{a}}(\alpha,\lambda)\Big)\Big],$$

where the subscript a numerates chiral matter superfields in irreducible representations of simple G_I,

$$\boldsymbol{T}_K(R) = \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K},$$

and we use the notation

$$\boldsymbol{T}_{\mathsf{a}K} = \begin{cases} \delta_{i_1}{}^{i_1} \ldots \delta_{i_{K-1}}{}^{i_{K-1}} T_K(R_{\mathsf{a}K})\, \delta_{i_{K+1}}{}^{i_{K+1}} \ldots \delta_{i_n}{}^{i_n} & \text{if } G_K \text{ is simple;} \\ \delta_{i_1}{}^{i_1} \ldots \delta_{i_{K-1}}{}^{i_{K-1}}\, q_{\mathsf{a}K}^2\, \delta_{i_{K+1}}{}^{i_{K+1}} \ldots \delta_{i_n}{}^{i_n} & \text{if } G_K = U(1). \end{cases}$$

For MSSM the all-order exact NSVZ β-functions are given by the equations

$$\frac{\beta_3(\alpha,\lambda)}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3+\mbox{tr}\Big(\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{D_I}(\alpha,\lambda)\Big)\Big];$$

$$\frac{\beta_2(\alpha,\lambda)}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1+\mbox{tr}\Big(\frac{3}{2}\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda) +\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11+\mbox{tr}\Big(\frac{1}{6}\gamma_{Q_I}(\alpha,\lambda)+\frac{4}{3}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{3}\gamma_{D_I}(\alpha,\lambda) +\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)+\gamma_{E_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big],$$

where the traces are taken with respect to the generation indices.

(In a different form) they were first presented in

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

and correctly reproduce the (scheme-independent) two-loop contributions.

Three-loop MSSM β-functions for an arbitrary supersymmetric renormalization prescription

Starting from the two-loop expressions for the anomalous dimensions of the matter superfields it is possible to find the three-loop MSSM β-functions for an arbitrary supersymmetric renormalization prescription supplementing the higher covariant derivative regularization

O.Haneychuk, V.Shirokova, K.S., JHEP **09** (2022), 189.

The result is very large and depends on both regularization parameters and finite constants fixing a subtraction scheme. For certain values of these finite constants it reproduces the $\overline{\text{DR}}$ result obtained earlier.

I.Jack, D.R.T.Jones, A.F.Kord, Annals Phys. **316** (2005), 213.

As an example, at the next slide we present the three-loop expression for the function $\widetilde{\beta}_3$.

Therefore, the higher covariant derivative regularization can really be used for making very complicated explicit multiloop calculations.

MSSM, three-loop β_3

$$\frac{\widetilde{\beta}_3(\alpha,Y)}{\alpha_3^2} = -\frac{1}{2\pi}\Big\{3 - \frac{11\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{7\alpha_3}{2\pi} + \frac{1}{8\pi^2}\,\mbox{tr}\Big(2\,Y_U^+ Y_U + 2\,Y_D^+ Y_D\Big) + \frac{1}{2\pi^2}\Big[\frac{137\alpha_1^2}{1200}$$

$$+\frac{27\alpha_2^2}{16} + \frac{\alpha_3^2}{6} + \frac{3\alpha_1\alpha_2}{40} - \frac{11\alpha_1\alpha_3}{60} - \frac{3\alpha_2\alpha_3}{4} + \frac{363\alpha_1^2}{100}\Big(\ln a_1 + 1 + \frac{A}{2} + b_{2,31} - b_{1,1}\Big) + \frac{9\alpha_2^2}{4}$$

$$\times\Big(-6\ln a_{\varphi,2} + 7\ln a_2 + 1 + \frac{A}{2} + b_{2,32} - b_{1,2}\Big) - 24\alpha_3^2\Big(3\ln a_{\varphi,3} - 2\ln a_3 + 1 + \frac{A}{2} + \frac{7}{16}b_{2,33}$$

$$-\frac{7}{16}b_{1,3}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_U Y_U^+\Big)\Big[\frac{3\alpha_1}{20} + \frac{3\alpha_2}{4} + 3\alpha_3 + \frac{13\alpha_1}{30}\Big(B - A + 2b_{2,3U} - 2j_{U1}\Big) + \frac{3\alpha_2}{2}$$

$$\times\Big(B - A + 2b_{2,3U} - 2j_{U2}\Big) + \frac{8\alpha_3}{3}\Big(B - A + 2b_{2,3U} - 2j_{U3}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_D Y_D^+\Big)\Big[\frac{3\alpha_1}{20} + \frac{3\alpha_2}{4}$$

$$+3\alpha_3 + \frac{7\alpha_1}{30}\Big(B - A + 2b_{2,3D} - 2j_{D1}\Big) + \frac{3\alpha_2}{2}\Big(B - A + 2b_{2,3D} - 2j_{D2}\Big) + \frac{8\alpha_3}{3}\Big(B - A$$

$$+2b_{2,3D} - 2j_{D3}\Big)\Big] - \frac{1}{(8\pi^2)^2}\Big[\frac{3}{2}\mbox{tr}\Big((Y_U Y_U^+)^2\Big)\Big(1 + 4b_{2,3U} - 4j_{UU}\Big) + \frac{3}{2}\mbox{tr}\Big((Y_D Y_D^+)^2\Big)\Big(1$$

$$+4b_{2,3D} - 4j_{DD}\Big) + 3\Big(\mbox{tr}(Y_U Y_U^+)\Big)^2\Big(1 + 2b_{2,3U} - 2j_{UtU}\Big) + 3\Big(\mbox{tr}(Y_D Y_D^+)\Big)^2\Big(1 + 2b_{2,3D}$$

$$-2j_{DtD}\Big) + \mbox{tr}\Big(Y_E Y_E^+\Big)\,\mbox{tr}\Big(Y_D Y_D^+\Big)\Big(1 + 2b_{2,3D} - 2j_{DtE}\Big) + \mbox{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)\Big(1 + 2b_{2,3U}$$

$$+2b_{2,3D} - 2j_{UD} - 2j_{DU}\Big)\Big]\Big\} + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).$$

- The higher covariant derivative regularization allows revealing some interesting features of supersymmetric theories and deriving some all-loop results.
- The integrals giving the β-function(s) of supersymmetric theories are integrals of double total derivatives with this regularization.
- RGFs defined in terms of the bare couplings satisfy the NSVZ relation in theories regularized by higher derivatives in all loops.
- Some all-order NSVZ schemes for RGFs defined in terms of the renormalized couplings are given by the HD+MSL prescription.
- The validity of the NSVZ equation with the higher covariant derivative regularization allows to essentially simplify some multiloop calculations.

Thank you for the attention!