On symmetry properties of correlation functions in various charts of Minkowski and de Sitter spaces

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Setup

- We consider massive real scalar field

$$S = \int d^d X \sqrt{|g|} \left[\frac{1}{2} (\partial \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{n!} \phi^n \right].$$

  in various charts (patches or wedges) of **Minkowski** and **de Sitter** space-times;
- We restrict considerations to the **Poincare** and **de Sitter isometry** invariant states. For such states the **Wightman propagator** is:

$$W(X, Y) \equiv \left\langle \phi(X) \phi(Y) \right\rangle = \mathfrak{F}\left[L_{XY}^2 - i\, \epsilon\, \mathrm{sign} \Delta X^0 \right].$$

  Here $\mathfrak{F}[Z]$ is the analytic function the the complex plane with the cut along time-like separations. From this correlation function one can construct any other propagator.

Why considering patches of entire space-times?

In studying Unruh effect one usually considers d-dimensional **right Rindler wedge**:

$$X^1 \geq \left|X^0\right|, \quad X^0 = e^{\xi}\sinh\tau, \quad X^1 = e^{\xi}\cosh\tau,$$

a quarter of the entire d-dimensional **Minkowski** space-time:

$$ds^2 = (dX^0)^2 - (dX^1)^2 - (dX^a)^2 = e^{2\xi}(d\tau^2 - d\xi^2) - (dX^a)^2.$$

For academic studies one also can consider other patches: left Rindler wedge, **upper or future wedge** and **lower or past** wedge.

Various wadges of Minkowski space-time

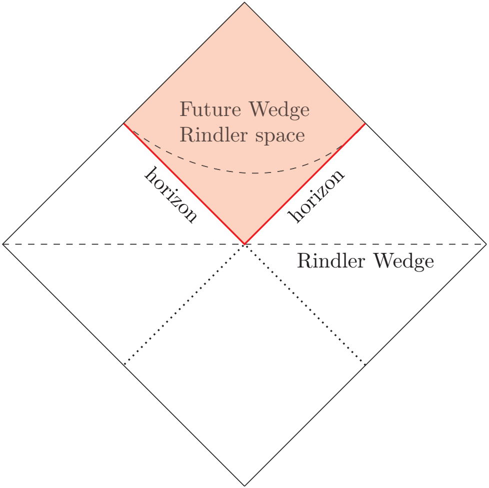

Figure: The dashed lines depict the Cauchy surfaces in various charts

Why considering patches of entire space-times?

In studying inflation one usually considers **Poincare patch**:

$$ds^2 = d\tau^2 - e^{2\tau}(dx^i)^2 = \frac{d\eta^2 - (dx^i)^2}{\eta^2}, \quad \eta = e^{-\tau},$$

which is a half, $X^0 > -X^d$, of the entire d-dimensional **de Sitter hyperboloid**

$$X_0^2 - X_a^2 - X_d^2 = -1, \quad H = 1.$$

The hyperboloid is embedded into $(d+1)$-dimensional ambient **Minkowski** space-time

$$ds^2 = dX_0^2 - dX_a^2 - dX_d^2.$$

We do not really know what was the initial state of the **Universe** (neither topology of Cauchy surfaces nor the Fock space state). Hence, it is worth studying also other patches of the **de Sitter space-time**.

What is the problem with the consideration of patches?

- To consider full QFT **in a patch** one has to do loop integrals. In the **vertexes** of the loop integrals one integrates over the patch;
- Namely, e.g. in the **right Rindler wedge** the measure of integration over a vertex Y in a loop integral contains:

$$dVol_Y = d^d Y \, \theta(Y^1 - Y^0) \, \theta(Y^1 + Y^0)$$

  The theta-functions violate the **Poincare isometry** of Minkowski space-time;
- Then, what about Poincare symmetry of the loop corrections? The same problem appears also in de Sitter patches.

- **Patches** of entire space-times have boundaries. Roughly speaking to quantize a theory in a patch one has to impose boundary and/or initial conditions at the boundaries. Then, instead of the **Feynman** one has to apply the **Schwinger-Keldysh** diagrammatic technique;
- In any case the **Feynman technique** does not provide invariant loop corrections for any of the listed above **patches**. Consider e.g. a vertex Y in the **right Rindler wedge** connected to the internal and/or external vertexes $X_1, \dots, X_n$. Then the loop integral contains:

$$I(X_1, \dots, X_n) = \int d^d Y \, \theta(Y^1 - Y^0) \, \theta(Y^1 + Y^0) \prod_{j=1}^{n} F(Y, X_j).$$

Under the transformation $Y^1 \to Y^1 + \epsilon$ which moves the right Rindler wedge $\delta_\epsilon I \neq 0$. Here $F(Y, X) \equiv \left\langle T\phi(X)\phi(Y)\right\rangle = \mathfrak{F}\left[L^2_{XY} - i\,\epsilon\right]$ is the **Feynman propagator**.

Schwinger-Keldysh vs. Feynman

- Time evolution of a correlation function:

$$\left\langle \hat{O} \right\rangle (t) \equiv \left\langle \psi_0 \left| \overline{T} e^{i \int_{t_0}^{t} dt' \, \hat{H}(t')} \, \hat{O} \, T e^{-i \int_{t_0}^{t} dt' \, \hat{H}(t')} \right| \psi_0 \right\rangle,$$

where $\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t)$. True both in **Srodinger and Heisenberg** representations.

- In the **interaction representation**:

$$\left\langle \hat{O} \right\rangle (t) = \left\langle \psi_0 \right| \hat{S}^{+}(t, t_0) \, \hat{O}_0(t) \, \hat{S}(t, t_0) \left| \psi_0 \right\rangle = \left\langle \psi_0 \right| \hat{S}^{+}(+\infty, t_0) \, T \left[\hat{O}_0(t) \, \hat{S}(+\infty, t_0) \right] \left| \psi_0 \right\rangle,$$

where $\hat{S}(t, t_0) = T e^{-i \int_{t_0}^{t} dt' \, \hat{V}_0(t')}$. The dependence on t_0 is of crucial importance here. In **Schwinger-Keldysh** technique one has to perturbatively expand both $\hat{S}$ and $\hat{S}^{+}$ and the dependence on the initial Cauchy surface t_0 is there.

- **Feynman technique** is applicable in the equilibrium:
  1. The normal ordered free Hamiltonian $\hat{H}_0$ is time independent and bounded from below;
  2. The expectation value should be taken over the ground state of $\hat{H}_0$: $|\psi_0\rangle = |0\rangle$, $\hat{H}_0\,|0\rangle = 0$;
  3. Interaction term, $\hat{V}$, is turned on adiabatically after t_0 and then switched off adiabatically after t. In effect we have to make the substitution as follows: $\hat{S}(+\infty, t_0) \to \hat{S}_{tt_0}(+\infty, -\infty)$.
- Then $\left|\left\langle 0 \left| \hat{S} \right| 0 \right\rangle\right| = 1$ and $\left\langle n \neq 0 \left| \hat{S} \right| 0 \right\rangle = 0$, where $\hat{S} \equiv \hat{S}_{tt_0}(+\infty, -\infty)$, and

$$\left\langle \hat{O} \right\rangle(t) = \sum_n \left\langle 0 \right| \hat{S}^+ \left| n \right\rangle \left\langle n \right| T\left[\hat{O}_0(t)\,\hat{S}\right] \left| 0 \right\rangle =$$

$$= \frac{\left\langle 0 \right| T\left[\hat{O}_0(t)\,\hat{S}\right] \left| 0 \right\rangle}{\left\langle 0 \right| \hat{S} \left| 0 \right\rangle}, \quad t_0 \text{ is disappeared.}$$

The Schwinger-Keldysh technique is causal

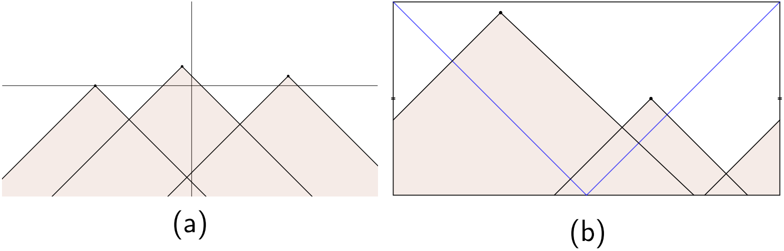

Figure: The union of past ligh cones of external points of a diagram: (a) in **Minkowski space-time**; (b) on the Penrose diagram of **de Sitter space-time**; the blue line shows the boundary between **Expanding and Contracting Poincare Patches**. Within the framework of the **Schwinger-Keldysh** technique one integrates in the loop integrals over these past light-cones.

Analytical continuation from the Rindler wedge to the Euclidian space

Consider e.g. a vertex Y in the **right Rindler wedge** connected to the internal and/or external vertexes $X_1, \ldots, X_n$. Then in the **Schwinger-Keldysh technique** the loop integrals contain:

$$I_K(X_1, \ldots, X_n) = \int d^d Y \, \theta(Y^1 - Y^0) \, \theta(Y^1 + Y^0) \times$$

$$\times \left[\prod_{j=1}^{k} F(Y, X_j) \prod_{j=k+1}^{n} \bar{W}(Y, X_j) - \prod_{j=1}^{k} W(Y, X_j) \prod_{j=k+1}^{n} \bar{F}(Y, X_j) \right].$$

Due to analytic properties of the propagators F and W as functions of **geodesic distances** one can show that $\delta_\epsilon I_K = 0$ under the shift of the patch $Y^1 \to Y^1 + \epsilon$. Moreover, one can **map** $I_K(X_1, \ldots, X_n)$ to the loop integral in the Matsubara technique by deforming contours in the complex plane of Y^0.

Consider the same integral over a half, $Y^0 > -Y^1$, of **Minkowski space-time**:

$$I_K(X_1, \dots, X_n) = \int d^d Y \, \theta(Y^1 + Y^0) \times$$
$$\times \left[\prod_{j=1}^{k} F(Y, X_j) \prod_{j=k+1}^{n} \bar{W}(Y, X_j) - \prod_{j=1}^{k} W(Y, X_j) \prod_{j=k+1}^{n} \bar{F}(Y, X_j) \right].$$

One can **map** such an integral even more straightforwardly to the loop integral in the Matsubara technique by deforming contours in the complex plane of Y^0. Recall **light-cone quantization**.

But due to causality property of the **Schwinger-Keldysh** technique such a loop integral for the points sitting in the **Rindler wedge** is equivalent to the integral over the wedge only.

- The situation in the **left Rindler wedge** is the same as in the **right one**. This **wedge** resides in the other half of entire **Minkowski space-time**;
- Due to causality property of the **Schwinger-Keldysh** technique the situation in the lower or past wedge, $Y^0 < |Y^1|$, is the same as in the entire **Minkowski space-time**;
- The situation in the **upper or the future wedge**, $Y^0 > |Y^1|$, is very much different:

$$I_K(X_1, \dots, X_n) = \int d^d Y \theta(Y^0 - Y^1)\theta(Y^0 + Y^1) \times$$

$$\left[\prod_{j=1}^{k} F(Y, X_j) \prod_{j=k+1}^{n} \bar{W}(Y, X_j) - \prod_{j=1}^{k} W(Y, X_j) \prod_{j=k+1}^{n} \bar{F}(Y, X_j)\right].$$

And $\delta_\epsilon I \neq 0$, because contours do not close.

A simple tree-level example in the future wedge

- Let us treat the mass term $\frac{m^2\phi^2}{2}$ as the **perturbation** in the massless theory. Then the **first correction** to the propagator in the **entire Minkowski** space-time in the Feynman technique is as follows:

$$F_M^{(1)}(0,X) = -im^2 \int d^4Y \frac{1}{(Y^2 - i\epsilon)((Y-X)^2 - i\epsilon)}.$$

where $X^\mu = (t,0,0,0)$.

- While in the **future wedge** the correction is:

$$F_F^{(1)}(0,X) = 2\pi m^2 \int_{|Y^1|<Y^0<t} d^4Y \frac{\delta[(Y-X)^2]}{(Y^2 - i\epsilon)}.$$

- The results are

$$F_M^{(1)}(0,X) = -\pi^2 m^2 \log\frac{\Lambda^2}{-t^2}, \quad F_F^{(1)}(0,X) = i\pi^3 m^2.$$

Metric in the future wedge $ds_F^2 = e^{2\tau}(d\tau^2 - d\xi^2) - (dX^a)^2$.

Various patches of the de Sitter space-time

- In the **expanding Poincare patch**, $Y^d > -Y^0$, which is a half of **entire de Sitter** space-time with the metric $ds^2 = d\tau^2 - e^{2\tau}(dx^i)^2$, for the Bunch-Davies state the situation is similar to the one in the **half of Minkowski** space-time. For such a state the propagators are maximally analytic functions in the complex plane of the geodesic distance.
- In the **static patch**, $Y^d > |Y^0|$, which is quarter of the **entire de Sitter** space-time with the static metric $ds^2 = \sin^2\theta\, dt^2 - d\theta^2 - \cos^2\theta\, d\Omega^2_{d-2}$, for the Bunch-Davies state the situation is similar to the one in the **right Rindler wedge**.
- In the **contracting Poincare patch** and in the **global (entire) de Sitter** space-time the situation is very much different. Somewhat similar to the future wedge, but with certain differences.

In contracting Poincare patch of de Sitter space–time

- The **contracting Poincare patch**, $ds^2 = d\tau^2 - e^{-2\tau}\, d\vec{x}^2$, is the time reversal of the **expanding Poincare patch**.
- Now **in the loops** one sees the secular divergences:

$$
\begin{array}{ll}
\lambda^2 \log\left(\frac{p\, e^{t}}{p\, e^{t_0}}\right) \sim \lambda^2\,(t - t_0) & p\, e^{t} < m, \\
\lambda^2 \log\left(\frac{\mu}{p\, e^{t_0}}\right) & p\, e^{t} > m.
\end{array}
$$

- Loop corrected propagator is not a function of the **geodesic distance** anymore. For any initial state!
- **Global de Sitter** contains both **expanding and contracting patches** simultaneously. The situation there is similar to the one in contracting patch.

Conclusions

Thus, even for Poincare and de Sitter invariant initial states one can encounter IR problems and the violation of the isometry in the loops in various patches of Minkowski and de Sitter space-times.

THANKS!