Concentrating Gravity on Branes

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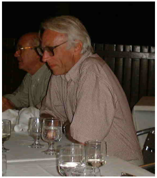

Work with Ben Crampton & Chris Pope;
Alex Harrold, Chris Erickson & Rahim Leung

Meetings and friendship with Andrei Slavnov

I believe that I first met Andrei at the first Moscow Seminar on Quantum Gravity, organised by Professor M.A. Markov in the Dom Uchenikh back in December 1978.

Since then, we met again many times, discussed, had dinners, drank wine and shared life stories: subsequent Quantum Gravity Seminars, several meetings in Alushta, Zvenogorod (1982), Yerevan (1983), meetings at JINR, FIAN and many more. We were pleased to host Andrei and his wife Slava when they visited us at Imperial College in 2010. In particular, I remember the festschrift meeting for Andrei's 70th birthday at MIAN, after which he invited a number of us to come out to his dacha near Zvenigorod for a marvellous afternoon and dinner: Russia as it should be.

The picture on the first page is from a conference in Sardinia in 1999, another great memory of a meeting with Andrei. His warm friendship and brilliance now are very much missed.

Some taxonomy about branes

Generalized membrane solutions of higher dimensional gravitational theories, or “branes” are characteristic solutions of supergravity theories. A taxonomy arises of different effective-theory behaviors that can occur in fluctuations away from such static brane “vacua”:

1) Consistent-truncation Kaluza-Klein reductions using the static brane as a “skeleton”. The resulting lower-dimensional worldvolume theory is a supergravity theory with the same degree of supersymmetry as that preserved by the brane skeleton “vacuum”. All fields share the underlying brane skeleton extension into the transverse higher dimensions.
2) Full localization of gravitational phenomena near the underlying brane, then falling off towards transverse infinity. This generates a massless effective lower-dimensional theory, but not via a technically consistent reduction.

The distinction between these two types depends on the choice of boundary conditions for fluctuations near the brane worldvolume.

Type 1: Ricci-flat branes and supergravity extensions

For every flat-worldvolume brane solution of a higher-dimensional supergravity it is possible to replace the flat worldvolume metric by a Ricci-flat metric. Brecher & Perry hep-th/9908018

Such constructions have been used to put black holes into dimensionally-reduced worldvolume theories. Chamblin, Hawking and Reall called such solutions “black strings” starting from 5D.

Chamblin, Hawking & Reall, hep-th/9909205

If there is surviving supersymmetry in the flat-worldvolume brane solution, Lü and Pope showed in a Randall-Sundrum context that one can make a consistent embedding of a corresponding supergravity theory into the asymptotic near-horizon geometry of the brane solution (*e.g.* $AdS_5 \times S^5$).

Lü & Pope, hep-th/0008050; Cvetič, Lü & Pope, hep-th/0009183

Such a near-horizon embedding can also be extended to a consistent embedding of the lower dimensional supergravity into a full brane solution, serving as a “skeleton” within the higher dimensional theory. R. Leung & KSS, arXiv/2205.13551

To see how one can embed gravity phenomena on the worldvolume of a p brane solution, consider first a model system in D spacetime dimensions consisting of a metric, a $(p+1)$ form gauge field and a scalar,

$$I_{\text{model}} = \int_{M_D} \left(R \star 1 - \frac{1}{2} d\phi \wedge \star d\phi - \frac{1}{2} e^{a\phi} F_{[p+2]} \wedge \star F_{[p+2]} \right)$$

Consider a warped-metric brane-solution ansatz

$$ds_d^2 = e^{2A(y)} \overline{g}_{\mu\nu}(x) dx^\mu dx^\nu + e^{2B(y)} \widetilde{g}_{ij}(y) dy^i dy^j \ .$$

For this, one obtains the D dimensional Ricci tensor components

$$\begin{aligned} R_{\mu\nu} &= \overline{R}_{\mu\nu} - e^{2(A-B)} \Big(\widetilde{\nabla}^2 A + \widetilde{g}^{ij} \partial_i A \big(d_e \partial_j A + d_m \partial_j B \big) \Big) \overline{g}_{\mu\nu} \ , \\ R_{ij} &= \widetilde{R}_{ij} - d_e \widetilde{\nabla}_i \widetilde{\nabla}_j A - d_m \widetilde{\nabla}_i \widetilde{\nabla}_j B + d_m \partial_i B \partial_j B - d_e \partial_i A \partial_j A \\ &\quad + 2 d_e \partial_{(i} A \partial_{j)} B - \Big(\widetilde{\nabla}^2 B + \widetilde{g}^{kl} \partial_k B \big(d_e \partial_l A + d_m \partial_l B \big) \Big) \widetilde{g}_{ij} \ . \end{aligned}$$

where $\overline{R}_{\mu\nu}$ and $\widetilde{R}_{ij}$ are the Ricci tensors of $\overline{g}_{\mu\nu}$ and $\widetilde{g}_{ij}$ respectively, $\widetilde{\nabla}_i$ is the covariant derivative with respect to $\widetilde{g}_{ij}$, and $d_e = p+1$, $d_m = d - p - 3$.

The "skeleton" brane solution will have $\overline{g}_{\mu\nu}(x) = \eta_{\mu\nu}$, but it is clear from the full $R_{\mu\nu}$, R_{ij} Ricci-tensor expressions that any $\overline{R}_{\mu\nu}$ Ricci-flat worldvolume metric $\overline{g}_{\mu\nu}(x)$ and $\widetilde{R}_{ij}$ Ricci-flat transverse metric $\widetilde{g}_{ij}$ will also yield a solution to the model system field equations, provided the A and B warp factors retain the same structure as in the vacuum solution.

The generalization incorporating both worldvolume $\overline{g}_{\mu\nu}(x)$ and transverse $\widetilde{g}_{ij}$ Ricci-flat metrics may be called "doubly Ricci-flat". In the following we will concentrate on the Minkowski-signature $\overline{g}_{\mu\nu}(x)$ Ricci-flat worldvolume generalization and the extension of it to a worldvolume supergravity.

Worldvolume supergravities

Provided the vacuum brane solution with $\overline{g}_{\mu\nu}(x) = \eta_{\mu\nu}$ preserves some degree of unbroken supersymmetry, the worldvolume Ricci-flat pure gravity can be generalized further to an arbitrary solution of a worldvolume supergravity theory with the vacuum's degree of supersymmetry. This creates a type of Kaluza-Klein *consistent truncation* ansatz for embedding lower-dimensional (in the model system example, $d = d_e = (p+1)$ dimensional) supergravity into the original D dimensional theory.

The key to constructing this reduction ansatz is to exploit the *interpolating soliton* structure of supersymmetric brane solutions, interpolating between flat space at infinity and another solution in the asymptotic horizon/brane worldvolume geometry.

A step along the way towards a consistent brane-skeleton ansatz was made by Lü and Pope who constructed an $N = 2$, $d = 5$ system of Randall-Sundrum type within the S^5 reduced geometry of Type IIB supergravity, using known techniques for Kaluza-Klein reductions on spheres. Lü & Pope, hep-th/0008050; Cvetič, Lü & Pope, hep-th/0009183

Since that system's $AdS_5 \times S^5$ vacuum solution is also the near-horizon geometry of the $\frac{1}{2}$ BPS Type IIB supergravity D3 brane, this motivates trying a KK ansatz for extending the pure gravity Ricci-flat solution to a general solution of a corresponding 16 supercharge (*i.e.* $N = 4$ in $d = 4$) supergravity.

Careful study of the $AdS_5 \times S^5$ reduction ansatz as well as the requirements for embedding in the flat-space limit at infinity motivate an ansatz for embedding $d = 4$, $N = 4$ supergravity into a KK ansatz with the D3 brane as the skeleton background.

R. Leung & KSS, arXiv/2205.13551

Here is the full D3 braneworld ansatz that works, producing the bosonic sector of $d = 4$, $N = 4$ supergravity, where the $D = 10$ Type IIB complex scalar is $\hat{\tau} = \hat{C}_0 + ie^{-\hat{\Phi}}$ and the coordinates are split into x^μ on the $d = 4$ worldvolume and y^Λ, $\Lambda = 1, \ldots, 6$ in the six transverse dimensions:

$$
\begin{aligned}
d\hat{s}^2 &= H^{-1/2} g_{\mu\nu}(x) dx^\mu dx^\nu + H^{1/2} dy^\Lambda dy^\Lambda \\
\hat{\Phi} &= \phi(x)\,, \quad \hat{C}_0 = -\chi(x) \\
\hat{H}_{(3)} &= \frac{1}{\sqrt{2}} \mathcal{F}^\Lambda_{(2)} \wedge dy^\Lambda\,, \quad \hat{F}_{(3)} = -\frac{1}{\sqrt{2}} e^{-\phi} *_4 \mathcal{F}^\Lambda_{(2)} \wedge dy^\Lambda \\
\hat{F}_{(5)} &= H^{-2} \,\mathrm{vol}_4 \wedge dH - *_6 dH\,, \quad H = 1 + \frac{4\pi N}{r^4}\,.
\end{aligned}
$$

where $\mathcal{F}^\Lambda_{(2)}$ is a two-form on the $d = 4$ subspace corresponding to the field strengths of the $\mathcal{N} = 4$ theory, $*_4$ is the Hodge dual computed with respect to the four-dimensional metric $g_{\mu\nu}$, vol_4 is the volume form associated with $g_{\mu\nu}$, and $*_6$ is the Hodge dual computed with respect to the flat metric $\delta_{\Lambda\Sigma}$ on the transverse $\mathbb{R}^6$.

Salam-Sezgin theory and its two taxonomies

Abdus Salam and Ergin Sezgin constructed in 1984 a version of 6D minimal (chiral, *i.e.* (1,0)) supergravity coupled to a 6D 2-form tensor multiplet and a 6D super-Maxwell multiplet that gauges the U(1) R-symmetry of the theory. Phys.Lett. B147 (1984) 47 This Einstein-tensor-Maxwell system has the bosonic Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{SS}} &= \tfrac{1}{2}R - \tfrac{1}{4g^2}e^{\bar{\phi}}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{6}e^{-2\bar{\phi}}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} - g^2e^{-\bar{\phi}} \\ G_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} + 3F_{[\mu\nu}A_{\rho]} \end{aligned}$$

Note the *positive* potential term for the scalar field $\bar{\phi}$. This is a key feature of R-symmetry-gauged models generalising the Salam-Sezgin model, leading to models with noncompact symmetries. For example, upon coupling to yet more vector multiplets, the sigma-model target space can have a structure $SO(p,q)/(SO(p)\times SO(q))$. Bergshoeff,Jong&Sezgin (2005); Pugh,Sezgin&KSS (2010)

The Salam-Sezgin theory does not admit a maximally symmetric 6D solution, but it does admit a $(\text{Minkowski})_4 \times S^2$ "vacuum" solution with the flux for a $U(1)$ monopole turned on in the S^2 directions

$$\begin{aligned} ds_6^2 &= dx^\mu dx^\nu \eta_{\mu\nu} + \frac{1}{4g^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \\ A_{(1)} &= -\frac{1}{\sqrt{2}g}\cos\theta\, d\varphi\,, \qquad G_{(3)} = 0\,, \qquad \bar{\phi} = 0\,. \end{aligned}$$

Within the $D = 10$ type IIA supergravity theory (or equivalently within type IIB, after a T-duality transformation), the Salam-Sezgin vacuum is a BPS solution preserving 8 unbroken supercharges.

Lifting the Salam-Sezgin vacuum back to $D = 10$

In the Einstein frame, the 6D SS vacuum is a 10D nonsingular type IIA solution (where $\mu = 0, 1, 2, 3$ correspond to the 4D subspace). This solution can be written

Cvetič, Gibbons & Pope, Nucl. Phys. B677 (2004) 164; Crampton, Pope & KSS, 1408.7072

$$
\begin{aligned}
d\hat{s}_{10}^2 &= H_{SS}^{-\frac{1}{4}}(dx^\mu dx_\mu + dy^2 + \tfrac{1}{4g^2}\,[d\psi + \operatorname{sech} 2\rho\,(d\chi + \cos\theta\, d\varphi)]^2) + H_{SS}^{\frac{3}{4}}\, ds_{EH}^2 \\
e^{\hat{\phi}} &= H_{SS}^{\frac{1}{2}}\,, \quad \hat{A}_2 = \frac{1}{4g^2}\left[d\chi + \operatorname{sech} 2\rho\, d\psi\right] \wedge (d\chi + \cos\theta\, d\varphi)
\end{aligned}
$$

where

$$
\begin{aligned}
ds_{EH}^2 &= \big(\cosh 2\rho d\rho^2 + \tfrac{1}{4}\cosh 2\rho(d\theta^2 + \sin\theta d\varphi^2) \\
&\qquad + \tfrac{1}{4}\sinh 2\rho \tanh 2\rho (d\chi + \cos\theta d\varphi)^2\big) \\
H_{SS} &= \operatorname{sech} 2\rho\ .
\end{aligned}
$$

The ds_{EH}^2 noncompact part of the transverse space metric is a form of the 4-dimensional Eguchi-Hanson metric. The coordinates y and ψ correspond to S^1 circles.

$N = 2$ supergravity on the Salam-Sezgin skeleton

As for the D3 brane, one can effect a consistent truncation of the $D = 10$ type IIB theory (T-dual to IIA) to a worldvolume supergravity theory with 8 supercharges (*i.e.* N=2 in $D = 4$):

$$d\hat{s}^2 = H^{-1/2} g_{\mu\nu} dx^\mu dx^\nu + H^{1/2}(ds^2_{\mathrm{EH}} + dz d\bar{z}) \,, \quad \hat{\Phi} = 0 \,, \quad \hat{C}_0 = 0 \,,$$

$$\hat{G}_{(3)} = \mathcal{F}_{(2)} \wedge d\bar{z} + \frac{1}{2}(F_{(2)} + i{*_4}F_{(2)}) \wedge dz \,,$$

$$\hat{F}_{(5)} = H^{-2} \,\mathrm{vol}_4 \wedge dH - \frac{i}{2} {*_{\mathrm{EH}}} dH \wedge dz \wedge d\bar{z} \,.$$

The success of this consistent braneworld gravity truncation, as for the D3 braneworld, suggests that such consistent truncations may be a general feature of supersymmetric braneworlds. So far, this is known from a proliferating set of examples. Obtaining a general proof remains an open problem.

Solutions within and without a consistently truncated theory

The consistent truncations to braneworld supergravities based on an original brane "skeleton" allow for arbitrary solutions of the truncated theory to be realised also as exact solutions of the parent higher-dimensional theory. Depending on the particulars of a given case, one can have supersymmetric or non-supersymmetric black holes, branes within the truncated braneworld, or any solution of the truncated theory.

But this works only for solutions purely within the truncated theory. As soon as one tries to couple to matter sources external to the truncated theory, one runs into the problem of a vanishing Newton constant identified by Hull and Warner for reductions on noncompact spaces. Hull & Warner, Class. Quant. Grav. **5** (1988) 1517 . Purely within the truncated theory, this problem does not occur because pure supergravities, like general relativity itself, possess a "trombone" scaling symmetry which makes Newton's constant irrelevant (until one couples to external sources, that is).

The other taxonomy: Salam-Sezgin genuinely concentrated braneworld gravity

An approach to obtaining a genuine concentration of gravity on a braneworld subspace is to look for a *normalizable* transverse-space wavefunction $\xi(\rho)$ for $h_{\mu\nu}(x,\rho) = h_{\mu\nu}(x)\xi(\rho)$.

Crampton, Pope & KSS, 1408.7072

General study of the fluctuation spectra about brane solutions shows that the mass spectrum of the spin-two fluctuations about a brane background is given by the spectrum of the scalar Laplacian in the transverse embedding space of the brane.

Csáki, Erlich, Hollowood & Shirman, Nucl.Phys. B581 (2000) 309; Bachas & Estes, JHEP 1106 (2011) 005

$$
\begin{aligned}
\Box_{(10)}F &= \frac{1}{\sqrt{-\det g_{(10)}}}\partial_M\left(\sqrt{-\det g_{(10)}}\,g^{MN}_{(10)}\partial_N F\right) \\
&= H_{\rm SS}^{\frac{1}{4}}(\Box_{(4)} + g^2\triangle_{\theta,\phi,y,\psi,\chi} + g^2\triangle) \\
H_{\rm SS} &= (\cosh 2\rho)^{-1}\ \text{warp factor;}\quad \triangle = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\tanh(2\rho)}\frac{\partial}{\partial\rho}
\end{aligned}
$$

The directions θ, ϕ, y, ψ & χ are all compact, and one can employ ordinary Kaluza-Klein methods for reduction on them, truncating to the invariant sector for these coordinates, but still allowing dependence on the noncompact coordinate ρ.

To handle the noncompact direction ρ, one needs to expand all fields in eigenmodes of $\triangle$:

$$\phi(x^\mu, \rho) = \sum_i \phi_{\omega_i}(x^\mu)\xi_{\omega_i}(\rho) + \int_\Lambda^\infty d\omega\, \phi_\omega(x^\mu)\xi_\omega(\rho)$$

where the ϕ_{ω_i} are discrete eigenmodes and the ϕ_ω are continuous Kaluza-Klein eigenmodes. Their eigenvalues give Kaluza-Klein masses $m_\omega = g\omega$ in 4D from solutions to the wave equation $\square_{(10)}\phi_\omega = 0$ using $\triangle_{\theta,\phi,y,\psi,\chi}\phi_\omega = 0$

$$\begin{aligned} \triangle\xi_\omega &= -\omega^2\xi_\omega \\ \square_{(4)}\phi_\omega &= (g^2\omega^2)\phi_\omega \,. \end{aligned}$$

The Schrödinger equation for $\mathcal{H}^{(2,2)}$ eigenfunctions

One can rewrite the $\triangle$ eigenvalue problem in terms of a Schrödinger equation by making the substitution

$$\Psi_\omega = \sqrt{\sinh(2\rho)}\xi_\omega$$

after which the eigenfunction equation takes the Schrödinger equation form

$$-\frac{d^2\Psi_\omega}{d\rho^2} + V(\rho)\Psi_\omega = \omega^2\Psi_\omega$$

where the potential is

$$V(\rho) = 2 - \frac{1}{\tanh^2(2\rho)} \, .$$

The SS Schrödinger equation potential $V(\rho)$ asymptotes to the value 1 for large ρ. In this limit, the Schrödinger equation becomes

$$\frac{d^2\Psi_\omega}{d\rho^2} + (\omega^2 - 1)\Psi_\omega = 0$$

giving "scattering-state" solutions for $\omega^2 > 1$:

$$\Psi_\omega(\rho) \sim \left(A_\omega e^{i\sqrt{\omega^2-1}\rho} + B_\omega e^{-i\sqrt{\omega^2-1}\rho}\right) \qquad \text{for large } \rho$$

while for $\omega^2 < 1$, one can have L^2 normalizable bound states. Recalling the ρ dependence of the measure $\sqrt{-g_{(10)}} \sim (\cosh(2\rho))^{\frac{1}{4}} \sinh(2\rho)$, one finds for large ρ

$$\int_{\rho_1 \gg 1}^{\infty} |\Psi_\omega(\rho)|^2 d\rho < \infty \;\Rightarrow\; \Psi_\omega \sim B_\omega e^{-\sqrt{1-\omega^2}\rho} \text{ for } \omega^2 < 1\,.$$

So for $\omega^2 < 1$ we can have candidate bound states, then a *mass gap* up to the edge of the scattering states' continuum spectrum.

The zero-mode bound state and massless 4D gravitons

The 1-D Schrödinger system with the $V(\rho) = 2 - \coth^2(2\rho)$ potential belongs to a special class of Pöschl-Teller integrable systems. Study of this system, and in particular of its self-adjointness properties, shows that it has a *unique* bound state separated by a mass gap before the onset of a continuum of delta-function-normalizable scattering states.

Happily, for $\omega = 0$ the Schrödinger equation can be solved exactly. The normalised result is

$$\Psi_0(\rho) = \sqrt{\sinh(2\rho)}\xi_0(\rho) = \frac{2\sqrt{3}}{\pi}\sqrt{\sinh(2\rho)}\log(\tanh\rho)\,.$$

Metric excitations $h_{\mu\nu}(x)\xi_0(\rho)$ around the 10D lifted SS background correspond, at the linearised level, to massless 4D gravitons on the 4D worldvolume subspacetime.

Near-field versus far-field behaviour

The gravity fluctuations with normalizable transverse wavefunction ξ_0 successfully concentrate massless gravity near the Salam-Sezgin braneworld surface. They interact with general sources localized on and near the braneworld in a hybrid fashion. For example, the near-field behaviour near a mass point approximates to Newtonian behaviour in the higher-dimensional theory, but the far-field behaviour asymptotes to Newtonian behaviour in the lower worldvolume dimension. This behaviour is seen in the gravitational Green function

$$G(0,r;\rho,\eta) = -\frac{\hat{\kappa}^2 gM}{4\pi r}\xi_0(\rho)\xi_0(\eta) - \int_1^\infty \frac{\hat{\kappa}^2 gM \exp(-g\omega r)}{4\pi r}\xi_\omega(\rho)\xi_\omega(\eta)d\omega$$

Near-field behaviour:

$$G(r,\rho) = -\frac{\hat{\kappa}^2 g^4 M}{2\pi\left(g^2r^2+\rho^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{R^2}\right) \ ,$$

where $R^2 = g^2r^2 + \rho^2$. This R dependence corresponds to $D = 6$, in a presentation where the theory is reduced on the naturally compact directions $\theta, \phi, y, \&\ \psi$. Note that the Eguchi-Hanson metric is a metric on the tangent bundle of S^2, and as one approaches the $\rho \to 0$ EH "nose", the SS vacuum solution has an asymptotic $\mathbb{R}^2 \times \{\text{compact}\}$ structure, with the ρ and χ coordinates in the $\mathbb{R}^2$ so the higher-dimensional structure naturally corresponds to $(D = 6) \times \{\text{compact}\}$ in such a presentation.

Far-field behaviour, on the other hand:

$$G(0,r;\rho,\eta) = -\frac{\hat{\kappa}^2 gM}{4\pi r}\xi_0(\rho)\xi_0(\eta) + \mathcal{O}\left(\exp(-gr)\right) \ ,$$

with the characteristic $1/r$ worldvolume $d = 4$ Newtonian structure.

Other gravity-concentrating systems

Analysis of the Sturm-Liouville systems corresponding to gravitational perturbations around a variety of braneworld vacua yields a number of other systems where gravitational fluctuations can be concentrated or "captured" near the braneworld.

Leung&KSS, in preparation

- Randall-Sundrum
- D3-branes on a resolved conifold over $S^5/\mathbb{Z}_3$
- D3-branes on resolved conifolds over $Y^{p,q}$
- D3-branes on a resolved cone over $T^{1,1}/\mathbb{Z}_2$

Examples where gravity is *not* concentrated near the braneworld are

- The simple D3 brane
- NS5-branes on Taub-NUT

Aside from the Randall-Sundrum case, the above gravity-concentrating cases have an asymptotic $\mathbb{R}^2 \times \{\text{compact}\}$ structure as one approaches the braneworld, and this gives rise to a logarithmic behaviour of the transverse wavefunction.

Overview

- Supersymmetric brane solutions give rise to a type of braneworld consistent-truncation Kaluza-Klein reduction with a fully interacting braneworld supergravity possessing the same supersymmetry as that of the "vacuum" brane, which serves as a "skeleton" for the KK ansatz.
- Alternately, obtaining *fully localizable* lower-dimensional gravitational behaviour requires a normalisable transverse wavefunction zero-mode like $\xi_0(\rho)$. Choosing a transverse zero-mode like $\xi_0(\rho)$ requires a modified choice of near-brane boundary conditions.
- Upon implementing such modified transverse boundary conditions near the worldvolume, one can achieve lower-dimensional *far-field* (*e.g.* $1/r$ for $d = 4$) gravitational behaviour, while preserving higher-dimensional *near-field* behaviour (*e.g.* $1/R^3$ for $D = 6$) as one gets close to a point-mass source.