Dressing Factors and TBA for $AdS_3 \times S^3 \times T^4$

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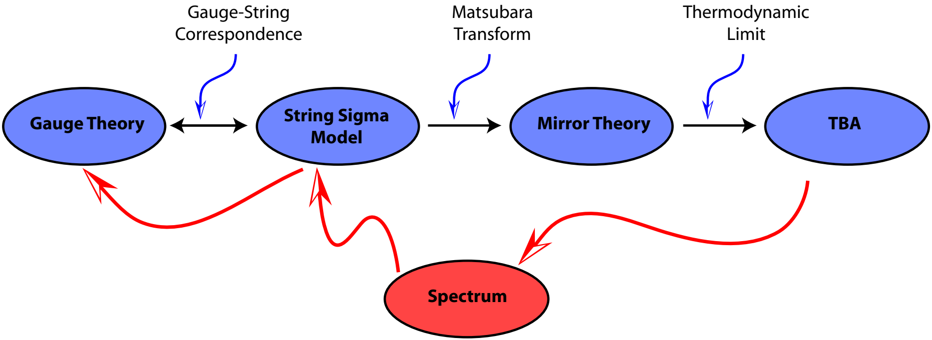
Outline

Mirror TBA approach to the AdS/CFT spectral problem



Thermodynamics of the *mirror* theory determines the finite-size spectrum of the original model

- Successful examples: $AdS_5 \times S^5$ and $AdS_4 \times CP^3$
- $AdS_3 \times S^3 \times T^4$ string sigma model on a plane was studied by

  Borsato, Sax, Sfondrini, Stefanski, Torrielli

- Symmetry algebra: $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2) \oplus \mathfrak{u}(1)^4$
- L.c.g. algebra: $\mathfrak{psu}(1|1)^{\oplus 2}_{\text{c.e.}} \oplus \mathfrak{psu}(1|1)^{\oplus 2}_{\text{c.e.}} \oplus \mathfrak{so}(4) \oplus \mathfrak{u}(1)^4$
- Fundamental particles transform in 4-dim short representations. Dispersion relations

$$E(p) = \sqrt{M^2 + 4h^2 \sin^2 \frac{p}{2}}, \quad -\pi \le p \le \pi$$

- The charge M for the RR background

$$M = \begin{cases} +1 & \text{“left”:} \quad (Y, \psi^\alpha, Z) \\ -1 & \text{“right”:} \quad (\bar{Z}, \bar{\psi}^\alpha, \bar{Y}) \\ 0 & \text{“massless”:} \quad (\chi^{\dot\alpha}, T^{\alpha\dot\alpha}, \tilde{\chi}^{\dot\alpha}) \end{cases}$$

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- Due to various discrete symmetries there are only four distinct blocks in the S-matrix of fundamental particles:
    1. $S_{LL}(p_1, p_2) = S_{RR}(p_1, p_2)$
    2. $S_{LR}(p_1, p_2) = S_{RL}(p_1, p_2)$
    3. $S_{L\circ}(p_1, p_2) = S_{R\circ}(p_1, p_2) = S^{-1}_{\circ L}(p_2, p_1) = S^{-1}_{\circ R}(p_2, p_1)$
    4. $S_{\circ\circ}(p_1, p_2)$
- Bethe-Yang eqs are derived
- Each block is fixed by symmetries up to a scalar dressing factor
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- The proposal has several disturbing features

1. $\sigma_{12}^{\bullet\bullet}$ and $\widetilde{\sigma}_{12}^{\bullet\bullet}$ violate the parity invariance of the model
2. $\sigma_{12}^{\circ\bullet}$ and $\sigma_{12}^{\circ\circ}$ contain the Arutyunov-Frolov-Staudacher (AFS) factor which is the leading-order term of the asymptotic expansion of the Beisert-Eden-Staudacher (BES) factor
3. $\sigma_{12}^{\circ\bullet}$ has additional apparent square-root branch points whose positions depend on the relative value of the momenta of the two scattered particles
4. It is not clear how to analytically continue the dressing factors to the mirror region
5. The dressing factors are not compatible with some of the perturbative computations in the literature

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S-matrix normalisation

Use the following scattering processes among highest-weight states in each representation

$$\mathbf{S}\,|Y_{p_1}Y_{p_2}\rangle = e^{+ip_1}e^{-ip_2}\frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{1-\frac{1}{x_1^- x_2^+}}{1-\frac{1}{x_1^+ x_2^-}}\left(\sigma_{12}^{\bullet\bullet}\right)^{-2}|Y_{p_1}Y_{p_2}\rangle$$

$$\mathbf{S}\,|Y_{p_1}\bar{Z}_{p_2}\rangle = e^{-ip_2}\frac{1-\frac{1}{x_1^- x_2^-}}{1-\frac{1}{x_1^+ x_2^+}}\frac{1-\frac{1}{x_1^- x_2^+}}{1-\frac{1}{x_1^+ x_2^-}}\left(\tilde{\sigma}_{12}^{\bullet\bullet}\right)^{-2}|Y_{p_1}\bar{Z}_{p_2}\rangle$$

$$\mathbf{S}\,|Y_{p_1}\chi^{\dot{\alpha}}_{p_2}\rangle = e^{+\frac{i}{2}p_1}e^{-ip_2}\frac{x_1^- - x_2}{1 - x_1^+ x_2}\left(\sigma_{12}^{\bullet\circ}\right)^{-2}|Y_{p_1}\chi^{\dot{\alpha}}_{p_2}\rangle$$

$$\mathbf{S}\,|\chi^{\dot{\alpha}}_{p_1}\bar{Z}_{p_2}\rangle = e^{+ip_1}e^{+\frac{i}{2}p_2}\frac{x_2^- - x_1}{1 - x_1 x_2^+}\left(\sigma_{12}^{\circ\bullet}\right)^{-2}|\chi^{\dot{\alpha}}_{p_1}\bar{Z}_{p_2}\rangle$$

$$\mathbf{S}\,|\chi^{\dot{\alpha}}_{p_1}\chi^{\dot{\beta}}_{p_2}\rangle = \left(\sigma_{12}^{\circ\circ}\right)^{-2}|\chi^{\dot{\alpha}}_{p_1}\chi^{\dot{\beta}}_{p_2}\rangle$$

Zhukovsky variables

- For massive particles

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{h}|M|\,, \qquad \frac{x^+}{x^-} = e^{ip}\,, \qquad M = \pm 1$$

$$x^\pm(p, |M|) = e^{\pm ip/2}\,\frac{|M| + \sqrt{M^2 + 4h^2\sin^2(p/2)}}{2h\sin(p/2)}$$

- For massless particles

$$x = x^+(p, 0) = e^{+ip/2}\,\frac{|\sin(p/2)|}{\sin(p/2)}\,, \qquad \Im(x) > 0$$

- Energy of massive and massless particles

$$E = \frac{h}{2i}\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right), \qquad E = \frac{h}{i}\left(x - \frac{1}{x}\right)$$

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Crossing equations: $\bar{x}^{\pm} = \frac{1}{x^{\pm}}, \bar{x} = \frac{1}{x}$

$$\left(\sigma^{\bullet\bullet}(x_1^{\pm}, x_2^{\pm})\right)^2 \left(\tilde{\sigma}^{\bullet\bullet}(\bar{x}_1^{\pm}, x_2^{\pm})\right)^2 = \left(\frac{x_2^-}{x_2^+}\right)^2 \frac{(x_1^- - x_2^+)^2}{(x_1^- - x_2^-)(x_1^+ - x_2^+)} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

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$$\left(\sigma^{\bullet\circ}(x_1^{\pm}, x_2)\right)^2 \left(\sigma^{\bullet\circ}(\bar{x}_1^{\pm}, x_2)\right)^2 = \frac{1}{(x_2)^4} \frac{f(x_1^+, x_2)}{f(x_1^-, x_2)},$$

$$\left(\sigma^{\circ\bullet}(x_1, x_2^{\pm})\right)^2 \left(\sigma^{\circ\bullet}(\bar{x}_1, x_2^{\pm})\right)^2 = \frac{f(x_1, x_2^+)}{f(x_1, x_2^-)}$$

$$\left(\sigma^{\circ\circ}(x_1, x_2)\right)^2 \left(\sigma^{\circ\circ}(\bar{x}_1, x_2)\right)^2 = -f(x_1, x_2)^2, \quad f(x, y) = i\,\frac{1 - xy}{x - y}$$

"Stripping out" the BES factors

$$\varsigma^{\bullet\bullet}(x_1^\pm,x_2^\pm)=\frac{\sigma^{\bullet\bullet}(x_1^\pm,x_2^\pm)}{\sigma_{\text{BES}}(x_1^\pm,x_2^\pm)}\,,\qquad \tilde{\varsigma}^{\bullet\bullet}(x_1^\pm,x_2^\pm)=\frac{\tilde{\sigma}^{\bullet\bullet}(x_1^\pm,x_2^\pm)}{\sigma_{\text{BES}}(x_1^\pm,x_2^\pm)}$$

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$\sigma_{\text{BES}}(x_1^\pm,x_2)$ is the massive-massless BES factor obtained from $\sigma_{\text{BES}}(x_1^\pm,x_2^\pm)$ in the limit $x_2^+\to x_2,\; x_2^-\to 1/x_2$

These BES factors satisfy

$$\sigma_{\text{BES}}(x_1^\pm,x_2^\pm)\sigma_{\text{BES}}(\bar{x}_1^\pm,x_2^\pm)=\frac{x_2^-}{x_2^+}\frac{x_1^--x_2^+}{x_1^--x_2^-}\frac{1-\frac{1}{x_1^+x_2^+}}{1-\frac{1}{x_1^+x_2^-}}$$

$$\sigma_{\text{BES}}(\bar{x}_1,x_2^\pm)\,\sigma_{\text{BES}}(x_1,x_2^\pm)=1\,,\qquad \sigma_{\text{BES}}(\bar{x}_1^\pm,x_2)\,\sigma_{\text{BES}}(x_1^\pm,x_2)=\frac{1}{x_2^2}\frac{f(x_1^+,x_2)}{f(x_1^-,x_2)}$$

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Crossing equations for the stripped out dressing factors

Let $\varsigma^{+}(x_1^\pm, x_2^\pm) = \varsigma^{\bullet\bullet}(x_1^\pm, x_2^\pm)\tilde{\varsigma}^{\bullet\bullet}(x_1^\pm, x_2^\pm)\,, \quad \varsigma^{-}(x_1^\pm, x_2^\pm) = \frac{\varsigma^{\bullet\bullet}(x_1^\pm, x_2^\pm)}{\tilde{\varsigma}^{\bullet\bullet}(x_1^\pm, x_2^\pm)}$

Crossing equations

$$\Big(\varsigma^{+}(\bar{x}_1^\pm, x_2^\pm)\Big)^{-2}\Big(\varsigma^{+}(x_1^\pm, x_2^\pm)\Big)^{-2} = \frac{f(x_1^+, x_2^+)\, f(x_1^-, x_2^-)}{f(x_1^+, x_2^-)\, f(x_1^-, x_2^+)}$$

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$$\Big(\varsigma^{\circ\circ}(\bar{x}_1, x_2)\Big)^{-2}\Big(\varsigma^{\circ\circ}(x_1, x_2)\Big)^{-2} = -\frac{1}{f(x_1, x_2)^2}$$

Monodromy equation

$$\frac{\Big(\varsigma^{-}(\bar{x}_1^\pm, x_2^\pm)\Big)^{-2}}{\Big(\varsigma^{-}(x_1^\pm, x_2^\pm)\Big)^{-2}} = \frac{(u_1 - u_2 + \frac{i}{h})(u_1 - u_2 - \frac{i}{h})}{(u_1 - u_2)^2}\,, \qquad x^\pm + \frac{1}{x^\pm} = u \pm \frac{i}{h}$$

γ-rapidity for massless particles

Fontanella, Torrielli '19

Consider the equation

$$S(\bar{x}_1, x_2)S(x_1, x_2) = \frac{1}{f(x_1, x_2)} = i\,\frac{x_1 - x_2}{x_1 x_2 - 1} = i \tanh \frac{\gamma_1 - \gamma_2}{2}$$

where the massless γ-rapidity is defined through

$$x = \frac{i - e^{\gamma}}{i + e^{\gamma}}\,, \quad i\,e^{\gamma} = \frac{x - 1}{x + 1}\,, \quad x\bar{x} = 1,\ \Im(\gamma) > 0 \text{ if } \gamma \in \mathbb{R}$$

Since

$$E(\gamma) = \frac{2h}{\cosh\gamma}\,, \quad e^{ip} = x^2$$

the crossing transformation $p \to -p$, $E \to -E$, $x \to \bar{x} = 1/x$ corresponds to either $\gamma \to \gamma + i\pi$ or $\gamma \to \gamma - i\pi$

To fix the sign we use that the real mirror momentum line is between the string and anti-string ones, and find

$$\tilde{E} = -i\,p(\gamma + \tfrac{i}{2}\pi) = -2\log\left|\tanh\frac{\gamma}{2}\right| > 0\,, \quad \tilde{p} = -i\,E(\gamma + \tfrac{i}{2}\pi) = -\frac{2h}{\sinh\gamma}$$

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To fix the sign we use that the real mirror momentum line is between the string and anti-string ones, and find

$$\tilde{E} = -i\,p(\gamma + \tfrac{i}{2}\pi) = -2\log\left|\tanh\frac{\gamma}{2}\right| > 0\,, \quad \tilde{p} = -i\,E(\gamma + \tfrac{i}{2}\pi) = -\frac{2h}{\sinh\gamma}$$

In terms of the γ-rapidity for massless particles

$$x = \frac{i - e^{\gamma}}{i + e^{\gamma}}, \quad x\bar{x} = 1, \ \Im(\gamma) > 0 \ \text{if} \ \gamma \in \mathbb{R}$$

The crossing transformation is $\gamma \to \bar{\gamma} = \gamma + i\pi$

The mirror transformation is $\gamma \to \gamma_m = \gamma + i\pi/2$

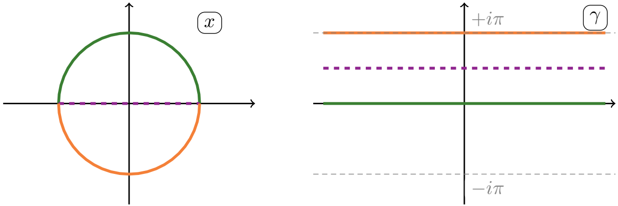

Figure 1. The string, mirror and anti-string region in the massless kinematics. In all three cases, the "region" is actually a line, corresponding to real momentum particles. We denote the string region by a solid green line (upper-half-circle in the x-plane), the mirror region by a dashed purple line (real segment in the x-plane), and the anti-string region by a solid orange line (lower-half-circle in the x-plane).

The Sine-Gordon factor

Equation

$$S(\gamma_1 + i\pi, \gamma_2)S(\gamma_1, \gamma_2) = i \tanh \frac{\gamma_{12}}{2}, \quad \gamma_{12} \equiv \gamma_1 - \gamma_2$$

has the following solution

$$S(\gamma_1, \gamma_2) = \Phi(\gamma_{12})$$

where Φ is the Sine-Gordon dressing factor

$$\Phi(\gamma) = \prod_{\ell=1}^{\infty} R(\ell, \gamma), \quad R(\ell, \gamma) = \frac{\Gamma^2(\ell - \frac{\gamma}{2\pi i})\Gamma(\frac{1}{2} + \ell + \frac{\gamma}{2\pi i})\Gamma(-\frac{1}{2} + \ell + \frac{\gamma}{2\pi i})}{\Gamma^2(\ell + \frac{\gamma}{2\pi i})\Gamma(\frac{1}{2} + \ell - \frac{\gamma}{2\pi i})\Gamma(-\frac{1}{2} + \ell - \frac{\gamma}{2\pi i})}$$

satisfying

$$\Phi(\gamma)\Phi(-\gamma) = 1, \quad \Phi(\gamma)^* = \frac{1}{\Phi(\gamma^*)}, \quad \Phi(\gamma)\Phi(\gamma + i\pi) = i \tanh \frac{\gamma}{2}$$

$$-\Phi(\gamma_{12})^4 = e^{2i\theta_{\text{HL}}(x_1, x_2)}$$

Massless-massless dressing factor

The crossing equation for $\varsigma^{\circ\circ}(\gamma_1,\gamma_2)$

$$\left(\varsigma^{\circ\circ}(\bar{\gamma}_1,\gamma_2)\right)^{-2}\left(\varsigma^{\circ\circ}(\gamma_1,\gamma_2)\right)^{-2}=-\left(i\tanh\frac{\gamma_{12}}{2}\right)^2$$

is solved by

$$\left(\tilde{\varsigma}^{\circ\circ}(\gamma_1,\gamma_2)\right)^{-2}=a(\gamma_{12})\left(\Phi(\gamma_{12})\right)^2$$

where

$$a(\gamma)\,a(\gamma+i\pi)=-1\,,\quad a(\gamma)\,a(-\gamma)=1\,,\quad a(\gamma)^*=\frac{1}{a(\gamma^*)}$$

$$a(\gamma)=-i\tanh\left(\frac{\gamma}{2}-\frac{i\pi}{4}\right)\,,\quad a(\mp\infty)=\pm i\,,\qquad a(0)=-1$$

Thus, the massless-massless dressing factor is

$$\left(\sigma^{\circ\circ}(\gamma_1,\gamma_2)\right)^{-2}=-i\tanh\left(\frac{\gamma_{12}}{2}-\frac{i\pi}{4}\right)\left(\Phi(\gamma_{12})\right)^2\left(\sigma_{\text{BES}}(x_1,x_2)\right)^{-2}$$

$\gamma^{\pm}$-rapidities for massive particles

Beisert, Hernandez, Lopez '06

Compare

$$i\, e^{\gamma} = \frac{x-1}{x+1}\,, \quad e^{iq} = \frac{x+1}{x-1}$$

This suggests to define the massive $\gamma^{\pm}$-rapidities through

$$x^{+} = \frac{i - e^{\gamma^{+}}}{i + e^{\gamma^{+}}}\,, \qquad x^{-} = \frac{i + e^{\gamma^{-}}}{i - e^{\gamma^{-}}}\,, \quad (\gamma^{+})^{*} = \gamma^{-} \;\; \text{if} \;\; p \in \mathbb{R}$$

The crossing transformation is $\gamma^{\pm} \to \bar{\gamma}^{\pm} = \gamma^{\pm} - i\pi$

The mirror transformation is $\gamma^{\pm} \to \gamma^{\pm}_{m} = \gamma^{\pm} - i\pi/2$

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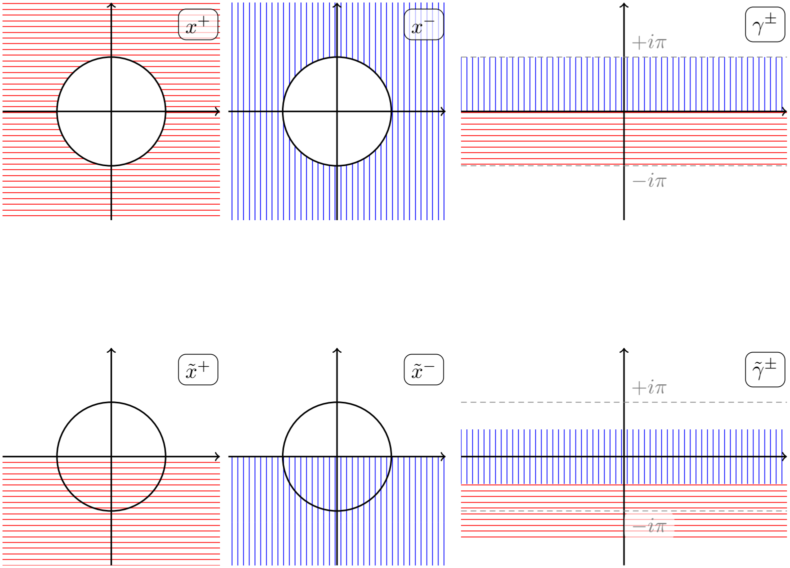
Physical string and mirror regions



Crossing equations for the stripped out dressing factors in terms of γ-rapidities

Notation

$$\gamma_{12}^{ab} = \gamma_1^a - \gamma_2^b\,, \quad a,b=\pm\,, \qquad \gamma_{12}^{\pm\circ} = \gamma_1^\pm - \gamma_2\,, \quad \gamma_{12}^{\circ\pm} = \gamma_1 - \gamma_2^\pm$$

Crossing equations

$$\left(\varsigma^{+}(\bar\gamma_1^\pm,\gamma_2^\pm)\right)^{-2}\left(\varsigma^{+}(\gamma_1^\pm,\gamma_2^\pm)\right)^{-2} = \coth\frac{\gamma_{12}^{++}}{2}\coth\frac{\gamma_{12}^{+-}}{2}\coth\frac{\gamma_{12}^{-+}}{2}\coth\frac{\gamma_{12}^{--}}{2}$$

$$\left(\varsigma^{\bullet\circ}(\bar\gamma_1^\pm,\gamma_2)\right)^{-2}\left(\varsigma^{\bullet\circ}(\gamma_1^\pm,\gamma_2)\right)^{-2} = \coth\frac{\gamma_{12}^{+\circ}}{2}\coth\frac{\gamma_{12}^{-\circ}}{2}$$

$$\left(\varsigma^{\circ\bullet}(\bar\gamma_1,\gamma_2^\pm)\right)^{-2}\left(\varsigma^{\circ\bullet}(\gamma_1,\gamma_2^\pm)\right)^{-2} = \tanh\frac{\gamma_{12}^{\circ+}}{2}\tanh\frac{\gamma_{12}^{\circ-}}{2}$$

Monodromy equation

$$\frac{\left(\varsigma^{-}(\bar\gamma_1^\pm,\gamma_2^\pm)\right)^{-2}}{\left(\varsigma^{-}(\gamma_1^\pm,\gamma_2^\pm)\right)^{-2}} = \frac{\sinh\gamma_{12}^{+-}\,\sinh\gamma_{12}^{-+}}{\sinh\gamma_{12}^{++}\,\sinh\gamma_{12}^{--}}$$

Mixed-mass dressing factors

$$\left(\varsigma^{\bullet\circ}(\gamma_1^{\pm},\gamma_2)\right)^{-2} = +i\,\frac{\tanh\frac{\gamma_{12}^{-\circ}}{2}}{\tanh\frac{\gamma_{12}^{+\circ}}{2}}\,\Phi(\gamma_{12}^{+\circ})\,\Phi(\gamma_{12}^{-\circ})$$

$$\left(\varsigma^{\circ\bullet}(\gamma_1,\gamma_2^{\pm})\right)^{-2} = -i\,\frac{\tanh\frac{\gamma_{12}^{\circ+}}{2}}{\tanh\frac{\gamma_{12}^{\circ-}}{2}}\,\Phi(\gamma_{12}^{\circ+})\,\Phi(\gamma_{12}^{\circ-})$$

Note that

$$\frac{\tanh\frac{\gamma_{12}^{+\circ}}{2}}{\tanh\frac{\gamma_{12}^{-\circ}}{2}}\left(\varsigma^{\bullet\circ}(x_1^{\pm},x_2)\right)^{-4} = e^{2i\theta_{\text{HL}}(x_1^{\pm},x_2)}$$

The massive-massless dressing factor is

$$\left(\sigma^{\bullet\circ}(x^{\pm},x_2)\right)^{-2} = +i\,\frac{\tanh\frac{\gamma_{12}^{-\circ}}{2}}{\tanh\frac{\gamma_{12}^{+\circ}}{2}}\,\Phi(\gamma_{12}^{+\circ})\,\Phi(\gamma_{12}^{-\circ})\left(\sigma(x_1^{\pm},x_2)\right)^{-2}$$

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The monodromy factor

Define

$$\widehat{\Phi}(\gamma) = e^{\frac{\gamma}{i\pi}} \prod_{\ell=1}^{\infty} \frac{\Gamma(\ell + \frac{\gamma}{i\pi})}{\Gamma(\ell - \frac{\gamma}{i\pi})} \, e^{\frac{2i}{\pi}\,\psi(\ell)\,\gamma}$$

where $\psi(z)$ is the Digamma function.
It satisfies

$$\frac{\widehat{\Phi}(\gamma \pm i\pi)}{\widehat{\Phi}(\gamma)} = i(2\sinh\gamma)^{\pm 1}\,, \quad \widehat{\Phi}(z)\,\widehat{\Phi}(-z) = 1\,, \quad \widehat{\Phi}(z)^* = \frac{1}{\widehat{\Phi}(z^*)}$$

The monodromy equation solution

$$\left(\varsigma^-(\gamma_1^\pm, \gamma_2^\pm)\right)^{-2} \sim \frac{\widehat{\Phi}(\gamma_{12}^{++})\,\widehat{\Phi}(\gamma_{12}^{--})}{\widehat{\Phi}(\gamma_{12}^{+-})\,\widehat{\Phi}(\gamma_{12}^{-+})} = \prod_{\ell=1}^{\infty} \frac{\widehat{R}(\ell, \gamma_{12}^{++})\,\widehat{R}(\ell, \gamma_{12}^{--})}{\widehat{R}(\ell, \gamma_{12}^{+-})\,\widehat{R}(\ell, \gamma_{12}^{-+})}$$

where

$$\widehat{R}(\ell, \gamma) = \frac{\Gamma(\ell + \frac{\gamma}{2\pi i})^2\,\Gamma(\ell + \frac{1}{2} + \frac{\gamma}{2\pi i})\,\Gamma(\ell - \frac{1}{2} + \frac{\gamma}{2\pi i})}{\Gamma(\ell - \frac{\gamma}{2\pi i})^2\,\Gamma(\ell + \frac{1}{2} - \frac{\gamma}{2\pi i})\,\Gamma(\ell - \frac{1}{2} - \frac{\gamma}{2\pi i})}$$

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Massive-massive dressing factors

$$\left(\varsigma^{+}(x_1^{\pm},x_2^{\pm})\right)^{-2} = -\frac{\tanh\frac{\gamma_{12}^{-+}}{2}}{\tanh\frac{\gamma_{12}^{+-}}{2}}\,\Phi(\gamma_{12}^{--})\Phi(\gamma_{12}^{++})\Phi(\gamma_{12}^{-+})\Phi(\gamma_{12}^{+-})$$

$$\left(\varsigma^{-}(x_1^{\pm},x_2^{\pm})\right)^{-2} = -\frac{\sinh\gamma_{12}^{-+}}{\sinh\gamma_{12}^{+-}}\,\frac{\widehat{\Phi}(\gamma_{12}^{++})\,\widehat{\Phi}(\gamma_{12}^{--})}{\widehat{\Phi}(\gamma_{12}^{+-})\,\widehat{\Phi}(\gamma_{12}^{-+})}$$

Note that

$$\left(\varsigma^{+}(x_1^{\pm},x_2^{\pm})\right)^{-2} = e^{2i\theta_{\text{HL}}(\gamma_1^{\pm},\gamma_2^{\pm})}$$

introduce

$$R_+(\ell,\gamma) = \frac{\Gamma(\ell+\frac{1}{2}+\frac{\gamma}{2\pi i})\,\Gamma(\ell-\frac{1}{2}+\frac{\gamma}{2\pi i})}{\Gamma(\ell+\frac{1}{2}-\frac{\gamma}{2\pi i})\,\Gamma(\ell-\frac{1}{2}-\frac{\gamma}{2\pi i})}\,, \qquad R_-(\ell,\gamma) = \frac{\Gamma^2(\ell-\frac{\gamma}{2\pi i})}{\Gamma^2(\ell+\frac{\gamma}{2\pi i})}$$

which satisfy

$$R_+(\ell,\gamma)\,R_-(\ell,\gamma) = R(\ell,\gamma)\,, \qquad \frac{R_+(\ell,\gamma)}{R_-(\ell,\gamma)} = \widehat{R}(\ell,\gamma)$$

Massive-massive dressing factors

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$$\left(\varsigma^-(x_1^\pm, x_2^\pm)\right)^{-2} = -\frac{\sinh\gamma_{12}^{-+}}{\sinh\gamma_{12}^{+-}}\,\frac{\widehat{\Phi}(\gamma_{12}^{++})\,\widehat{\Phi}(\gamma_{12}^{--})}{\widehat{\Phi}(\gamma_{12}^{+-})\,\widehat{\Phi}(\gamma_{12}^{-+})}$$

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Massive-massive dressing factors

The massive-massive dressing factors are

$$\left(\sigma^{\bullet\bullet}(x_1^\pm, x_2^\pm)\right)^{-2} = -\frac{\sinh\frac{\gamma_{12}^{-+}}{2}}{\sinh\frac{\gamma_{12}^{+-}}{2}}\,\Phi^{\bullet\bullet}(\gamma_1^\pm,\gamma_2^\pm)\,\left(\sigma_{\text{BES}}(x_1^\pm,x_2^\pm)\right)^{-2},$$

$$\left(\widetilde{\sigma}^{\bullet\bullet}(x_1^\pm, x_2^\pm)\right)^{-2} = +\frac{\cosh\frac{\gamma_{12}^{+-}}{2}}{\cosh\frac{\gamma_{12}^{-+}}{2}}\,\widetilde{\Phi}^{\bullet\bullet}(\gamma_1^\pm,\gamma_2^\pm)\,\left(\sigma_{\text{BES}}(x_1^\pm,x_2^\pm)\right)^{-2}.$$

$$\Phi^{\bullet\bullet}(\gamma_1^\pm,\gamma_2^\pm) = \prod_{\ell=1}^{\infty} R_+(\ell,\gamma_{12}^{--})R_+(\ell,\gamma_{12}^{++})R_-(\ell,\gamma_{12}^{-+})R_-(\ell,\gamma_{12}^{+-})$$

$$\widetilde{\Phi}^{\bullet\bullet}(\gamma_1^\pm,\gamma_2^\pm) = \prod_{\ell=1}^{\infty} R_-(\ell,\gamma_{12}^{--})R_-(\ell,\gamma_{12}^{++})R_+(\ell,\gamma_{12}^{-+})R_+(\ell,\gamma_{12}^{+-})$$

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$$\widetilde{\Phi}^{\bullet\bullet}(\gamma_1^{\pm},\gamma_2^{\pm}) = \prod_{\ell=1}^{\infty} R_-(\ell,\gamma_{12}^{--})R_-(\ell,\gamma_{12}^{++})R_+(\ell,\gamma_{12}^{-+})R_+(\ell,\gamma_{12}^{+-})$$

We can also write

$$\Phi^{\bullet\bullet}(\gamma_1^{\pm},\gamma_2^{\pm}) = \Phi_+(\gamma_{12}^{--})\Phi_+(\gamma_{12}^{++})\Phi_-(\gamma_{12}^{-+})\Phi_-(\gamma_{12}^{+-})$$

$$\widetilde{\Phi}^{\bullet\bullet}(\gamma_1^{\pm},\gamma_2^{\pm}) = \Phi_-(\gamma_{12}^{--})\Phi_-(\gamma_{12}^{++})\Phi_+(\gamma_{12}^{-+})\Phi_+(\gamma_{12}^{+-})$$

$$\Phi_+(\gamma) = \frac{1}{\pi}\mathcal{R}(\gamma-\pi i)^2 \cosh\frac{\gamma}{2}\,, \qquad \Phi_-(\gamma) = \frac{1}{\mathcal{R}(\gamma)^2}$$

$$\mathcal{R}(\gamma) \equiv \frac{G(1-\frac{\gamma}{2\pi i})}{G(1+\frac{\gamma}{2\pi i})} = \left(\frac{e}{2\pi}\right)^{+\frac{\gamma}{2\pi i}} \prod_{\ell=1}^{\infty} \frac{\Gamma(\ell+\frac{\gamma}{2\pi i})}{\Gamma(\ell-\frac{\gamma}{2\pi i})}\, e^{-\frac{\gamma}{\pi i}\psi(\ell)}\,, \qquad \mathcal{R}(-\gamma) = \frac{1}{\mathcal{R}(\gamma)}$$

where $G(x)$ is Barnes G-function

$$G(1+x) = \Gamma(x)G(x)$$

Perturbative expansion

was studied by Rughoonauth, Sundin, Wulff '12; Sundin, Wulff '13; Engelund, McKeown, Roiban '13; Roiban, Sundin, Tseytlin, Wulff '14; Bianchi, Hoare '14; Sundin, Wulff '16;

1. At tree-level, our factors match with these computations
2. At one-loop in the near-BMN limit, our factor ς_{12}^{-} disagrees

$$\log(\varsigma_{12}^{-})^{-2}\Big|_{\text{ours}} - \log(\varsigma_{12}^{-})^{-2}\Big|_{\text{theirs}} = \frac{i}{2\pi h^2}\left(\omega_1 p_2 - \omega_2 p_1\right) p_1 p_2 + O(h^{-3})$$

This can arise from a local counter-term

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Perturbative expansion

1. In the mixed-mass sector we got

$$\log\langle Y_1\chi_2^{\dot\alpha}|\mathbf{S}|Y_1\chi_2^{\dot\alpha}\rangle_{1-\text{loop}} = -\frac{ip_1^2}{2\pi h^2}(\omega_1 - p_1)p_2 \log\left(\frac{-(\omega_1-p_1)p_2}{4h}\right)$$

while Sundin and Wulff obtained

$$\log\langle Y_1\chi_2^{\dot\alpha}|\mathbf{S}_{\text{SW}}|Y_1\chi_2^{\dot\alpha}\rangle_{1-\text{loop}} = -\frac{ip_1^2}{2\pi h^2}(\omega_1 - p_1)p_2 \left[1+\log\left(\frac{\omega_1-p_1}{-2p_2}\right)\right]$$

- The rational part of the one-loop result does not match

$$\frac{i}{2\pi h^2}(\omega_1 - p_1)p_1^2 p_2$$

  This can arise from a local counter-term
- The logarithmic piece is quite different.

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Perturbative expansion

1. Mixed-mass sector

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The logarithmic piece is quite different.

- The reason is in the order of limits
- In SW a UV regulator was removed first, and then the IR
- The correct order is opposite
- One does not remove UV divergent terms
- The exact dispersion relation implies that a natural UV regularisation for the model is a lattice one with the propagator replaced by $1/(m^2 + 4h^2\sin^2 p/2h)$
- The UV regulator, is identified with the coupling constant h
- At one-loop this leads to the appearance of a $\log h$ term

Perturbative expansion

1. Mixed-mass sector

$$\log\langle Y_1\chi_2^{\dot\alpha}|\mathbf{S}|Y_1\chi_2^{\dot\alpha}\rangle_{1-\text{loop}} = -\frac{ip_1^2}{2\pi h^2}(\omega_1 - p_1)p_2\,\log\left(\frac{-(\omega_1-p_1)p_2}{4h}\right)$$

$$\log\langle Y_1\chi_2^{\dot\alpha}|\mathbf{S}_{\text{SW}}|Y_1\chi_2^{\dot\alpha}\rangle_{1-\text{loop}} = -\frac{ip_1^2}{2\pi h^2}(\omega_1 - p_1)p_2\,\left[1+\log\left(\frac{\omega_1-p_1}{-2p_2}\right)\right]$$

The logarithmic piece is quite different.

- The reason is in the order of limits
- In SW a UV regulator was removed first, and then the IR
- The correct order is opposite
- One does not remove UV divergent terms
- The exact dispersion relation implies that a natural UV regularisation for the model is a lattice one with the propagator replaced by $1/(m^2+4h^2\sin^2 p/2h)$
- The UV regulator, is identified with the coupling constant h
- At one-loop this leads to the appearance of a $\log h$ term

Perturbative expansion

1. in the massless sector we get

$$\log\left(\sigma^{\circ\circ}(p_1,p_2)\right)^{-2} = -\frac{i}{h}p_1p_2 - \frac{i}{h^2}\frac{p_1p_2}{8} - i\frac{p_1p_2}{4\pi h^2}\left(\log\frac{-p_1p_2}{16h^2} - 1\right) + O(h^{-3})$$

while Sundin and Wulff obtained

$$\log\langle\chi_1^{\dot\alpha}\chi_2^{\dot\beta}|\mathbf{S}_{SW}|\chi_1^{\dot\alpha}\chi_2^{\dot\beta}\rangle = -\frac{i}{h}p_1p_2 + i\frac{p_1p_2}{4\pi h^2}\left(\log(-4p_1p_2) - 1\right) + O(h^{-3})$$

- At one-loop, the coefficient of the logarithm does not match; the sign is opposite
- The argument of the logarithm is sensitive to the UV cutoff, which in our case is provided by the coupling constant h and in the case of Sundin and Wulff has been removed. As such, the finite pieces of the one-loop result can be removed by a change in the UV cutoff.
- The first $O(h^{-2})$ term is precisely due to $-ia(\gamma)$, which we may eliminate in the opposite-chirality scattering

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Mirror BYE

Excitations appearing in the mirror BYE for fundamental particles

1. N_1 "left" momentum carrying modes
2. $N_{\bar{1}}$ "right" momentum carrying modes
3. $N_0^{(\dot{\alpha})}$, $\dot{\alpha} = 1, 2$ momentum carrying massless modes which are in a doublet of an external $su(2)$.

   The total number of massless excitations is $N_0 = N_0^{(1)} + N_0^{(2)}$
4. $N_0^{(\alpha)}$, $\alpha = 1, 2$ auxiliary Bethe roots, or y-roots

Mirror BYE

$$1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1\\ j\neq k}}^{N_1} S^{11}_{sl}(u_k,u_j) \prod_{j=1}^{N_{\bar{1}}} \widetilde{S}^{11}_{sl}(u_k,u_j) \prod_{\dot{\alpha}=1}^{2} \prod_{j=1}^{N_0^{(\dot{\alpha})}} S^{10}(u_k,u_j^{(\dot{\alpha})}) \prod_{\alpha=1}^{2} \prod_{j=1}^{N_y^{(\alpha)}} S^{1y}(u_k,y_j^{(\alpha)})$$

$$1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1\\ j\neq k}}^{N_{\bar{1}}} S^{11}_{su}(u_k,u_j) \prod_{j=1}^{N_1} \widetilde{S}^{11}_{su}(u_k,u_j) \prod_{\dot{\alpha}=1}^{2} \prod_{j=1}^{N_0^{(\dot{\alpha})}} \overline{S}^{10}(u_k,u_j^{(\dot{\alpha})}) \prod_{\alpha=1}^{2} \prod_{j=1}^{N_y^{(\alpha)}} \overline{S}^{1y}(u_k,y_j^{(\alpha)})$$

$$-1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1\\ j\neq k}}^{N_0^{(1)}} S^{00}(u_k^{(1)},u_j^{(1)}) \prod_{j=1}^{N_0^{(2)}} S^{00}(u_k^{(1)},u_j^{(2)}) \prod_{j=1}^{N_1} S^{01}(u_k,u_j) \prod_{j=1}^{N_{\bar{1}}} \overline{S}^{01}(u_k^{(1)},u_j)$$

$$\times \prod_{\alpha=1}^{2} \prod_{j=1}^{N_y^{(\alpha)}} S^{0y}(u_k^{(1)},y_j^{(\alpha)})$$

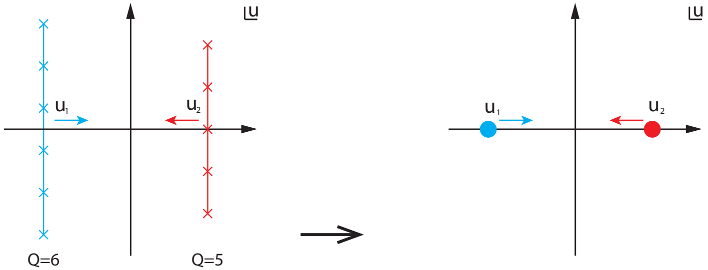
$$-1 = \prod_{j=1}^{N_1} S^{y1}(y_k^{(\alpha)},u_j) \prod_{j=1}^{N_{\bar{1}}} \bar{S}^{y1}(y_k^{(\alpha)},u_j) \prod_{\dot{\alpha}=1}^{2} \prod_{j=1}^{N_0^{(\dot{\alpha})}} S^{y0}(y_k^{(\alpha)},u_k^{(\dot{\alpha})})$$

Left-left bound states, or Q-particles

$$1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1 \\ j\neq k}}^{N_1} S_{sl}^{11}(u_k, u_j)\,, \qquad S_{sl}^{11}(u_k, u_j) = \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\, \frac{1 - \frac{1}{x_k^- x_j^+}}{1 - \frac{1}{x_k^+ x_j^-}} \left(\sigma_{kj}^{\bullet\bullet}\right)^{-2}$$

$$x_1^- = x_2^+\,, \quad x_2^- = x_3^+\,, \quad \ldots\,, \quad x_{Q-1}^- = x_Q^+$$

$$x_j^\pm = x\left(u_j \pm \frac{i}{h}\right), \quad u_j = u + \frac{(Q+1-2j)\,i}{h}\,, \qquad j = 1, \ldots Q\,, \quad u \in \mathbb{R}$$



Bethe strings → Q-particles with real rapidities

Right-right bound states, or $\overline{Q}$–particles

$$1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1\\ j\neq k}}^{N_{\bar{1}}} S^{11}_{su}(u_k,u_j) \prod_{\alpha=1}^{2} \prod_{j=1}^{N_y^{(\alpha)}} \overline{S}^{1y}(u_k, y_j^{(\alpha)})\,, \qquad -1 = \prod_{j=1}^{N_{\bar{1}}} \bar{S}^{y1}(y_k^{(\alpha)}, u_j)$$

$$S^{11}_{su}(u_k,u_j) = e^{+ip_k} e^{-ip_j} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - \frac{1}{x_k^- x_j^+}}{1 - \frac{1}{x_k^+ x_j^-}} \left(\sigma^{\bullet\bullet}_{kj}\right)^{-2}$$

$$\bar{S}^{1y}(u_k,y_j) = e^{-\frac{i}{2}p_k} \frac{x_k^+ - \frac{1}{y_j}}{x_k^- - \frac{1}{y_j}} = \frac{1}{S^{1y}(u_k, 1/y_j)}\,, \qquad \bar{S}^{y1}(y,u) = \frac{1}{\bar{S}^{1y}(u,y)}$$

$$x_1^- = x_2^+\,, \quad x_2^- = x_3^+\,, \ldots \quad x_{\overline{Q}-1}^- = x_{\overline{Q}}^+$$

$$y_j^{(1)} = y_j^{(2)} = \frac{1}{x_j^-} = \frac{1}{x(u + (\overline{Q}-2j)\frac{i}{h})}\,, \qquad j = 1,2,\ldots,\overline{Q}-1$$

$$\frac{1}{(y_j^{(\alpha)})^*} = \left(x(u + (\overline{Q}-2j)\frac{i}{h})\right)^* = \frac{1}{x(u - (\overline{Q}-2j)\frac{i}{h})} = y^{(\alpha)}_{\overline{Q}-j}$$

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$$1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1\\ j\neq k}}^{N_{\bar{1}}} S_{su}^{11}(u_k,u_j) \prod_{\alpha=1}^{2} \prod_{j=1}^{N_y^{(\alpha)}} \overline{S}^{1y}(u_k, y_j^{(\alpha)}) , \quad -1 = \prod_{j=1}^{N_{\bar{1}}} \bar{S}^{y1}(y_k^{(\alpha)}, u_j)$$

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String hypothesis for the mirror model

In the thermodynamic limit $R \to \infty$ with
$N_L^{(Q)}/R\,,\ N_R^{(\overline{Q})}/R\,,\ N_0^{(\dot\alpha)}/R\,,\ N_{y_\epsilon}^{(\alpha)}/R$ fixed
solutions of the BYE arrange themselves into

eight different classes of Bethe strings

1. Q-particles with $x_a^{\pm} = x\big(u_a \pm \frac{i}{h} Q_a\big)$, and $\sum_{Q=1}^{\infty} N_L^{(Q)} = N_L$
2. $\overline{Q}$-particles with $x_a^{\pm} = x\big(u_a \pm \frac{i}{h} \overline{Q}_a\big)$, and $\sum_{\overline{Q}=1}^{\infty} N_R^{(\overline{Q})} = N_R$
3. $N_0^{(\dot\alpha)}$ massless particles, $\dot\alpha = 1, 2$, with
   $-1 < x_k^{(\dot\alpha)} = x(u_k + i0) = 1/x_s(u_k) < 1, \quad |u_k| > 2$
4. $N_{y_\epsilon}^{(\alpha)}$ "auxiliary particles" $y_\epsilon^{(\alpha)}$, with $\alpha = 1, 2$, $\epsilon = \pm$, $\Im(y_\epsilon^{(\alpha)}) = \epsilon$,
   $y_-^{(\alpha)} = x(u) = x_s(u - i0)$, $y_+^{(\alpha)} = 1/x(u) = x_s(u + i0)$, $|u| < 2$

The mirror energy of the resulting state is given by

$$\widetilde{\mathcal{E}} = \sum_{a=1}^{N_L} \widetilde{\mathcal{E}}_{Q_a}(u_a) + \sum_{a=1}^{N_R} \widetilde{\mathcal{E}}_{\overline{Q}_a}(u_a) + \sum_{\dot\alpha=1}^{2} \sum_{k=1}^{N_0^{(\dot\alpha)}} \widetilde{\mathcal{E}}_0(u_k^{(\dot\alpha)})$$

Bethe-Yang eqs for Bethe strings

N_a strings with rapidities $u_{a,k}\,,\;\; k=1,\ldots,N_a$

$$(-1)^{\varphi_a} = e^{i\delta_a \widetilde{p}_{a,k} R} \prod_b \prod_{n=1}^{N_b} S_{ab}(u_{a,k}, u_{b,n})$$

Here $\delta_a = 1$ for strings carrying momentum and $\delta_a = 0$ otherwise, φ_a are constants, S_{ab} is scattering matrix of an a-string with a b-string.

	$y_+^{(1)}$	$y_-^{(1)}$	Q	$0^{(1)}$	$0^{(2)}$	$\overline{Q}$	$y_-^{(2)}$	$y_+^{(2)}$
$y_+^{(1)}$			•	•	•	•		
$y_-^{(1)}$			•	•	•	•		
Q	•	•	•	•	•	•	•	•
$0^{(1)}$	•	•	•	•	•	•	•	•
$0^{(2)}$	•	•	•	•	•	•	•	•
$\overline{Q}'$	•	•	•	•	•	•	•	•
$y_-^{(2)}$			•	•	•	•		
$y_+^{(2)}$			•	•	•	•		

Thermodynamic limit

$$(-1)^{\varphi_a} = e^{i\delta_a \widetilde{p}_{a,k} R} \prod_b \prod_{n=1}^{N_b} S_{ab}(u_{a,k}, u_{b,n})$$

- Roots become dense as $R \to \infty$
  $\rho_a(u)\,\mathrm{d}u$: # roots for string of type a in $\mathrm{d}u$
- Integral eqs in the thermodynamic limit

$$\rho_a + \bar{\rho}_a = \frac{R}{2\pi}\delta_a \frac{d\widetilde{p}_a}{du} + K_{ab} \star \rho_b$$

In the $AdS_3 \times S^3 \times T^4$ case $\widetilde{p}_a$ vanish only for y-particles.

- Star operation is defined as

$$K_{ab} \star \rho_b(u) = \int \mathrm{d}u'\, K_{ab}(u,u')\rho_b(u')\,, \quad K_{ab}(u,v) = \frac{1}{2\pi i}\frac{d}{du}\log S_{ab}(u,v)$$

$$\rho_b \star K_{ba}(u) = \int \mathrm{d}u'\, \rho_b(u')K_{ba}(u',u)$$

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Free energy and canonical TBA equations

Free energy: $\mathcal{F}_{\vec{\mu}}(J) = \int \mathrm{d}u \sum_a \left[\widetilde{H}_a \rho_a + \frac{\mu_a}{J}\,\rho_a - \frac{1}{J}\mathfrak{s}(\rho_a)\right]\,, \quad T = 1/J$

Variations of the densities of particles and holes are subject to

$$\delta\rho_a + \delta\bar{\rho}_a = K_{ab} \star \delta\rho_b$$

Using the extremum condition $\delta\mathcal{F}_{\vec{\mu}}(J) = 0$, one gets

The canonical TBA eqs and the ground state energy

$$\log \mathcal{Y}_a = -J\widetilde{\mathcal{E}}_a - \mu_a + \log\left(1 + \mathcal{Y}_b\right) \star K_{ba}$$

$$E_{\vec{\mu}}(J) - J = \lim_{R\to\infty} \frac{J}{R}\mathcal{F}(J) = -\sum_a{}' \int \frac{\mathrm{d}\widetilde{p}_a}{2\pi} \log\left(1 + \mathcal{Y}_a\right)$$

where each Bethe string leads to a Y–function

$$\mathcal{Y}_a = \frac{\rho_a}{\bar{\rho}_a}$$

Canonical TBA equations

In the $AdS_3 \times S^3 \times T^4$ case

$$Y_Q = \mathcal{Y}_Q\,, \quad \overline{Y}_Q = \overline{\mathcal{Y}}_Q\,, \quad Y_0^{(\dot\alpha)} = \mathcal{Y}_0^{(\dot\alpha)}\,, \quad Y_\pm^{(\alpha)} = -\frac{e^{i\mu_\alpha}}{\mathcal{Y}_\pm^{(\alpha)}}$$

where $\mu_\alpha = (-1)^\alpha \mu$, and μ is a twist parameter

$$\begin{aligned} -\log Y_Q = {} & L\widetilde{\mathcal{E}}_Q - \log\left(1+Y_{Q'}\right)\star K_{sl}^{Q'Q} - \log\left(1+\overline{Y}_{Q'}\right)\star \widetilde{K}_{su}^{Q'Q} \\ & - \sum_{\dot a=1,2} \log\left(1+Y_0^{(\dot a)}\right) \check{\star} K^{0Q} \\ & - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right) \hat{\star} K_+^{yQ} - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right) \hat{\star} K_-^{yQ} \end{aligned}$$

$$\begin{aligned} -\log \overline{Y}_Q = {} & L\widetilde{\mathcal{E}}_Q - \log\left(1+\overline{Y}_{Q'}\right)\star K_{su}^{Q'Q} - \log\left(1+Y_{Q'}\right)\star \widetilde{K}_{sl}^{Q'Q} \\ & - \sum_{\dot a=1,2} \log\left(1+Y_0^{(\dot a)}\right) \check{\star} \widetilde{K}^{0Q} \\ & - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right) \hat{\star} K_-^{yQ} - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right) \hat{\star} K_+^{yQ} \end{aligned}$$

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& - \sum_{\dot a=1,2} \log\left(1+Y_0^{(\dot a)}\right) \check\star\, K^{0Q} \\
& - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right) \hat\star\, K_+^{yQ} - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right) \hat\star\, K_-^{yQ} \\
-\log \overline{Y}_Q = {}& L\widetilde{\mathcal{E}}_Q - \log\left(1+\overline{Y}_{Q'}\right)\star K_{su}^{Q'Q} - \log\left(1+Y_{Q'}\right)\star \widetilde{K}_{sl}^{Q'Q} \\
& - \sum_{\dot a=1,2} \log\left(1+Y_0^{(\dot a)}\right) \check\star\, \widetilde{K}^{0Q} \\
& - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right) \hat\star\, K_-^{yQ} - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right) \hat\star\, K_+^{yQ}
\end{aligned}$$

Canonical TBA equations

$$-\log Y_0^{(\dot\alpha)} = L\,\widetilde{\mathcal{E}}_0 - \sum_{\dot\beta=1,2} \log\left(1+Y_0^{(\dot\beta)}\right) \check{\star}\, K^{00} - \log\left(1+Y_Q\right) \star K^{Q0} - \log\left(1+\overline{Y}_Q\right) \star \widetilde{K}^{Q0}$$

$$- \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right) \hat{\star}\, K^{y0} - \sum_{\alpha=1,2} \log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right) \hat{\star}\, K^{y0}$$

$$\log Y_-^{(\alpha)} = -\log\left(1+Y_Q\right) \star K_-^{Qy} + \log\left(1+\overline{Y}_Q\right) \star K_+^{Qy} + \sum_{\dot a=1,2} \log\left(1+Y_0^{(\dot a)}\right) \check{\star}\, K^{0y}$$

$$\log Y_+^{(\alpha)} = -\log\left(1+Y_Q\right) \star K_+^{Qy} + \log\left(1+\overline{Y}_Q\right) \star K_-^{Qy} - \sum_{\dot a=1,2} \log\left(1+Y_0^{(\dot a)}\right) \check{\star}\, K^{0y}$$

Ground-state energy

$$E(L) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \frac{d\widetilde{p}^Q}{du} \log\left(\left(1+Y_Q\right)\left(1+\overline{Y}_Q\right)\right)$$

$$- \int_{|u|>2} \frac{\mathrm{d}u}{2\pi} \frac{d\widetilde{p}^0}{du} \log\left(\left(1+Y_0^{(1)}\right)\left(1+Y_0^{(2)}\right)\right)$$

Simplified TBA equations

- introduce the kernel inverse to the kernel $K_{NQ} + \delta_{NQ}$

$$(K+1)^{-1}_{MN} = \delta_{MN} - s\left(\delta_{M+1,N} + \delta_{M-1,N}\right)\,, \qquad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}}$$

$$\sum_{N=1}^{\infty}(K_{QN} + \delta_{QN}) \star (K+1)^{-1}_{NM} = \delta_{QM} = \sum_{N=1}^{\infty}(K+1)^{-1}_{MN} \star (K_{NQ} + \delta_{NQ})$$

- Using various identities it satisfies, we find

$$-\log Y_Q = \log\left(1 + \frac{1}{Y_{Q-1}}\right)\left(1 + \frac{1}{Y_{Q+1}}\right) \star s\,, \qquad Q \geq 2$$

$$-\log \overline{Y}_Q = \log\left(1 + \frac{1}{\overline{Y}_{Q-1}}\right)\left(1 + \frac{1}{\overline{Y}_{Q+1}}\right) \star s$$

  - Eqs for $\overline{Y}_Q$ and Y_Q decouple
  - $Y_{\pm}^{(\alpha)}$ do not appear
  - The same eqs as in $AdS_5 \times S^5$

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  - Eqs for $\overline{Y}_Q$ and Y_Q decouple
  - $Y_{\pm}^{(\alpha)}$ do not appear
  - The same eqs as in $AdS_5 \times S^5$

Simplified TBA equations

- Equations for Y_1 and $\overline{Y}_1$

$$-\log Y_1 = \log\left(1+\frac{1}{Y_2}\right)\star s - \log\left(1-\frac{e^{-i\mu}}{Y_-^{(1)}}\right)\left(1-\frac{e^{i\mu}}{Y_-^{(2)}}\right)\hat{\star}\, s + F_1 \check{\star}\, s$$

$$-\log \overline{Y}_1 = \log\left(1+\frac{1}{\overline{Y}_2}\right)\star s - \log\left(e^{-i\mu}-Y_+^{(1)}\right)\left(e^{i\mu}-Y_+^{(2)}\right)\hat{\star}\, s + \overline{F}_1 \check{\star}\, s$$

- F_1 and $\overline{F}_1$ depend on all Y-functions and various kernels
- In the equation for Y_1 the functions $Y_+^{(\alpha)}$ appear only in F_1
- In the equation for $\overline{Y}_1$ the functions $Y_-^{(\alpha)}$ appear only in $\overline{F}_1$
- Y_1 and $\overline{Y}_1$ couple to each other only through F_1 and $\overline{F}_1$

Y-system equations

- introduce the operator s^{-1} which is the right inverse of s

$$(f \star s^{-1})(u) = \lim_{\epsilon \to 0^+} \left[f(u + \frac{i}{g} - i\epsilon) + f(u - \frac{i}{g} + i\epsilon) \right]$$

- Applying s^{-1} to the simplified equations, we find

$$\frac{Y_Q^+ Y_Q^-}{Y_{Q-1} Y_{Q+1}} = \frac{1}{(1 + Y_{Q-1})(1 + Y_{Q+1})}, \quad Q \geq 2$$

$$\frac{\overline{Y}_Q^+ \overline{Y}_Q^-}{\overline{Y}_{Q-1} \overline{Y}_{Q+1}} = \frac{1}{(1 + \overline{Y}_{Q-1})(1 + \overline{Y}_{Q+1})}, \quad u \in \mathbb{R}$$

$$\frac{Y_1^+ Y_1^-}{Y_2} = \frac{\left(1 - \frac{e^{-i\mu}}{Y_-^{(1)}}\right)\left(1 - \frac{e^{i\mu}}{Y_-^{(2)}}\right)}{1 + Y_2},$$

$$\frac{\overline{Y}_1^+ \overline{Y}_1^-}{\overline{Y}_2} = \frac{\left(e^{-i\mu} - Y_+^{(1)}\right)\left(e^{i\mu} - Y_+^{(2)}\right)}{1 + \overline{Y}_2}, \quad -2 < u < 2$$

- There are no Y-system equations of the standard form for the remaining Y-functions

Summary and open questions

- New solution to the crossing equations for $AdS_3 \times S^3 \times T^4$.

  The general structure of our dressing factors is such that they all include a BES factor times a piece which depends on the γ-rapidities in a simple way.

  - Reconsider the computation of the one-loop dressing factors by using the lattice regularisation with $h = 1/a$ playing the role of the UV cutoff, removing the IR regulator and keeping $\log h$-divergent terms.
  - Use this approach, based on splitting off a BES factor from a rapidity-difference part of the crossing equations, to find solutions for other AdS_3 worldsheet S matrices, e.g. the pure-RR $AdS_3 \times S^3 \times S^3 \times S^1$ background, mixed-flux $AdS_3 \times S^3 \times T^4$, and η-deformed $AdS_3 \times S^3$ models. The main problem is to generalise properly the BES factor. γ-parametrisation will be different too.

Summary and open questions

- Derived mirror TBA equations for $AdS_3 \times S^3 \times T^4$
  - Use them to study the spectrum of excited states numerically for finite h, and analytically (if possible) for small h.
  - Apply them to semi-classical strings to see whether the disagreement found by Abbott and Aniceto is resolved.
  - Use the TBA equations to derive quantum spectral curve equations, and compare them with the recent educated guess

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THANK YOU!