(NPB 985, 116019,2022; arXiv:2207.01665)

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Five-Point Correlation Numbers in Minimal Liouville Gravity

24 декабря 2022 г.

There exist a few approaches to 2d Quantum gravity.

We will consider the Polyakov's continuous approach.
This approach uses the functional integration over 2d Riemannian metrics, as well as over the matter fields of some model of 2d CFT.

After fixing the conformal gauge, we arrive at the so-called theory of Liouville gravity.

If the matter sector is described by one of the BPZ Minimal models of the Conformal field theory, then we get the so-called **Minimum Liouville gravity (MLG)**. MLG was introduced and partially solved by Knizhnik, Polyakov and A. Zamolodchikov.

Correlation numbers in MLG are defined as vacuum averages of the product of physical (BRST-closed) operators.

Such operators are either local fields of the conformal dimension 0 or integrals of local densities (fields of the dimension $(1,1)$) over the world surface.

We will study correlators of local operators $W_{m,n}$ with ghost number one and integrals of their corresponding local densities $U_{m,n}(x)$ with ghost number zero.

Due to the anomaly of the ghost number, for such a correlator on the sphere to be non-vanishing, it must contain three W-type operators and any numbers of such integrals.

Hence the three-point functions should not contain integrals over the world surface at all.
To find them it is sufficient to know the structure constants in MLG and DOZZ structure constants in Liouville field theory.

The four-point correlator contains **one integral** over the position of the $U_{m,n}(x)$ field. The method for calculating such a correlator was proposed by AB-AlZ, using the so-called **The key relation** in Liouville's minimal gravity.
The key relation (KR) is a consequence of the **Higher equations of motion Al. Zamolodchikov (HEM)** in the quantum Liouville field theory.

This method allows, using the Stokes formula, to reduce the integral $\int d^2x$ to the boundary contributions from the neighborhoods of the insertion points $x_i, i = 1, 2, 3$ of the operators W and the point $x = \infty$.

An important fact used here is the BRST invariance of 3 other fields $W_{m,n}$ under the correlator sign and the consequent of it possibility to drop BRST exact terms in the **Key Relation** for the integrand corresponding the the first of fields.

This is no longer true for correlators with more than one density integral $\int d^2 y_i \, U_{m,n}(y_i)$ — such integrals are BRST-invariant only up to boundary terms that cannot be discarded.

In our work , for the case of a five-point correlator, we show that taking into account the $\mathcal{Q}$-exact terms in the KR applied to one of the integrable densities $\int d^2 y_1 \, U_{m,n}(y_1)$, reduces the integral over y_1 to boundary contributions in the form of a four-point correlator.

A similar assertion generalizes to N-point correlators with $N > 4$.

Minimal Liouville gravity is a Conformal field theory with a total central charge 0, consisting of a Liouville field theory describing gravity, a minimal CFT model as a sector of matter, and B, C system of reparameterization BRST ghosts with a central charge -26.

$$A_{MLG} = A_L + A_{\mathcal{M}_{q,q'}} + \underbrace{\frac{1}{\pi}\int d^2x\ \left(C\bar{\partial}B + \overline{C}\partial\overline{B}\right)}_{A_{ghost}}$$

The central charge of the Liouville theory is determined by the condition $c_L + c_M + c_{gh} = 0$ and, accordingly, by the characteristics of the minimal model —numbers (q', q).

The Minimal model of CFT is determined by a pair of mutual numbers q, q'; Its space of states consists of the set of the representations of the Virasoro algebra consisting of primary fields of $\ Phi_{m,n}$ with 1 $leqm < q$ and 1 $leqn < q'$ and their descendants. Denote for $q/q' = b$ 2, then the central charge of $\ mathcalm_{q',q}$

$$c = 1 - 6(b^{-1} - b)^2$$

the conformal dimensions of $\Phi_{m,n}$

$$\Delta^M_{m,n} = -(b^{-1} - b)^2/4 + \lambda^2_{m,-n},\ \lambda_{m,n} = (mb^{-1} + nb)/2$$

We will also denote these primary fields by Φ_α, where the parameter α is related to the dimension as $\Delta^{(M)}_\alpha = \alpha(\alpha - b^{-1} + b)$.

Liouville field theory The gravitational sector is described by a quantum version of the classical field theory based on the Liouville action. This is a conformal field theory with a central charge parameterized by the variable b or $Q = b^{-1} + b$

$$c_{\mathsf{L}} = 1 + 6Q^2$$

In the MLG, it follows from the requirement that the total central charge be zero that the parameter b in the Liouville theory must be equal to $\sqrt{q/q'}$, as it was defined in the previous section. It enters the Lagrangian of the theory as follows:

$$\mathcal{L}_{\mathsf{L}} = \frac{1}{4\pi}\left(\partial_a \phi\right)^2 + \mu e^{2b\phi}$$

Here μ is the parameter of the theory, the cosmological constant, and ϕ is the degree of freedom remaining after fixing the gauge in the integral over the metrics.

Primary fields in the theory are exponential operators $V_a \equiv \exp(2a\phi)$, parametrized by the parameter a; corresponding conformal dimension

$$\Delta_a^{(L)} = a(Q-a)$$

Two kinds of Liouville operators will be important for the construction of physical fields in the MLG.
The first one is the degenerate fields $V_{m,n} \equiv V_{a_{m,n}}$ with

$$a_{m,n} = -b^{-1}\frac{(m-1)}{2} - b\frac{(n-1)}{2}$$

They obey the equations $D_{m,n}^{(L)} V_{m,n} = \bar{D}_{m,n}^{(L)} V_{m,n} = 0$, which are similar to those in the Minimal models.
The second type is the $V_{m,-n}$ fields used to construct physical fields in the MLG.

Three-point function in Liouville theory
$C_{\mathrm{L}}(a_1, a_2, a_3) = \langle V_{a_1}(0) V_{a_2}(1) V_{a_3}(\infty) \rangle_{\mathrm{L}}$ explicitly known (DOZZ):

$$C_{\mathrm{L}}(a_1, a_2, a_3) = \left(\pi\mu\gamma(b^2) b^{2-2b^2}\right)^{(Q-a)/b} \frac{\Upsilon_b(b)}{\Upsilon_b(a-Q)} \prod_{i=1}^{3} \frac{\Upsilon_b(2a_i)}{\Upsilon_b(a-a_i)}$$

where $a = a_1 + a_2 + a_3$ и $\Upsilon_b(x)$ — is a special function.
The operator prodact expansion (OPE) in Liouville theory is given by the following formula:

$$V_{a_1}(x) V_{a_2}(0) = \int' \frac{dP}{4\pi} C^{(\mathrm{L})Q/2+iP}_{a_1,a_2} (x\bar{x})^{\Delta^{(\mathrm{L})}_{Q/2+iP} - \Delta^{(\mathrm{L})}_{a_1} - \Delta^{(\mathrm{L})}_{a_2}} \left[V_{Q/2+iP}(0)\right]$$

where the structure constant $C^{(\mathrm{L})p}_{a_1,a_2} = C_{\mathrm{L}}(g, a, Q-p)$.

Integration here is along the imaginary axis if a_1 and a_2 lie in the region

$$|Q/2 - \operatorname{Re} a_1| + |Q/2 - \operatorname{Re} a_2| < Q/2$$

The form in other parameter regions is determined from the analytic continuation, i.e. the integral must be supplemented separately with the residues at the poles crossing the contour.

These additional contributions («discrete terms») are especially important for the formula to reproduce the operator expansion with degenerate fields $V_{m,n}$.

Ghosts and BRST invariance. The ghost sector is a fermionic (B, C)-system with field dimensions $(2, -1)$

$$A_{\mathrm{gh}} = \frac{1}{\pi} \int (C\bar{\partial}B + \bar{C}\partial\bar{B}) d^2x$$

and a central charge equal to -26. It appears as a result of fixing the calibration by the Faddeev-Popov method.

An odd BRST symmetry appears in the MLG, generated by a (holomorphic) charge

$$\mathcal{Q} = \oint (CT + C\partial CB) \frac{dz}{2\pi i}.$$

where T is the energy-momentum tensor of matter and the Liouville theory. By definition, physical states in the MLG are classes of BRST cohomology $\mathcal{Q}$ and antiholomorphic charge $\bar{\mathcal{Q}}$.

Physical fields and their correlators. The simplest cohomology representatives with ghost number zero can be obtained by «clothing» primary fields $\Phi_{m,n}$ with Liouville operators $V_{m,-n}$; the total dimension $U_{m,n} \equiv V_{m,-n}\Phi_{m,n}$ is equal to $(1,1)$.
BRST variation $U_{m,n}$ is given

$$\mathcal{Q}U_{m,n} = \partial(CU_{m,n})$$

therefore integrals of $U_{m,n}$ over the sphere are BRST-invariant.

Local physical fields with spiritual number one.
Instead of integration, we can consider the $(0,0)$-form $W_{m,n} \equiv C\overline{C}U_{m,n}$; it is BRST-closed:
$\mathcal{Q}W_{m,n} = \overline{\mathcal{Q}}W_{m,n} = 0$.
We will also parametrize such fields by the number a:
$W_a = V_a\Phi_{a-b}$.

In Minimal Gravity, there is an additional set of BRST-closed ghost zero fields, forming the so-called «Ring of discrete states». They are built from descendants of degenerate fields in both sectors and have the general form

$$O_{m,n}(x) = H_{m,n}\overline{H}_{m,n}\Theta_{m,n}, \quad \Theta_{m,n} \equiv V_{m,n}\Phi_{m,n}.$$

where $H_{m,n}$ is a polynomial of degree $mn-1$ from the Virasoro modes L_k^M, L_k, and ghosts B and C.
The general explicit form of $H_{m,n}$ is unknown, but it can be found in each individual case by requiring $\mathcal{Q}$-closure of $O_{m,n}$. These operators form a ring. They play an important role in the future derivation of the so-called Key relation.

Properties of operators from the Ring of discrete states:

1. the independence of correlators from their position in the sense that
$$\partial O_{m,n} = \text{BRST-exact}.$$
2. a simple form of merging with each other (in cohomology, i.e. up to $\mathcal{Q}$-exact terms)
$$O_{m,n}(x)O_{m',n'}(0) = \sum_{r=|m-m'|+1:2}^{m+m'+1} \sum_{s=|n-n'|+1:2}^{n+n'+1} G_{r,s}^{(m,n)|(m',n')} O_{r,s}(0)$$
3. and also with the operators W_a of the ghost number 1
$$O_{m,n}W_a = \sum_{r=-m+1:2}^{m-1} \sum_{s=-n+1:2}^{n-1} A_{r,s}^{(m,n)}(a)\, W_{a+\frac{rb^{-1}+sb}{2}}$$

By renormalizing the operators $O_{m,n}$ and $W_{m,n}$ the coefficients $G_{r,s}^{(m,n)|(m',n')}$ and $A_{r,s}^{(m,n)}(a)$ can be made equal to one in these formulas.

Higher equations of motion (HEM) and MLG. HEM by Alexei Zamolodchikov in the Liouville theory include the so-called logarithmic operators V'_a

$$V'_a(x) = \frac{1}{2}\frac{\partial}{\partial a}V_a(x).$$

$V'_{m,n}$ denote such derivatives with respect to the parameter a calculated at the point $a = a_{m,n}$.

The Higher equations of motion have the form

$$D^{(L)}_{m,n}\overline{D}^{(L)}_{m,n}V'_{m,n} = B_{m,n}V_{m,-n}, \tag{1}$$

where

$$B_{m,n} = (\pi\mu\gamma(b^2)b^{2-2b^2})^n \frac{\Upsilon'_b(2\alpha_{m,n})}{\Upsilon_b(2\alpha_{m,-n})} \tag{2}$$

The following important relations can be derived from HEM; We will call them **Key Relations (KR)**:

$$U_{m,n} = B^{-1}_{m,n}(\overline{\partial} - \bar{\mathcal{Q}}\bar{B}_{-1})(\partial - \mathcal{Q}B_{-1})O'_{m,n}.$$

here $O'_{m,n} := H_{m,n}\overline{H}_{m,n}\Theta'_{m,n}$ and $\Theta'_{m,n} := \Phi_{m,n}V'_{m,n}$,
B_{-1} —is the mode of the ghost field $B(z)$.

The four-point correlator.

Let's apply the CS to the four-point correlator

$$C_4(a_1, a_2, a_3|m, n) \equiv \frac{1}{Z_L}\left\langle \int d^2x\, \frac{U_{m,n}(x)}{\mathcal{N}(a_{m,-n})}\mathcal{W}_{a_1}(x_1)\mathcal{W}_{a_2}(x_2)\mathcal{W}_{a_3}(x_3)\right\rangle$$

Rewrite $U_{m,n}$ according to KR and take into account that the BRST-exact terms do not contribute because the W-operators are $\mathcal{Q}$-closed. Then

$$Z_L\, C_4(a_1, a_2, a_3|m, n) = \frac{1}{\pi}\left\langle \int d^2x\, \partial\bar{\partial}\mathcal{O}'_{m,n}\,\mathcal{W}_{a_1}\mathcal{W}_{a_2}\mathcal{W}_{a_3}\right\rangle$$

Boundary contributions of the neighborhoods $x = x_i$ and $x = \infty$ $O'_{m,n}$ are nonzero, since when O' is merged with $W(x_i)$, the multipliers arise such as $log((x - x_i)(\overline{x - x_i}))$ which give delta functions after differentiation.

Logarithmic factors in OPE O' with W can only arise from differentiation with respect to a of power factors $(x\overline{x})$ in discrete terms in the Liouville part.
For example, for the case of OPE $V'_{1,2}V_a$ we get

$$\log(x\overline{x})\Big(q_{0,1}^{(1,2)}(a)(x\overline{x})^{ab}C_L^{+}(a)[V_{a-b/2}(0)]+ \\ +q_{0,-1}^{(1,2)}(a)(x\overline{x})^{1-ab+b^2)}C_L^{-}(a)[V_{a+b/2}(0)]\Big)$$

$$q_{r,s}^{(m,n)} \equiv |a-\lambda_{r,s}-\frac{Q}{2}|-\lambda_{m,n}$$

This result coincides with the OPE of the usual primary field $V_{1,2}$, differing only by the factor $q_{r,s}^{(1,2)}$.
The same is true for the general case $V'_{m,n}$.

Multiplying the OPE in the Liouville and minimal model sectors and acting by $H_{m,n}\overline{H}_{m,n}$, we obtain an expression of the form

$$\mathcal{O}'_{m,n}(x)\mathcal{W}_a(0) = \log(x\overline{x}) \sum_{r=-m+1:2}^{m-1} \sum_{s=-n+1:2}^{n-1} q_{r,s}^{(m,n)}(a)\mathcal{W}_{a-\lambda_{r,s}} + \dots$$

The corresponding contributions to the four-point correlator (with the additional sign «-» due to the opposite orientation of the boundary contours for the neighborhoods of x_i and ∞) are

$$-\sum_{i=1}^{3} \sum_{r=-m+1:2}^{m-1} \sum_{s=-n+1:2}^{n-1} q_{r,s}^{(m,n)}(a_i)\langle \mathcal{W}_{a_i-\lambda_{r,s}} \dots \rangle$$

There is also an additional contribution from infinity. It is taken from the fact that $V'_{m,n}$ for $x \to \infty$ behaves like

$$V'_{1,2}(x) \sim -\Delta'_{m,n} \log(x\bar{x}) V_{1,2}(0),$$
$$\Delta'_{m,n} \equiv 2\lambda_{m,n} = mb^{-1} + nb$$

$O'_{m,n}$ behaves similarly; at infinity we can replace it with $O_{m,n}$ with the same coefficient and logarithm. The corresponding boundary contribution is given by

$$-2\lambda_{m,n} \left\langle \mathcal{O}_{m,n}(0) \mathcal{W}_{a_1}(x_1) \mathcal{W}_{a_2}(x_2) \mathcal{W}_{a_3}(x_3) \right\rangle$$

This correlator does not depend on the position of $O_{m,n}$ (this follows from the properties described earlier), so we can move it to any of the $\mathcal{W}$ fields and perform an operator expansion, obtaining, for example,

$$-2\lambda_{m,n} \sum_{r=-m+1:2}^{m-1} \sum_{s=-n+1:2}^{n-1} \left\langle \mathcal{W}_{a_1+\lambda_{r,s}}(x_1) \mathcal{W}_{a_2}(x_2) \mathcal{W}_{a_3}(x_3) \right\rangle$$

Collecting all contributions together and using expressions for normalized divisions on $\mathcal{N}(a)$ three-point correlators, we arrive at the final answer

$$C_4(a_1, a_2, a_3|m, n) = (b^{-6} - b^{-2})$$
$$\left[2mn\lambda_{mn} + \sum_{i=1}^{3} \sum_{r=-m+1:2}^{m-1} \sum_{s=-n+1:2}^{n-1} q_{r,s}^{(m,n)}(a_i)\right].$$

Let us apply the method used above to the case of a five-point correlator.

$$C_5(a_1, a_2, a_3|k_1, k_2) =$$
$$\left\langle \int d^2x \, \frac{U_{1,k_1+1}(x)}{\mathcal{N}(a_{1,-1-k_1})} \int d^2y \, \frac{U_{1,k_2+1}(y)}{\mathcal{N}(a_{1,-1-k_2})} \mathcal{W}_{a_1}(x_1)\mathcal{W}_{a_2}(x_2)\mathcal{W}_{a_3}(x_3) \right\rangle .$$

Now, applying CS to one of the $U_{m,n}$ fields, we cannot discard the BRST-exact terms, because in addition to the $\mathcal{Q}$-closed W-fields, there are also non - $\mathcal{Q}$-closed field $U_{m,n}$.

Let's start the calculation by applying the KR for the field U_{k_1}. The term with the second derivative $\partial\bar{\partial}\mathcal{O}'_{m,n}$ can be reduced to boundary contributions in the neighborhoods of x_i, y and ∞.

Contributions for $x \sim x_i$**.** Apply OPE $\mathcal{O}'$ c $\mathcal{W}_a(x_i)$. As before, only the logarithmic terms are important for this purpose. In total, the contributions received will give

$$-\sum_{i=1}^{3}\sum_{s=-k_1:2}^{k_1} q_{0,s}^{(1,k_1+1)}(a_i)\left\langle \int d^2y\, \frac{U_{1,k_2+1}(y)}{\mathcal{N}(a_{1,-1-k_2})}\, \mathcal{W}_{a_i-\lambda_{0,s}}(x_i)\ldots \right\rangle$$

that is, they are expressed in terms of the previously calculated 4-correlator.

Contribution at $x \to \infty$.
Using the asymptotic behavior of $O'(x)$ as $x \to \infty$, we get the corresponding boundary contribution

$$-2\lambda_{1,k_1+1} \int d^2y \left\langle \mathcal{O}_{1,k_1+1}(0) \frac{U_{1,k_2+1}(y)}{\mathcal{N}(a_{1,-1-k_2})} \prod_{i=1}^{3} \mathcal{W}_{a_i} \right\rangle$$

To calculate it, we apply KR now for $U_{1,k_2+1}(y)$. $\mathcal{Q}$-exact terms in KR are no important here, because all other operators are $\mathcal{Q}$-closed. So we get

$$-\frac{2\lambda_{1,k_1+1}}{\pi} \int d^2y \, \partial\overline{\partial} \left\langle \left(\mathcal{O}'_{1,1+k_2}(y)\right) \mathcal{O}_{1,k_1+1}(0) \prod_{i=1}^{3} \mathcal{W}_{a_i} \right\rangle$$

Now the integral over y is reduced to the boundary terms in the vicinity of $x_i, 0, \infty$. The new moment appearing here is the logarithmic contribution to OPE from $O'_{1,1+k_2}(y)O_{1,k_1+1}(0)$. Since the fields included in the logarithmic part in the OPE $V'_{1,k}$ and V_a are the same as in the OPE of the ordinary primary field $V_{1,k}$ and V_a (besides the factors $q_{r,s}^{(m,n)}$), then it is enough to add these factors to it:

$$\mathcal{O}'_{1,k_2+1}(y)\mathcal{O}_{1,k_1+1}(x) = \log|y-x|^2 \sum_{s=k_2-k_1}^{k_2+k_1} q_{0,s-k_1}^{(1,k_2+1)}(a_{1,k_1+1})\mathcal{O}_{1,1+s} + \ldots$$

The four-point correlator $\langle OWWW \rangle$ can now be computed as before.

Contribution for $x \sim y$.
These contributions are the most problematic to calculate for two reasons. First, such boundary terms arise not only from $\partial\bar{\partial}O'_{1,1+k_1}$, but also from $\mathcal{Q}$-exact terms in the CK.

Secondly, due to the lack of additional ghosts C, the logarithmic terms in OPE $O'_{1,1+k_1}(x)$ c $U_{1,1+k_2}$ are different from those in OPE O and W.

However, it turns out that these two problems cancel each other out. To see this, we must take the following steps.

First, we rewrite the product of local operators $U_{1,1+k_1}(x)U_{1,1+k_2}(y)$ in the expression for the path integral for 5 - point function using Key Equation as

$$B_{1,1+k_1}U_{1,1+k_1}(x)U_{1,1+k_2}(y) =$$

$$(\overline{\partial}\partial - \bar{\mathcal{Q}}\bar{B}_{-1}\partial - \overline{\partial}\mathcal{Q}B_{-1} + \bar{\mathcal{Q}}\bar{B}_{-1}\mathcal{Q}B_{-1})O'_{m,n}(x)U_{1,1+k_2}(y). (3)$$

Second, we move the action of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ from $O'_{m,n}(x)$ to $U_{1,1+k_2}(y)$ and get

$$\begin{aligned}&\overline{\partial}\partial O'_{m,n}(x)U_{1,1+k_2}(y) - \overline{\partial}B_{-1}O'_{m,n}(x)\mathcal{Q}U_{1,1+k_2}(y) - \\ &\partial\bar{B}_{-1}O'_{m,n}(x)\bar{\mathcal{Q}}U_{1,1+k_2}(y) + \bar{B}_{-1}B_{-1}O'_{m,n}(x)\bar{\mathcal{Q}}\mathcal{Q}U_{1,1+k_2}(y).\end{aligned} \quad (4)$$

Next, using $\mathcal{Q}U_{1,1+k_2}(y) = \partial_y(CU_{1,1+k_2}(y))$ we get the following expression for the product of the fields in this part of the 5-point correlator

$$\int d^2y\, U_{k_2}(y) \int d^2x\, \partial_x\overline{\partial}_x \left(H_{1,1+k_1}\overline{H}_{1,1+k_1}\Theta'_{1,1+k_1}\right) \prod_{i=1}^{3}\mathcal{W}_{a_i}(x_i) \tag{5}$$

$$-\int d^2y \int d^2x\, \overline{\partial}_x \left(R_{1,1+k_1}\overline{H}_{1,1+k_1}\Theta'_{1,1+k_1}\right) \partial_y(CU_{k_2}(y)) \prod_{i=1}^{3}\mathcal{W}_{a_i}(x_i) \tag{6}$$

$$-\int d^2y \int d^2x\, \partial_x \left(\overline{R}_{1,1+k_1}H_{1,1+k_1}\Theta'_{1,1+k_1}\right) \overline{\partial}_y(\overline{C}U_{k_2}(y)) \prod_{i=1}^{3}\mathcal{W}_{a_i}(x_i) \tag{7}$$

$$+\int d^2y \int d^2x\, R_{1,1+k_1}\overline{R}_{1,1+k_1}\Theta'_{1,1+k_1}\partial_y\overline{\partial}_y \left(C\overline{C}U_{k_2}(y)\right) \prod_{i=1}^{3}\mathcal{W}_{a_i}(x_i), \tag{8}$$

All lines contain integrals that reduce to boundary contributions from the region where x is close to y, which can be calculated from the Stokes theorem.

These contributions are equal to the residues at the poles arising from differentiating the logarithmic factors $\log(x - y)$ in the Operator Product Expansion of the logarithmic field at y with the primary field at x.

The remaining terms resulting from differentiation do not contain first-order poles and do not give any boundary contributions.

It is noteworthy that, in sum, the boundary terms from the neighborhood y reduce to an expression similar to that obtained from the neighborhood x_i.
Namely, they look like

$$-\sum_{s=-k_1:2}^{k_1} \frac{q_{0,s}^{(1,k_1+1)}(a_{1,-k_2-1})}{\mathcal{N}(a_{1,-1-(k_2-s)})} \left\langle \int d^2y\, U_{1,(k_2-s)+1}(y)\mathcal{W}_{a_i}(x_i)\dots \right\rangle . \tag{9}$$

As a result, by collecting all the contributions, we obtain the following explicit expression for the 5-point correlator

$$C_5(a_1,a_2,a_3|k_1,k_2) = (b^{-6} - b^{-2}\,[\Sigma_1 + \Sigma_2 + \Sigma_3]).$$

$$\Sigma_1 = \sum_{s=-k_1:2}^{k_1} q_{0,s}^{(1,k_1+1)}(a_{1,-k_2-1}) \times \Bigg[2(1+k_2-s)\lambda_{1,1+k_2-s} + \sum_{i=1}^{3} \sum_{l=-k_2+s:2}^{k_2-s} q_{0,l}^{(1,1+k_2-s)}(a_i)\Bigg]$$

$$\Sigma_2 = \sum_{i=1}^{3} \sum_{s=-k_1:2}^{k_1} q_{0,s}^{(1,k_1+1)}(a_i)\Bigg[2(1+k_2)\lambda_{1,1+k_2} + \sum_{l=-k_2:2}^{k_2} \left(q_{0,l}^{(1,k_2+1)}(a_i - \lambda_{0,s}) + \sum_{j\neq i} q_{0,l}^{(1,k_2+1)}(a_j) \right)\Bigg]$$

$$\Sigma_3 = 2\lambda_{1,1+k_1} \Bigg[\sum_{s=-k_1:2}^{k_1} \sum_{l=-k_2:2}^{k_2} \left(\sum_{i=1}^{3} q_{0,l}^{(1,1+k_2)}(a_i) + 2\lambda_{1,k_2+1} \right) + \sum_{s=k_2-k_1}^{k_2+k_1} q_{0,s-k_1}^{(1,k_2+1)}(a_{1,k_1+1})(1+s) \Bigg].$$

Generalization to **the case of N-point correlator** in MLG with $N > 5$ can be done directly.
We just need to do the same steps. Namely, choosing of one of the $N-3$ fields of the type $U_{k_1}(y_1)$ we must integrate over the variable y_1 and use then the Key relation for this field.
In the result we obtain boundary contributions of three non-integrated fields $W_a(x_i)$, the contribution of ∞ and contributions of the vicinities of y_j, positions of the other $U_{k_j}(y_j), j > 1$.
The final expression for the original N-point correlator will reduce to a sum of N'-point correlators with $N' < N$.