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Solving symplectic groupoid: quantization and integrability

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- Groupoid of upper-triangular matrices: solution in terms of directed networks
- Special quivers and moduli spaces of **closed** Riemann surfaces
- Perspectives: DAHA, knot invariants, and many more

L.Ch., M.Shapiro arXiv:2003.07499v2;
L.Ch. arXiv:2012.10982;
L.Ch, M.Shapiro, H.Shibo arXiv:2101.10323;
previous joint papers with M.Mazzocco and V.Rubtsov

Groupoid of upper triangular matrices

Let $\mathcal{A} \subseteq gl_n$ be a subspace of unipotent upper-triangular matrices in some basis of V_n. We identify elements of this subspace, matrices $\mathbb{A}$, with matrices of bilinear forms in V_n. The matrix $B \in GL_n$ that is a matrix of a change of a basis in V_n transforms $\mathbb{A}$ into $B\mathbb{A}B^{\mathrm{T}}$. We introduce the space of *morphisms* identified with admissible pairs of matrices $(B, \mathbb{A})$ such that

$$\mathcal{M} = \{(B, \mathbb{A}) \,\big|\, B \in GL(V),\ \mathbb{A} \in \mathcal{A},\ B\mathbb{A}B^{\mathrm{T}} \in \mathcal{A}\}.$$

The pair $(B, \mathbb{A})$ admits a standard symplectic structure [Karasev, Weinstein] compatible with the groupoid composition law. This structure indices the Poisson quantum structure on $\mathcal{A}$ (Bondal, 2000) (Nelson, Regge, Ugaglia)

$$\{a_{ik}, a_{jl}\} = 0,\ [a^{\hbar}_{ik}, a^{\hbar}_{jl}] = 0,\ i<k<j<l, \text{ and } i<j<l<k,$$

$$\{a_{ik}, a_{jl}\} = 2\left(a_{ij}a_{kl} - a_{il}a_{kj}\right),\ [a^{\hbar}_{ik}, a^{\hbar}_{jl}] = (q - q^{-1})\left(a^{\hbar}_{ij}a^{\hbar}_{kl} - a^{\hbar}_{il}a^{\hbar}_{kj}\right),\ i<j<k<l,$$

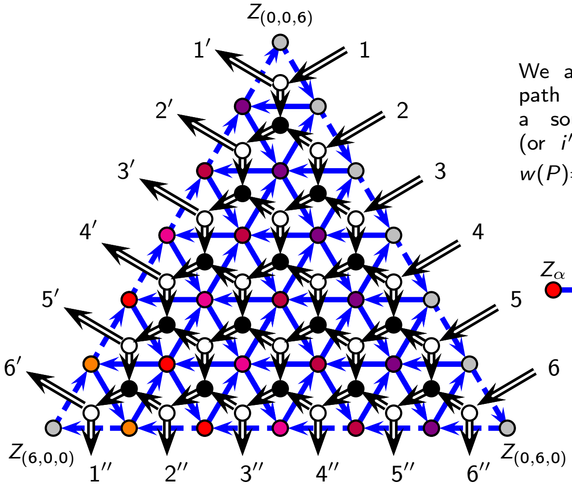
$$\{a_{ik}, a_{kl}\} = a_{ik}a_{kl} - 2a_{il},\ [a^{\hbar}_{kl}, a^{\hbar}_{ik}]_q = (q - q^{-1})a^{\hbar}_{il}\ i<k<l,$$

$$\{a_{ik}, a_{jk}\} = -a_{ik}a_{jk} + 2a_{ij},\ [a^{\hbar}_{ik}, a^{\hbar}_{jk}]_q = (q - q^{-1})a^{\hbar}_{ij}\ i<j<k,$$

$$\{a_{ik}, a_{il}\} = -a_{ik}a_{il} + 2a_{kl},\ [a^{\hbar}_{ik}, a^{\hbar}_{il}]_q = (q - q^{-1})a^{\hbar}_{kl}\ i<k<l.$$

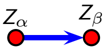
$[X, Y]_q := q^{1/2}XY - q^{-1/2}YX$, **Goldman bracket!**

Directed planar networks: Fock–Goncharov–Shen B_6–Borel subgroup of SL_6

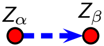


We assign to every oriented path $P : j \rightsquigarrow i'$ from a source j to a sink i' (or i'') the *quantum weight*

$$w(P) = :\prod_{\text{faces } \alpha \text{ lie to the right of the path } P} Z_\alpha :$$



$$Z_\alpha Z_\beta = q^2 Z_\beta Z_\alpha$$



$$Z_\alpha Z_\beta = q Z_\beta Z_\alpha$$

Quantum algebras for planar networks

For **any** planar directed network $\mathcal{N}$, define *transport elements*

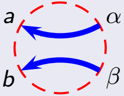
$$\mathcal{M}_{i,j} := \sum_{\text{all paths } j \rightsquigarrow i} (-1)^{\#\,\text{self-intersections}} w(P_{j \rightsquigarrow i})$$

where the sum ranges all paths from the source j to the sink i. This sum is finite for acyclic networks and can be infinite for networks containing cycles.

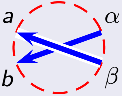
Theorem (Ch-Shapiro'20)

For any (acyclic or cyclic) planar network, we have

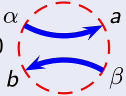
$$[\mathcal{M}_{a,\alpha}, \mathcal{M}_{b,\beta}] = (q - q^{-1})\mathcal{M}_{a,\beta}\mathcal{M}_{b,\alpha}$$



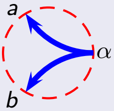
$$[\mathcal{M}_{a,\beta}, \mathcal{M}_{b,\alpha}] = 0$$



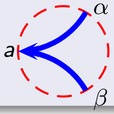
$$[\mathcal{M}_{a,\alpha}, \mathcal{M}_{b,\beta}] = 0$$



$$\mathcal{M}_{a,\alpha}\mathcal{M}_{b,\alpha} = q^{-1}\mathcal{M}_{b,\alpha}\mathcal{M}_{a,\alpha}$$



$$\mathcal{M}_{a,\beta}\mathcal{M}_{a,\alpha} = q^{-1}\mathcal{M}_{a,\alpha}\mathcal{M}_{a,\beta}$$



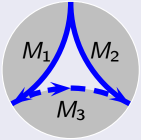
Semiclassical version of these relations, see Gekhtman, Shapiro and Vainstein'09

Quantum algebras for planar networks and groupoid condition

Theorem (Ch-Mazzocco-Rubtsov-M.Shapiro-A.Shapiro-G.Shrader'2018-19)

Quantum monodromy matrices are invariant under quantum MCG transformations and satisfy R-matrix commutation relations of Fock–Rosly type. All these relations follow from elementary commutation relations in $\Sigma_{0,1,3}$:

$$M_i = QS\mathcal{M}_i, \quad S = \sum_i (-1)^i e_{i,n+1-i}$$
$$\overset{1}{M}_1 \otimes \overset{2}{M}_2 = \overset{2}{M}_2 \otimes \overset{1}{M}_1 R_{12}(q),$$
$$R_{12}^T(q) \overset{1}{M}_{1,2} \otimes \overset{2}{M}_{1,2} = \overset{2}{M}_{1,2} \otimes \overset{1}{M}_{1,2} R_{12}(q)$$

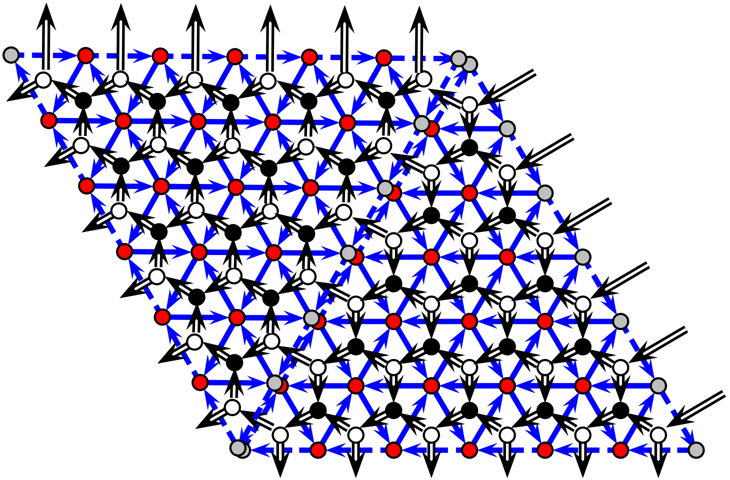


Here M_1 and M_2 are matrices of monodromies for transitions between triangle sides and $R_{12}(q)$ is the quantum R-matrix,

$$R_{12}(q) = q^{-1/k} \left[\sum_{i,j} \overset{1}{e}_{ii} \otimes \overset{2}{e}_{jj} + \sum_i (q-1) \overset{1}{e}_{ii} \otimes \overset{2}{e}_{ii} + \sum_{j>i} (q - q^{-1}) \overset{1}{e}_{ij} \otimes \overset{2}{e}_{ji} \right]$$

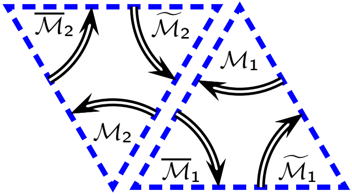
and we have the **quantum groupoid condition** *$M_3 M_1 = M_2$ for sl_n systems, which ensures consistency in the groupoid of paths for monodromies of Fuchsian systems on $\Sigma_{g,s,m}$ [Ch.-Mazzocco-Rubtsov]*

Main example: Fomin–Zelevinsky sl_n network (1998, 2BeforeClusters):



Solving groupoid condition

Let $\mathcal{M}_1$, $\widetilde{\mathcal{M}}_1$, and $\overline{\mathcal{M}}_1$ be (upper-triangular) transport matrices in the left triangle and let $\mathcal{M}_2$, $\widetilde{\mathcal{M}}_2$, and $\overline{\mathcal{M}}_2$ be (lower-triangular) transport matrices in the right triangle. Cluster variables in different triangles Poisson commute.



Theorem (Ch-Shapiro'23)

For the matrix B given by the product of transport matrices $B = \mathcal{M}_2\mathcal{M}_1$, the groupoid condition that both $\mathbb{A}$ and $\widetilde{\mathbb{A}} := B\mathbb{A}B^T$ are upper-triangular is resolved by taking

$$\mathbb{A} = \mathcal{M}_1^{-1} S \widetilde{\mathcal{M}}_2 \widetilde{\mathcal{M}}_1^T S.$$

Then

$$\widetilde{\mathbb{A}} := B\mathbb{A}B^T = S[\overline{\mathcal{M}}_2]^{-1}\left[\overline{\mathcal{M}}_1^T\right]^{-1} S \mathcal{M}_2^T$$

is automatically upper triangular itself.

$A - B$ Poisson algebra

It was proven [Ch.-Mazzocco'18] that the Lie-Poisson algebra of B under the symplectic groupoid condition generates the whole set of algebraic relations. The same relations can be derived directly from the expressions for A, $\widetilde{A}$ in terms of transport matrices. So, we have a theorem

Theorem (Ch-Shapiro'23)

For the matrix B given by the product of transport matrices $B = \mathcal{M}_2\mathcal{M}_1$, and $\mathbb{A}$, $\widetilde{\mathbb{A}}$ as above, the complete set of Poisson relations on B, $\mathbb{A}$, $\widetilde{\mathbb{A}}$ reads

$$\left\{\overset{1}{B}\underset{,}{\otimes}\overset{2}{B}\right\} = -r\overset{1}{B}\overset{2}{B} + \overset{1}{B}\overset{2}{B}r, \qquad \left\{\overset{1}{B}\underset{,}{\otimes}\overset{2}{\mathbb{A}}\right\} = \overset{1}{B}r^T\overset{2}{\mathbb{A}} - \overset{1}{B}\overset{2}{\mathbb{A}}r^{t_2},$$

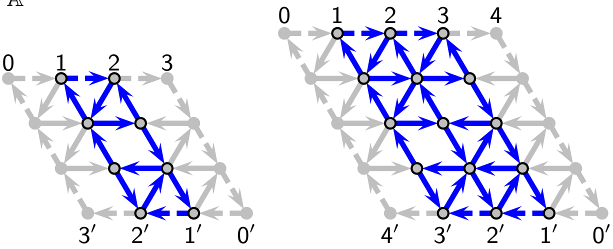
$$\left\{\overset{1}{B}\underset{,}{\otimes}\overset{2}{\widetilde{\mathbb{A}}}\right\} = r^T\overset{1}{B}\overset{2}{\widetilde{\mathbb{A}}} - \overset{2}{\widetilde{\mathbb{A}}}r^{t_2}\overset{1}{B}, \qquad \left\{\overset{1}{\mathbb{A}}\underset{,}{\otimes}\overset{2}{\widetilde{\mathbb{A}}}\right\} = 0,$$

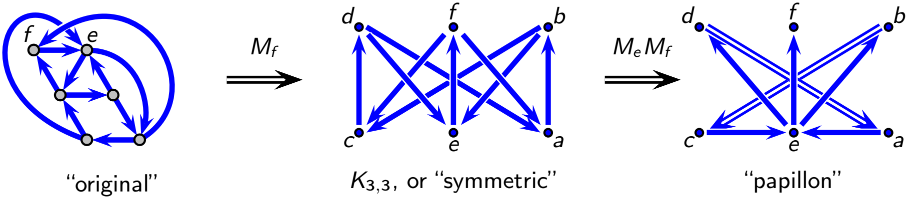
$$\left\{\overset{1}{\mathbb{A}}\underset{,}{\otimes}\overset{2}{\mathbb{A}}\right\} = r\overset{1}{\mathbb{A}}\overset{2}{\mathbb{A}} - \overset{1}{\mathbb{A}}\overset{2}{\mathbb{A}}r - \overset{1}{\mathbb{A}}r^{t_2}\overset{2}{\mathbb{A}} + \overset{2}{\mathbb{A}}r^{t_2}\overset{1}{\mathbb{A}},$$

$$\left\{\overset{1}{\widetilde{\mathbb{A}}}\underset{,}{\otimes}\overset{2}{\widetilde{\mathbb{A}}}\right\} = -r\overset{1}{\widetilde{\mathbb{A}}}\overset{2}{\widetilde{\mathbb{A}}} + \overset{1}{\widetilde{\mathbb{A}}}\overset{2}{\widetilde{\mathbb{A}}}r + \overset{1}{\widetilde{\mathbb{A}}}r^{t_2}\overset{2}{\widetilde{\mathbb{A}}} - \overset{2}{\widetilde{\mathbb{A}}}r^{t_2}\overset{1}{\widetilde{\mathbb{A}}},$$

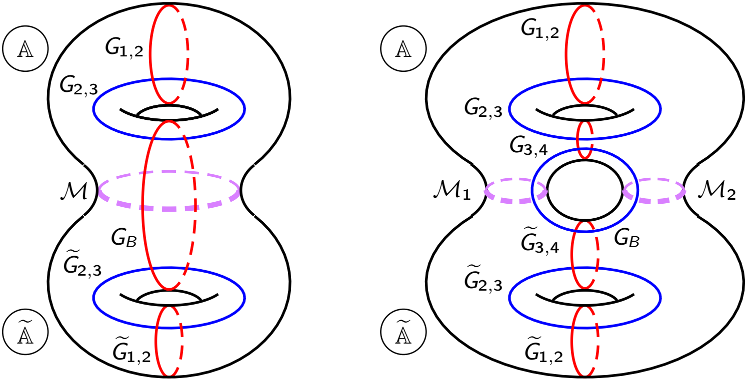
Geometry: moduli spaces of closed Riemann surfaces

Quivers for $\mathbb{A}$

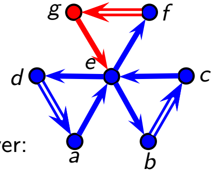




Geometry: moduli spaces of closed Riemann surfaces



Adding a new vertex gives the X_7 quiver:



Theorem (Ch-Shapiro'23)

The Teichmüller space of smooth Riemann surfaces of genus two is the space of real positive cluster variables of the X_7 quiver restricted by $e^2 abcdfg = 1$.

- *All geodesic functions are elements of an upper cluster algebra (positive Laurent polynomials) of the X_7 quiver variables. They are generated via skein and Poisson relations, by five elements:*

$$G_{1,2} = (da)^{1/2}\left(1 + 1/d + 1/(da)\right), \quad \widetilde{G}_{1,2} = (bc)^{1/2}\left(1 + 1/b + 1/(bc)\right),$$
$$G_B = (fg)^{1/2}\left(1 + 1/f + 1/(fg)\right)$$
$$G_{2,3} = (gd)^{-1/2}(1 + 1/a) + (f/a)^{1/2}\widetilde{G}_{1,2} + (gd)^{1/2}(1 + f),$$
$$\widetilde{G}_{2,3} = (gb)^{-1/2}(1 + 1/c) + (f/c)^{1/2} G_{1,2} + (gb)^{1/2}(1 + f)$$

- *Dehn twists = mutations preserving X_7. All these mutations are Poisson morphisms.*

Perspectives

- Analogous constructions work for Teichmüller spaces of smooth Riemann surfaces of higher genus.
- Probing DAHA structures for $g = 2$ Riemann surfaces: Toda Hamiltonians and conjugate cycles; Macdonald polynomials (work in progress with M.Shapiro, A.Shapiro, G. Schreder, R. Kedem, P. Di Francesco)
- Liouville theory: conformal blocks and relation to DOZZ formula
- quantum knot invariants and Vassiliev invariants of hyperbolic knots
- A-polynomials and Topological Recursion relations...